Geometrization of fundamental principles through the eyes of a physical observable

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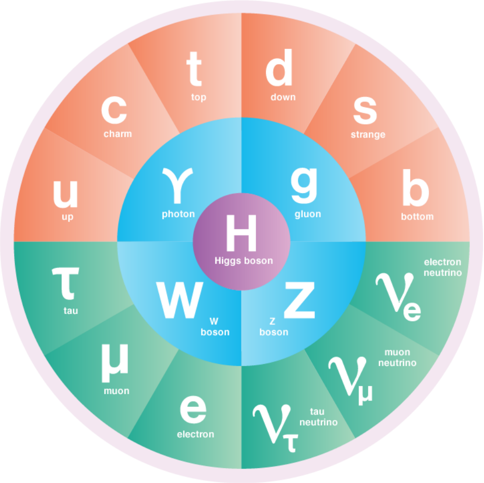
Why is our world so boring?



+ *gravity*

At the fundamental scale simply replicas of the same thing we only have spin 0, $\frac{1}{2}$, 1, and 2

$\sim\sim$ why does nature appears to lack imagination ?

The conventional answer: It is hard to construct a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi + \text{Polynomials}(\phi, \partial\phi)$$

such that

- It is Lorentz invariant
- It is Local (all interactions happen at a point)
- It is Unitary, introduction of gauge invariance, the absence of ghosts, the definite positivity of the Hilbert space, e.t.c

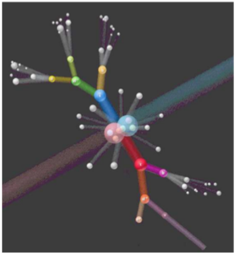
So it is natural that we only have limited cases.

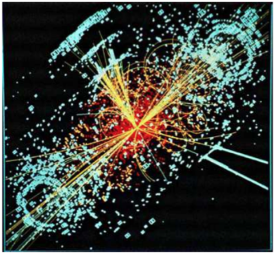
But

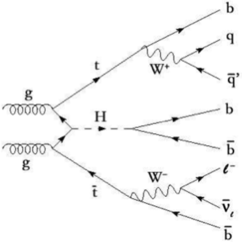
- hard is an adjective, we should be able to say no
- $\mathcal{L}$ is not physical !

$\leftarrow$ *what are we suppose to say*??

Can we discuss the constraint of *Lorentz* invariance and *Unitary* on physical observables? Like the S-matrix



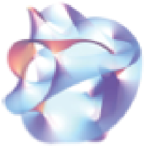




Completely on-shell, “language” independent statements!

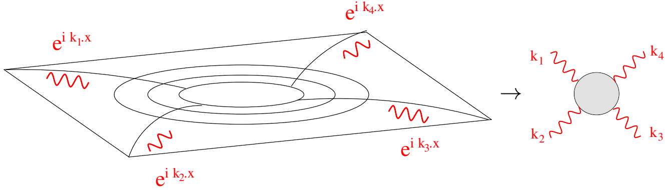
Why S-matrix?

$\langle \phi(x_1)\,, \phi(x_1), \cdots \phi(x_1) \rangle +$

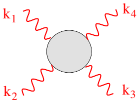


$\leftarrow$ Not gauge invariant!!

Instead, we can consider the scattering of “quantized ripple of space-time” $g^{\mu\nu}$



We can impose constraints directly on the S-matrix as analytic requirements

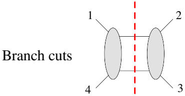


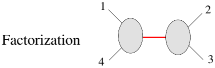
$\sim M_4$

- Lorentz invariance:

$$M_4(p_i \cdot p_j, \epsilon_i \cdot p_j)$$

- Locality+Unitarity:
  Branch cuts and singularities in $(k_a + k_b + \cdots + k_c)^2$ (Locality)





Discontinuities and residues are products $M_n \times M_p$ (Unitarity $i(T^\dagger - T) = TT^\dagger$)

Exercise:

By staring at the S-matrix, can we show that it is impossible to have fundamental particles with spins$>$2?

Fundamental particles means that it has a massless limit

$$k^2 = 0 \quad \to k^{\alpha\dot{\alpha}} = \begin{pmatrix} k^0 - k^3 & k^1 + ik^2 \\ k^1 - ik^2 & k^0 + k^3 \end{pmatrix} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$$

$\alpha, \dot{\alpha} \in$ SL(2,C)∼SO(1,3). The momenta is invariant under

$$\lambda \to t\lambda, \qquad \tilde{\lambda} \to t^{-1}\tilde{\lambda}$$

This is the SO(2)∼U(1) little group that characterize the particle

$$h_i = 0, \pm\frac{1}{2}, \pm 1, \cdots$$

We have the Lorentz invariance building blocks $[12] \sim \sqrt{p_1 \cdot p_2}$

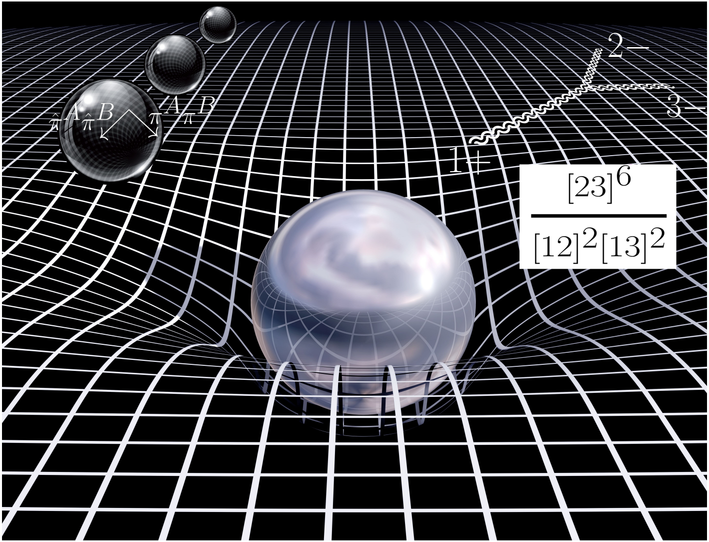
- Lorentz invariance

$$M(1^{h_1}, 2^{h_2}, 3^{h_3}) = [12]^{d_1}[23]^{d_2}[31]^{d_3}, \quad [12] \equiv \lambda_1^\alpha \lambda_2^\beta \epsilon_{\alpha\beta} \qquad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

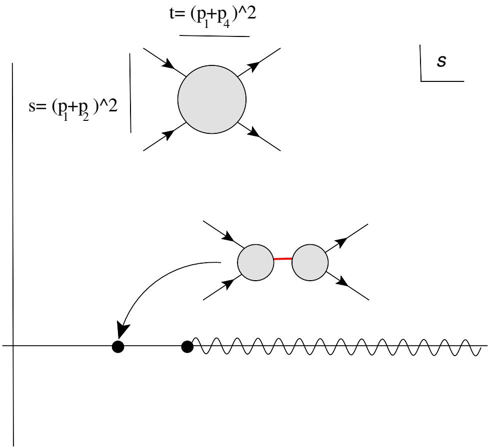
- Little group fixes the amplitude!

$$\text{spin 1} \quad M_3(2^{-1}, 3^{-1}, 1^{+1}) = \frac{[23]^3}{[31][12]}$$

$$\text{spin 2} \quad M_3(2^{-2}, 3^{-2}, 1^{+2}) = \frac{[23]^6}{[31]^2[12]^2}$$


$\hat{\pi}^A \hat{\pi}^B$
$\pi^A \pi^B$
$1+$
$2-$
$3-$
$\frac{[23]^6}{[12]^2[13]^2}$

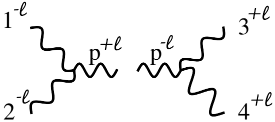
Unitarity is a statement of factorisation: The residues of the pole in the four-point S-matrix must be given by the three-point interaction



$$s + t + u = 0$$

If the product of three-point amplitudes contains poles, consistency with other channels imposes stringent constraint!

Consider the four point amplitude $A(1^{-\ell}2^{-\ell}3^{+\ell}4^{+\ell})$, constructed from the s-channel gluing:



$$\sim \left(\frac{\langle 12\rangle^3}{\langle 1p\rangle\langle p2\rangle}\frac{[34]^3}{[3p][p4]}\right)^{\ell} = \left(\frac{\langle 12\rangle^2[34]^2}{t}\right)^{\ell} \qquad (1)$$

Note the present of t-poles in the denominator.

- For $\ell = 1$, this implies that the amplitude must be written as

$$\frac{\langle 12\rangle^2[34]^2}{st}$$

- For $\ell = 2$, although there is a double pole in t^2, it can be viewed as the degenerate limit for tu since $u + s + t = 0$ in the limit $s = 0$.

$$\frac{\langle 12\rangle^4[34]^4}{stu}$$

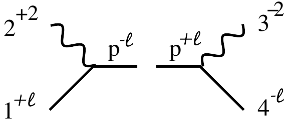
- For $\ell > 2$, one simply have too high a power of poles and one does not have a self interacting theory in four-dimensions.

Rules out weakly coupled massless higher spin theories

Are free-theories safe?

no

In the presence of gravity one must have minimal coupling $\rightarrow A(1^{-\ell}2^{-2}3^{+2}4^{+\ell})$.
Again from the *s*-channel we have:



$$\sim \frac{[12]^{2+2\ell}}{[p1]^2[2p]^{2\ell-2}} \frac{\langle 34\rangle^{2+2\ell}}{\langle p4\rangle^2\langle 3p\rangle^{2\ell-2}} = \frac{[12]^{2\ell}\langle 34\rangle^4}{\langle 24\rangle^2[24]^{2\ell-2}} \quad (2)$$

The factor $\langle 24\rangle^2$ in the denominator implies a $1/u^2$ pole, reflecting the exchange in u channel as well as the graviton exchange in the t-channel. However, beyond spin-2 one has extra factors of [24] in the denominator beyond necessary for the $1/u^2$ pole. Thus one concludes that free massless higher-spin theory is inconsistent in the presence of gravity.

Similarly charged $\ell > 1$ massless particles cannot exist.

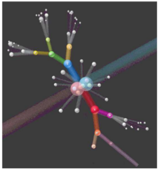
The fact that all particles couple to gravity, we can only have:

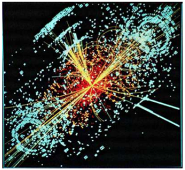
$$0, \quad \frac{1}{2}, \quad 1, \quad \frac{3}{2}, \quad 2$$

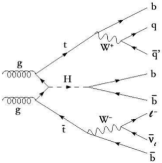
Nature does not lack imagination, it has its hands tied!

The important lessons:

- Consequence of physical principles can be most straight forwardly extracted by imposing or framing it on physical observables



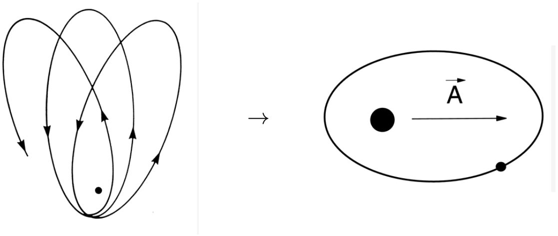




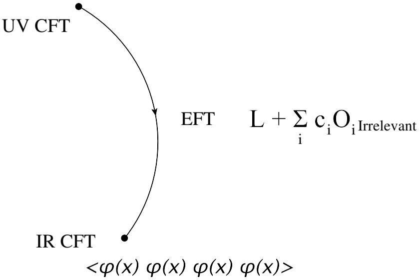
- The space of possible theories = The space of possible physical observables ← constrained by the desired principles (unitarity, locality... e.t.c)

We are then forced to ask:

- What is this space?
- How does symmetry, locality, and unitarity manifest itself in this space?
- Is it possible that these properties are unified?
- Is there new structure that was hidden before? Ample of examples: from the Runge-Lentz vector, to the recent dual conformal symmetry of maximal super Yang-Mills



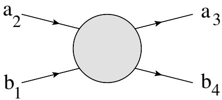
What is this space?



- EFT: S-matrix, coefficient of higher-dimensional (irrelevant) operators
- CFT: Four point correlation function, conformal dimension and three-point coupling of primary operators.

EFT

In the IR the UV degrees of freedom are encoded in the higher dimensional operators. These information are encoded in the four-point function as



$$M(s,t) = \sum_{i,j} g_{i,j} s^i t^j$$

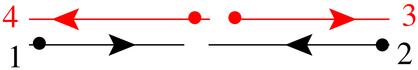
where $s = (p_1 + p_2)^2$ $t = (p_1 + p_4)^2$. For example:

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + a(\partial\phi\cdot\partial\phi)^2 \to M(s,t) = a(s^2 + st + t^2)$$

The space of possible theories = The space of possible $g_{i,j}$

A simple model

Lets consider the scattering amplitude of four-particles in $1+1$-dimensions



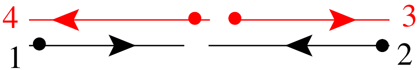
Irrespective of the degrees of freedom of the ultimate fundamental theory, the scattering process at low energy can be approximated as:

$$M(E) = g_0 + g_1 E + g_2 E^2 + \cdots = \sum_{i=0}^{\infty} g_i E^i$$

Question: given a set of $\{g_i\}$s, is there a way to see the underlying theory is unitary ?

A simple model

Lets consider the scattering amplitude of four-particles in $1+1$-dimensions



Irrespective of the degrees of freedom of the ultimate fundamental theory, the scattering process at low energy can be approximated as:

$$M^{IR}(E) = g_0 + g_1 E + g_2 E^2 + \cdots = \sum_{i=0}^{\infty} g_i E^i$$

In the UV, from unitarity, the function should take the form:

$$M(E) = \sum_a -\frac{c_a}{E - m_a^2} \quad \text{unitarity} : c_a > 0$$

From this we see that the low energy amplitude must be expressible as

$$M^{IR}(E) = \sum_a \frac{c_a}{m_a^2}\left(1 + \frac{E}{m_a^2} + \left(\frac{E}{m_a^2}\right)^2 + \cdots\right)$$

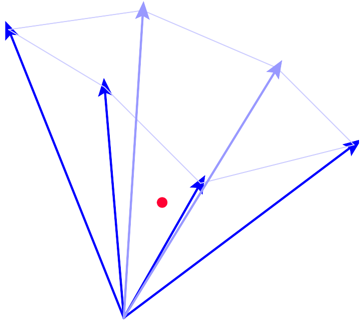
A simple model

We must have

$$M^{IR}(E) = g_0 + g_1 E + g_2 E^2 + \cdots = \sum_a \frac{c_a}{m_a^2}\left(1 + \frac{E}{m_a^2} + \left(\frac{E}{m_a^2}\right)^2 + \cdots\right)$$

In other words the point $\vec{g} = \{g_0, g_1, g_2, \cdots\}$

$$\vec{g} = \sum_a c_a \left(\frac{1}{m_a^2}, \frac{1}{m_a^4}, \frac{1}{m_a^6}, \cdots\right) \equiv \sum_a c_a \vec{v}_a$$



The space of allowed $\vec{g}$ is inside a polytope with the vertices given by $\vec{v}_a$s!

A simple model

Is this polytope special ? Yes! the vertices are given by points on a moment curve

$$\vec{v}_a = \left(\frac{1}{m_a^2}, \frac{1}{m_a^4}, \frac{1}{m_a^6}, \cdots \right) \sim (1, t, t^2, t^3, \cdots)$$

Cyclic polytope

From Wikipedia, the free encyclopedia

In mathematics, a **cyclic polytope**, denoted $C(n,d)$, is a convex polytope formed as a convex hull of n distinct points on a rational normal curve in $\mathbf{R}^d$, where n is greater than d. These polytopes were studied by Constantin Carathéodory, David Gale, Theodore Motzkin, Victor Klee, and others. They play an important role in polyhedral combinatorics: according to the upper bound theorem, proved by Peter McMullen and Richard Stanley, the boundary $\Delta(n,d)$ of the cyclic polytope $C(n,d)$ maximizes the number f_i of i-dimensional faces among all simplicial spheres of dimension $d - 1$ with n vertices.

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2 Gale evenness condition
3 Neighborliness
4 Number of faces
5 Upper bound theorem
6 See also
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Definition [edit]

The moment curve in $\mathbb{R}^d$ is defined by

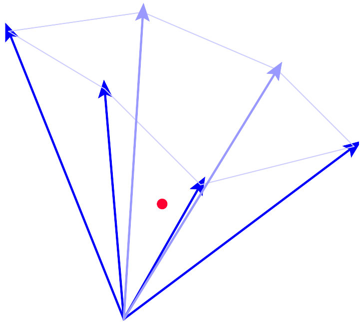
$\mathbf{x} : \mathbb{R} \to \mathbb{R}^d, \mathbf{x}(t) := [t, t^2, \ldots, t^d]^T$.[1]

A simple model

This geometry is also useful !

Let's say given n vectors $\vec{v}_a$, to compute the region of the polytope we need to

- Determine which one of these $\vec{v}_a$s are vertices
- Amongst the vertices, determine all the set that constitute boundary facets



The complexity is n^d

A simple model

This geometry is also useful !

However, since our vectors $\vec{v}_a$ are points on a moment curve,

$$\{0.2, (0.2)^2, (0.2)^3, \cdots\}, \quad \{14.6, (14.6)^2, (14.6)^3, \cdots\}$$

We know that

- All $\vec{v}_a$s are vertices
- The boundaries are all known.

A simple model

The constraint on the couplings becomes simple. Organizing the couplings into the Hankel matrix

$$K(g') = \begin{pmatrix} 1 & g'_1 & \cdots & g'_{p-1} \\ g'_1 & g'_2 & \cdots & g'_p \\ \vdots & \vdots & \vdots & \vdots \\ g'_{p-1} & g'_p & \cdots & g'_{2p-2} \end{pmatrix},$$

Then we simply have

$$i \in even: \quad Det \begin{pmatrix} 1 & g'_1 & \cdots & g'_{\frac{i}{2}} \\ g'_1 & g'_2 & \cdots & g'_{\frac{i}{2}+1} \\ \vdots & \vdots & \vdots & \vdots \\ g'_{\frac{i}{2}} & g'_{\frac{i}{2}+1} & \cdots & g'_i \end{pmatrix} \geq 0, \qquad i \in odd: \quad Det \begin{pmatrix} g'_1 & g'_2 & \cdots & g'_{\frac{i+1}{2}} \\ g'_2 & g'_3 & \cdots & g'_{\frac{i+3}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ g'_{\frac{i+1}{2}} & g'_{\frac{i+3}{2}} & \cdots & g'_i \end{pmatrix} \geq 0$$

A simple model

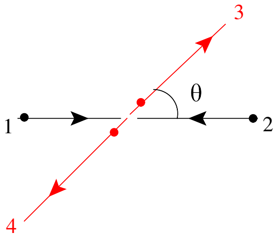
Summary:

The statement of unitarity for $1 + 1$-dimensional scattering is translated into the geometric property that the coefficients of the expansion in s (center of mass energy) lies within the convex hull of points on moment curves.

What about higher D and Lorentz symmetry?

General dimensions

What about higher D and Lorentz symmetry?

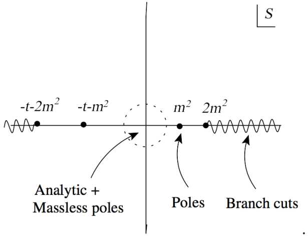


The scattering amplitude is now a function of two variables: $M(s,t)$

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_4)^2 = -\frac{s}{2}(1 - \cos\theta)$$

General dimensions

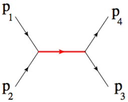
For fixed *t* the analytic structure of M(s,t) is



From basic unitarity constraint, we know the residue and discontinuity

General dimensions

- Lorentz invariance + Unitarity dictates



$$\rightarrow \quad A_3(\phi_1, \phi_2, h^\ell) \sim i c_\ell (p_1 - p_2)^{\mu_1} (p_1 - p_2)^{\mu_2} \cdots (p_1 - p_2)^{\mu_\ell} \epsilon_{\mu_1 \mu_2 \cdots \mu_\ell}$$

The residue must take the form ($X \equiv p_1 - p_2, \; Y \equiv p_3 - p_4$):

$$X^{\mu_1} X^{\mu_2} \ldots X^{\mu_\ell} \mathcal{P}_{\mu_1 \cdots \mu_\ell \nu_1 \cdots \nu_\ell} Y^{\nu_1} Y^{\nu_2} \ldots Y^{\nu_\ell}$$

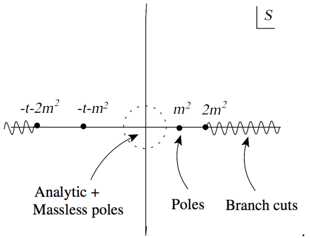
where $\mathcal{P}_{\mu_1 \cdots \mu_\ell \nu_1 \cdots \nu_\ell}$ is symmetric traceless. This implies

$$\Box_X f(X, Y) = \delta^{D-1}(X - Y) \rightarrow \frac{1}{|1 - \cos\theta t + t^2|^{D-3/2}} = \sum_\ell t^\ell G_\ell^{\frac{D-3}{2}}(\cos\theta)$$

set $\alpha \equiv \frac{D-3}{2}$

General dimensions

For fixed *t* the analytic structure of M(s,t) is



From unitarity constraint, we know the residue and discontinuity

$$Res_{s\to m^2} M(s,t) = \sum_\ell c^2_{m^2\ell} G^\alpha_\ell(\cos\theta = 1 + 2\frac{t}{m^2})$$

$$Dis_{s\to 2m^2} M(s,t) = \sum_\ell c^2_{m^2\ell} G^\alpha_\ell(\cos\theta = 1 + 2\frac{t}{m^2}),$$

General dimensions

This implies that N. Arkani-Hamed Y-t Huang, T-z Huang

$$M^{IR}(s,t) = -\left\{\sum_i \left[\sum_{\ell_i} \frac{c_{\ell_i}^2 G_{\ell_i}^{\alpha}(1+2\frac{t}{m_i^2})}{s-m_i^2} + \int_{2m_i^2} ds' \frac{|f_{\ell_i}(s')|^2 G_{\ell_i}^{\alpha}(1+2\frac{t}{s'})}{s-s'}\right] + \{u\} + B\right\}$$

B is the boundary term, which is bounded by causality to be $< s^2$

Now $M^{IR}(s,t)$ has a polynomial representation,

$$M^{IR}(s,t) = \sum_{i,j} g_{i,j} s^i t^j$$

where $g_{i,j}$ encodes the information of the coefficients of higher dimension operators:

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + a(\partial\phi\cdot\partial\phi)^2 \rightarrow M^{IR}(s,t) = a(s^2+st+t^2)$$

General dimensions

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$$M^{IR}(s,t) = -\left\{\sum_i \left[\sum_{\ell_i} \frac{c^2_{\ell_i} G^\alpha_{\ell_i}(1+2\frac{t}{m_i^2})}{s-m_i^2} + \int_{2m_i^2} ds' \frac{|f_{\ell_i}(s')|^2 G^\alpha_{\ell_i}(1+2\frac{t}{s'})}{s-s'}\right] + \{u\} + B\right\}$$

B is the boundary term, which is bounded by causality to be $< s^2$

We have an identity relating the coefficients of the EFT to the polynomial expansion of the RHS with

$$G^\alpha_\ell(1+x) = \sum_{i=0}^{\ell} v^\alpha_{\ell,i} x^i$$

where the vector $\vec{v}^\alpha_\ell = (v^\alpha_{\ell,0},\ v^\alpha_{\ell,1},,\ v^\alpha_{\ell,2},\cdots)$ take the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 \\ 0 & 0 & \frac{3}{2} & \frac{15}{2} & \frac{45}{2} & \frac{105}{2} & 105 & 189 \\ 0 & 0 & 0 & \frac{5}{2} & \frac{35}{2} & 70 & 210 & 525 \\ 0 & 0 & 0 & 0 & \frac{35}{8} & \frac{315}{8} & \frac{1575}{8} & \frac{5775}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{63}{8} & \frac{693}{8} & \frac{2079}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{231}{16} & \frac{3003}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{429}{16} \end{pmatrix}.$$

All v is positive !

General dimensions

This implies that N. Arkani-Hamed Y-t Huang, T-z Huang

$$M^{IR}(s,t) = -\left\{\sum_i \left[\sum_{\ell_i} \frac{c^2_{\ell_i} G^{\alpha}_{\ell_i}(1+2\frac{t}{m_i^2})}{s-m_i^2} + \int_{2m_i^2} ds' \frac{|f_{\ell_i}(s')|^2 G^{\alpha}_{\ell_i}(1+2\frac{t}{s'})}{s-s'}\right] + \{u\} + B\right\}$$

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$$G^{\alpha}_{\ell}(1+x) = \sum_{i=0}^{\ell} v^{\alpha}_{\ell,i} x^i$$

We have

$$-\sum_a \left(\frac{1}{s-m_a^2}\right) c_a^2 G^{\alpha}_{\ell_a}(1+2t/m_a^2) = \sum_a c_a^2 \left[\sum_{p,q} \frac{s^p}{m_a^{2(p+1)}} v_{\ell_a,q} \left(\frac{2t}{m_a^2}\right)^q\right]$$

both c^2 and v is positive

General dimensions

$$-\sum_a \left(\frac{1}{s-m_a^2}\right) c_a^2 G_{\ell_a}^{\alpha}(1+2t/m_a^2) = \sum_a c_a^2 \left[\sum_{p,q} \frac{s^p}{m_a^{2(p+1)}} v_{\ell_a,q} \left(\frac{2t}{m_a^2}\right)^q\right]$$

both c^2 and v is positive.

Let's consider the following two scenarios:

- Fixed q: higher dimension operators of generated by a lower bound in spin

  $$g_{0,4}\, t^4 + g_{1,4}\, st^4 + g_{2,4}\, s^2 t^4 + g_{3,4}\, st^4 \quad \to \quad \vec{g} = \{g_{0,4}, g_{1,4}, g_{2,4}, g_{3,4}\}$$

  $\vec{g}$ must lie in the polytope built from points on a moment curve, just as the $1+1$-dimension toy model!

- Fixed $p+q=k$: higher dimension operators of fixed mass-dimension

$$g_{4,0}\, s^4 + g_{3,1}\, s^3 t + g_{2,2}\, s^2 t^2 + g_{1,3}\, st^3 + g_{0,4}\, t^4 \quad \to \quad \vec{\alpha} = \{g_{4,0}, g_{3,1}, g_{2,2}, g_{1,3}, g_{0,4}\}$$

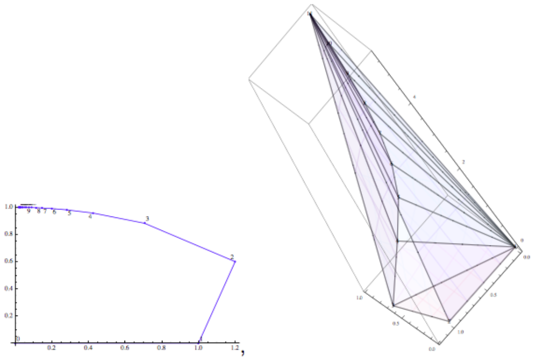
General dimensions

- Fixed $p + q = k$: higher dimension operators of fixed mass-dimension

$$\vec{\alpha} = \sum_a c_a^2 \vec{v}_{\ell_a}$$

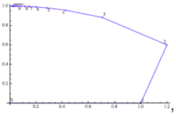
The constraint is more practically given by

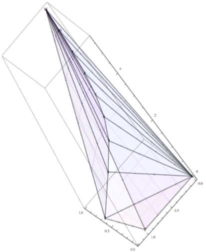
$$\vec{\alpha} \cdot \vec{\mathcal{W}}_i > 0, \forall\, i$$



General dimensions

$$\vec{\alpha} \cdot \vec{\mathcal{W}}_i > 0, \forall\, i$$





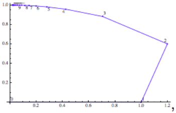
Naively, this is complicated, since there is in principle an infinite number of vertices. However, we show that the polytope is in fact again a cyclic polytope !

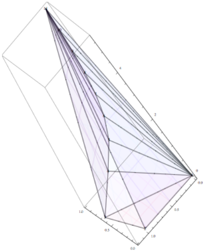
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 \\ 0 & 0 & \frac{3}{2} & \frac{15}{2} & \frac{45}{2} & \frac{105}{2} & 105 & 189 \\ 0 & 0 & 0 & \frac{5}{2} & \frac{35}{2} & 70 & 210 & 525 \\ 0 & 0 & 0 & 0 & \frac{35}{8} & \frac{315}{8} & \frac{1575}{8} & \frac{5775}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{63}{8} & \frac{693}{8} & \frac{2079}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{231}{16} & \frac{3003}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{429}{16} \end{pmatrix}.$$

$$det[\vec{v}^{\alpha}_{\ell_1}\, \vec{v}^{\alpha}_{\ell_2} \cdots] > 0, \quad \text{for} \ell_1 > \ell_2 > \cdots$$

General dimensions

$$\vec{\alpha}\cdot\vec{\mathcal{W}}_i > 0, \forall\, i$$





Since it is a cyclic polytope, we know all facets:

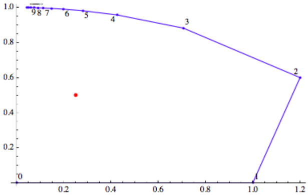
$$k \in even, \quad \langle\vec{\alpha} i, i{+}1\cdots j, j+1\rangle > 0, \qquad k\in odd, \quad \langle\vec{\alpha}, 0, i, i+1\cdots\rangle > 0\,,$$

For example

$$\langle\vec{\alpha}01\rangle > 0 \to \alpha_2 > 0, \quad \langle\vec{\alpha}12\rangle > 0 \to \frac{3}{2} - \frac{3\alpha_1}{2} + 2\alpha_2 > 0, \quad \langle\vec{\alpha}23\rangle > 0 \to \frac{27}{2} - 6\alpha_1 + 3\alpha_2 > 0$$
$$\langle\vec{\alpha}34\rangle > 0 \to 60 - 15\alpha_1 + 4\alpha_2 > 0, \quad \langle\vec{\alpha}45\rangle > 0 \to \frac{375}{2} - 30\alpha_1 + 5\alpha_2 > 0\,. \qquad (4.21)$$

A simple example, consider string theory in flat space:

$$\frac{\Gamma[-\alpha s]\Gamma[-\alpha t]}{\Gamma[1-\alpha s-\alpha t]} = \cdots + \frac{\pi^4}{90}\left(s^2 + \frac{st}{4} + t^2\right) + \mathcal{O}(\alpha^3)$$



Most importantly, we've seen that

- Lorentz-symmetry: In the form of fixing the residue basis to be

$$G_\ell^D(\cos\theta)$$

- Unitarity: In the form of residue having positive coefficients
- Locality: In the form of

$$\frac{1}{s-m_a}, \quad \text{or} \int ds' \, \frac{1}{s-s'}$$

Is unified to a simple geometric statement:

In the space of all possible couplings for higher-dimensional (irrelevant) operators, a consistent theory is bounded by cyclic polytopes.

Generalizations

Spinning polytopes:

The same structure is found for when the external states are massless with spins: photons, gauge bosons, and gravitons.

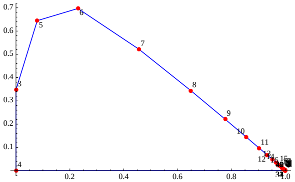
- Lorentz-symmetry: In the form of fixing the residue basis to be Wigner $d^j_{m',m}(\theta) = \langle j, m'|e^{-i\theta \mathcal{J}_y}|j, m\rangle$, or equivalently

$$d^j_{m',m}(\theta) = \mathcal{J}(\ell + 4h, 0, -4h, \cos\theta)$$

- Unitarity: In the form of residue having positive coefficients
- Locality: In the form of

$$\frac{1}{s - m_a}, \quad \text{or} \int ds' \frac{1}{s - s'}$$

Out[120]=



Generalizations

Remarkably, the same polytope is present for CFT four-point function! w Nima, Shu-Heng Shao

Consider the a 1D four-point function:

$$\langle \phi(1)\phi(2)\phi(3)\phi(4) \rangle \equiv F(z)$$

$$F(z) = \sum_{\Delta} p_{\Delta}^2 C_{\Delta}(z), \quad C_{\Delta}(z) = z^{\Delta}\, {}_2F_1(\Delta, \Delta, 2\Delta, z)$$

We can again expand the four-point function, say around $z = \frac{1}{2}$

$$F\left(\frac{1}{2} + y\right) = \sum_{q=0}^{\infty} F_q y^q$$

The 1-D blocks also yield an infinite set of vectors

$$C_{\Delta}\left(\frac{1}{2} + y\right) = \sum_{q=0}^{\infty} c_{\Delta,q} y^q$$

Unitarity then requires that

$$\mathbf{F} = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{L-1} \end{pmatrix} \subset \sum_{\Delta} p_{\Delta}^2 \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix}$$

Generalizations

Now crossing is just

$$z^{-2\Delta_\phi} F(z) = (1-z)^{-2\Delta_\phi} F(1-z) \to F(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi} F(1-z)$$

Again expanded around $z = \frac{1}{2}$ we find

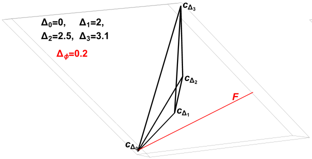
$$\sum_q F_q y^q = \left(\frac{1+2y}{1-2y}\right)^{2\Delta_\phi} \sum_q (-)^q F_q y^q$$

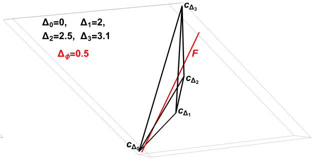
This tells us that **F** must lie within the crossing plane X

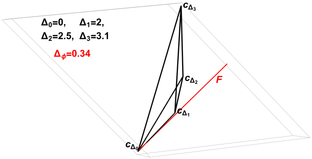
We have the polytope $P(\Delta_i) = \sum_i p^2_{\Delta_i} c_{\Delta_i}$ and a crossing plane $X(\Delta_\phi)$, and they must intersect. $P(\Delta_i)$ is a cylic polytope! See Nima's talk

Generalizations

For example:



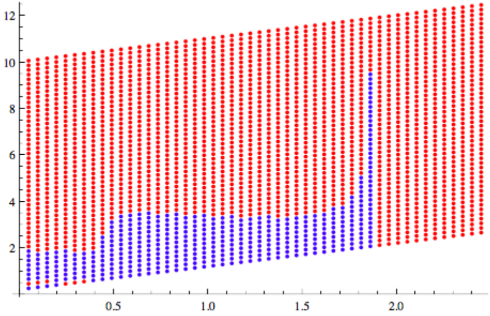




Generalizations

The allows us to “carve” out the space of consistent CFTs.

Exp, given $\Delta_\phi = 0.3$, in the space of possible lowest first two operators (Δ_1, Δ_2) are given by:



Conclusions

- It pays to (speak) and (think) in terms of physical observables !
- Unitarity $\oplus$ Locality $\oplus$ Symmetries unifies into a geometric property
- The low energy expansion of the four-point function must live in an infinite dimension space whose boundaries are those of cyclic polytopes.
- The expansion of the four-point correlation function for CFTs, must live in an infinite dimension space whose boundaries are those of cyclic polytopes.
- Similar structure must exists in other observables, exp: cosmological correlators?
- Utilize these constraints to bound possible BSM physics, and test various conjecture with respect to quantum gravity (weak gravity)