

Generalized symmetries and their applications to topological phases of matter

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Phys.Rev.Lett. 136 (2026) 4, 040402, with Masahito Yamazaki (UTokyo & Kavli IPMU) and Linhao Li (PSU)



2024/12



2026/06



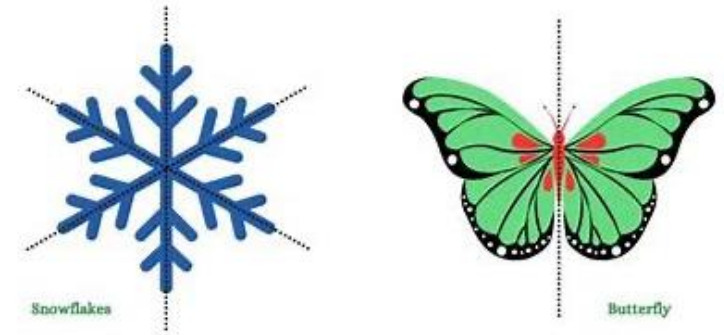
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AND TECHNOLOGY

date: 2026/04/21

1. Review of recent developments on generalized symmetries
2. My work on noninvertible symmetries and related phases
3. Conclusion and some Outlook

Symmetry

- A pattern invariant under certain manipulations (reflection, rotation, translation...)
- Transformations that leave the Lagrangian/action invariant
- Operators that commutes with the Hamiltonian
- ...



—How to characterize ordinary (**internal**) symmetry?

1. Symmetry action on **whole space** → symmetry operators live on a **co-dimensional one** manifold
2. Symmetry **commute** with the Hamiltonian / energy-momentum tensor → symmetry operators are **topological** in space
3. The **composition** of two symmetry actions is another symmetry action → symmetry operators are labelled by **group** elements

Ordinary symmetry → **Co-dimension one topological** operators with **group** multiplication laws

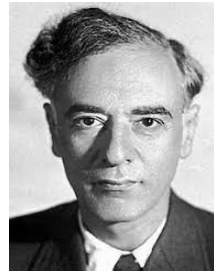
Power of Ordinary Symmetry



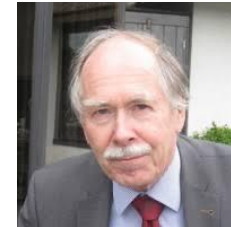
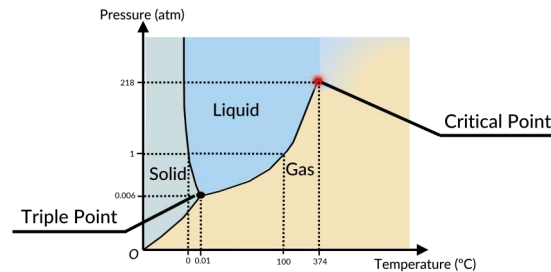
Noether's theorem
Conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0,$$

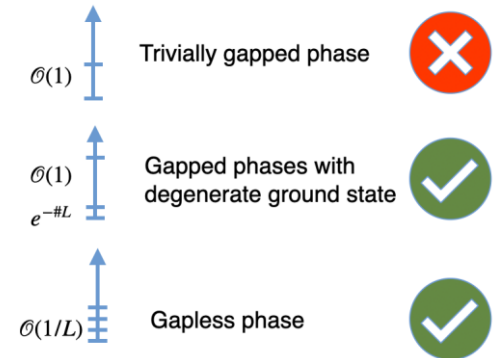
$$Q = \int dx^3 \rho(x)$$



Landau's paradigm
Characterization of Phases
of matter



't Hooft anomaly
Constraints in the RG flow



Ordinary symmetry is already very powerful.

But do we exhaust all possible types of symmetries?

In recent years, new generalized symmetries are indeed been found, together with plenty of applications

Generalized symmetry

Ordinary symmetry \rightarrow **Co-dimension one topological** operators with **group** multiplication laws



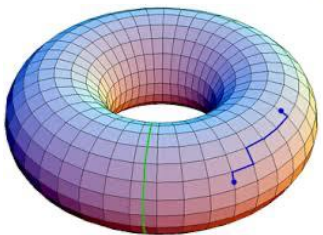
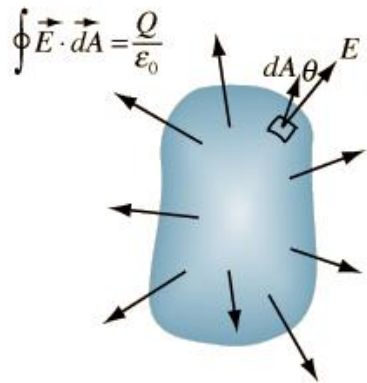
Higher form symmetry

Topological operators are defined on co-dimension- $(p + 1)$ surface.

Example:

1. Gauss law in free Maxwell equation \rightarrow conservation law for $U(1)$ one form symmetry

2. Toric code, as \mathbb{Z}_2 one form symmetry breaking phases, is stable under local perturbations
 \rightarrow Suitable for topological quantum computation.

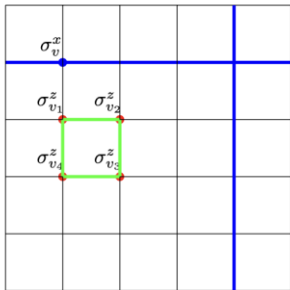


Generalized symmetry

Ordinary symmetry → **Co-dimension one topological** operators with **group** multiplication laws

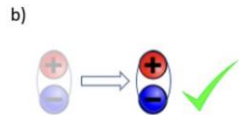
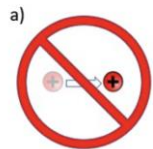


Modulated symmetry



Symmetry operators are topological in the time direction, but is **not topological** in space direction

Example: Dipole symmetry, subsystem symmetry, fractal symmetry

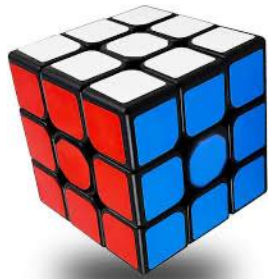


Fracton phases of matter: 1. Large ground state degeneracy;
2. Quasiparticles with restricted mobility

→ theoretically proposed as hardware for quantum computer.

Generalized symmetry

Ordinary symmetry \rightarrow *Co-dimension one topological* operators with *group* multiplication laws



Noninvertible symmetry

Symmetry operators may not have an inverse

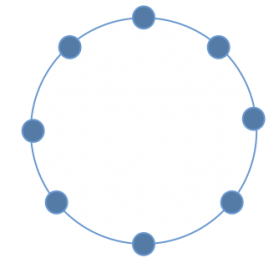
$$\mathcal{L}_a \otimes \mathcal{L}_b = \bigoplus_c N_{ab}^c \mathcal{L}_c, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

Example: $\text{Rep}(G)$ fusion category for a non-abelian group G

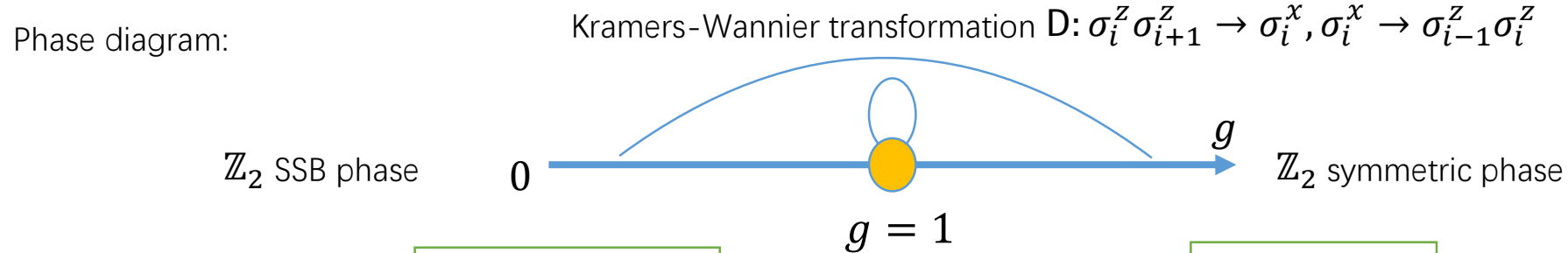
Object: irreducible representations (irreps) of G

Fusion: Tensor product of irreps

The **simplest** noninvertible symmetry on lattice



— Transverse field Ising model: $H(g) = -\sum_i (g^{-1} \sigma_i^z \sigma_{i+1}^z + g \sigma_i^x)$ w/ \mathbb{Z}_2 spin flip symmetry: $U = \prod_{j=1}^L \sigma_j^x$



Ground state constraints:

$$\sigma_i^z \sigma_{i+1}^z = 1, \forall i$$

$$|\uparrow\uparrow\dots\uparrow\rangle$$

$$|\downarrow\downarrow\dots\downarrow\rangle$$

$$\sigma_i^x = 1, \forall i$$

$$|+\dots+\rangle$$

$$|+\rangle := \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$$

D is noninvertible: $D|\uparrow\uparrow\dots\uparrow\rangle = D|\downarrow\downarrow\dots\downarrow\rangle = |+\dots+\rangle \rightarrow D|\psi\rangle = 0, |\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\dots\uparrow\rangle - |\downarrow\downarrow\dots\downarrow\rangle)$

D becomes a **noninvertible symmetry** at the critical point $g = 1$

$$D \times D^\dagger = 1 + U$$

The **simplest** noninvertible symmetry on lattice



Statistics of the Two-Dimensional Ferromagnet. Part I

H. A. KRAMERS, *University of Leiden, Leiden, Holland*

AND

G. H. WANNIER, *University of Texas, Austin, Texas*¹

(Received April 7, 1941)

It is shown that these matrices possess a **symmetry property** which permits location of the Curie temperature if it exists and is unique. It lies at

$$J/kT_c = 0.8814$$

if we denote by J the coupling energy between neighboring spins. The **symmetry** relation also excludes certain forms of singularities at T_c , as, e.g., a jump in the specific heat.

Kramers and Wannier already mentioned about this symmetry in their 1941 paper.

But the ***ubiquitous existence*** of noninvertible symmetry in other areas, such as *CFT*, *TQFT*, *QFT* and *lattice systems* are only appreciated recently

Ubiquitous existence of noninvertible symmetry



Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories

*J. Kaidi, K. Ohmori, Y. Zheng, **Phys.Rev.Lett.** 128 (2022) 11, 111601*



Noninvertible Global Symmetries in the Standard Model

*Y. Choi, H. T. Lam, S. Shao, **Phys.Rev.Lett.** 129 (2022) 16, 161601*



Categorical Landau Paradigm for Gapped Phases

*L. Bhardwaj, L. E Botany, D. Pajer, S. Schafer-Nameki, **Phys.Rev.Lett.** 133 (2024) 16, 161601*



Noninvertible Symmetries Act Locally by Quantum Operations

*M. Okada, Y. Tachikawa, **Phys.Rev.Lett.** 133 (2024) 19, 191602*

High energy
theory

High energy
pheno

Condensed matter
theory

Quantum
information

Ubiquitous existence of noninvertible symmetry



[Clay Córdova](#)



[Thomas Dumitrescu](#)



[Shu-Heng Shao](#)



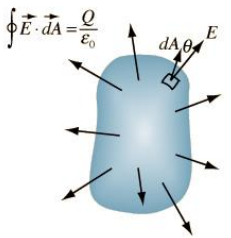
[Yifan Wang](#)

2026 New Horizons in Physics Prize

For generalizing the notion of symmetry in various ways, and for exploring the consequences of these generalized symmetries, in quantum field theory, particle physics, condensed matter physics, string theory, and quantum information theory.

Generalized symmetry

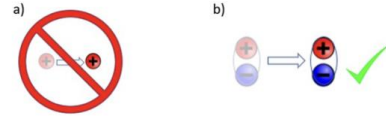
Ordinary symmetry \rightarrow **Co-dimension one topological** operators with **group** multiplication laws



Higher form symmetry

Modulated symmetry
(Subsystem symmetry, dipole symmetry)

Noninvertible symmetry

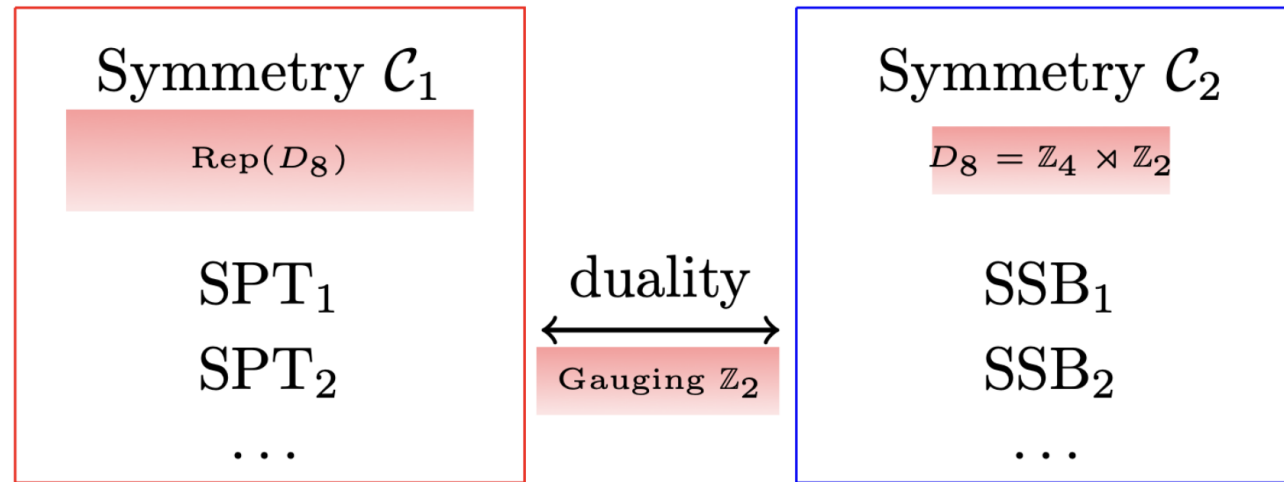


Applications:

Spectrum degeneracies. Selection rules. Constraints on RG flow. New phases of matter.

- My focus (1) Find new generalized **symmetries**
(2) classify new **phases** with generalized symmetries
(3) construct **lattice models** to realize these phases

Take home message: Duality method

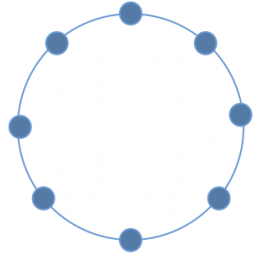


- (1). **Duality/gauging** can link a group symmetry to a noninvertible symmetry
- (2). Duality works as a “dictionary” by keeping the phase diagram structure (one-to-one correspondence of phases on each side)

Well-known example: Kramers-Wannier transformation (\mathbb{Z}_2 Sym \leftrightarrow \mathbb{Z}_2 SSB)

1. Review of recent developments on generalized symmetries
2. My work on noninvertible symmetries and related phases
3. Conclusion and some Outlook

Application 1: find new symmetries



Kramers-Wannier transformation as gauging \mathbb{Z}_2

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z + \sigma_i^x, \quad \text{w/ } \mathbb{Z}_2 \text{ global symmetry: } U = \prod_{j=1}^L \sigma_j^x$$

$$H_{\text{gauged}} = -\sum_i \sigma_i^z \tau_{i+1/2}^x \sigma_{i+1}^z + \sigma_i^x, \quad \text{w/ gauge symmetry: } G_i = \tau_{i-1/2}^z \sigma_i^x \tau_{i+1/2}^z, \forall i$$

Only the gauge invariant sector is physical, that is we require $G_i = 1, \forall i$

(This is the lattice analog of Gauss Law constraints)

We can further do a unitary transformation to simplify the constraints

$$\mathcal{U} = \prod_i CZ_{i-1/2,i} CZ_{i,i+1/2} : \sigma_j^x \rightarrow \tau_{j-1/2}^z \sigma_j^x \tau_{j+1/2}^z, \tau_{j+1/2}^x \rightarrow \sigma_j^z \tau_{j+1/2}^x \sigma_{j+1}^z$$

$$G_i \rightarrow \sigma_i^x, \forall i$$

$$H_{\text{gauged}} \rightarrow -\sum_i \tau_{i+1/2}^x + \tau_{i-1/2}^z \sigma_i^x \tau_{i+1/2}^z \rightarrow -\sum_i \tau_{i+1/2}^x + \tau_{i-1/2}^z \tau_{i+1/2}^z,$$

Controlled Z gate:

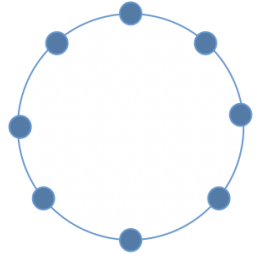
$$CZ_{i,j} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

If two qubits *both* pointing down, give a minus sign

$$CZ_{i,j}: \sigma_j^x \rightarrow \sigma_j^x \sigma_i^z,$$

$$\sigma_i^x \rightarrow \sigma_i^x \sigma_j^z$$

Kramers-Wannier transformation as gauging \mathbb{Z}_2



$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z + \sigma_i^x, \quad \text{w/ } \mathbb{Z}_2 \text{ global symmetry: } U = \prod_{j=1}^L \sigma_j^x$$

Gauging \mathbb{Z}_2 symmetry

$$\tilde{H} = -\sum_i \tau_{i+1/2}^x + \tau_{i-1/2}^z \tau_{i+1/2}^z, \quad \text{w/ a dual } \mathbb{Z}_2 \text{ global symmetry: } \tilde{U} = \prod_{j=1}^L \tau_{i-1/2}^x$$

— After redefining the variables, we arrive at the Kramers-Wannier transformation

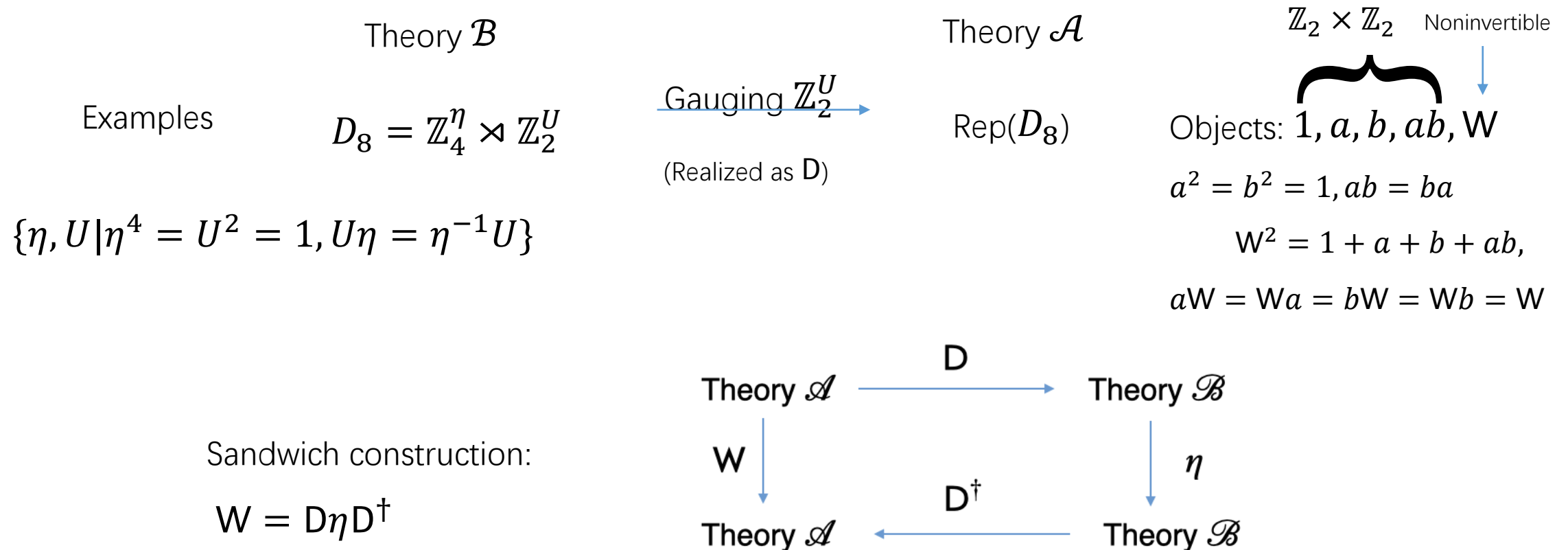
—Gauging \mathbb{Z}_2 symmetry in $(1+1)d$ will lead to a **dual \mathbb{Z}_2 symmetry**

—In general gauging G symmetry in $(1+1)d$ will lead to a **dual $\text{Rep}(G)$ symmetry**

—We can also gauge a **non-normal subgroup**, which also yields noninvertible symmetries

Example: noninvertible symmetry from gauging

—Gauging a non-normal subgroup yields noninvertible symmetries



Lattice realization

$$\text{Rep}(D_8) \left\{ \begin{array}{l} a = \prod_{k=1}^{L/2} \sigma_{2k}^x, b = \prod_{k=1}^{L/2} \sigma_{2k-1}^x \\ W = D\eta D^\dagger \end{array} \right.$$

$$D_8: \quad U = \prod_{j=1}^L \sigma_j^x, \eta = \prod_{j=1}^L \exp\left(\frac{\pi i}{4}(1 - \sigma_j^z)\right)$$

$$H_{IZZ} = \sum_{j=1}^L \sigma_{i-1}^z \sigma_{i+1}^z + \sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z + \Delta \sigma_i^x \quad \begin{array}{c} \text{KW} \\ \longleftrightarrow \\ \sigma_i^z \sigma_{i+1}^z \rightarrow \sigma_i^x, \sigma_i^x \rightarrow \sigma_{i-1}^z \sigma_i^z \end{array} \quad H_{XXZ} = \sum_{j=1}^L \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z$$

Check of fusion rule

$$W \times W = (D\eta D^\dagger)(D\eta D^\dagger) = D\eta(1 + U)\eta D^\dagger = D\eta^2 D^\dagger + DD^\dagger = 1 + a + b + ab$$

With Kramers-Wannier transformation (or gauging \mathbb{Z}_2 symmetry), we can **construct lattice models**

w/ Rep(D_8) symmetry from lattice models w/ D_8 symmetry.

Noninvertible symmetry in general dimensions

Weiguang Cao, M. Yamazaki, L. Li, *Phys.Rev.Lett.* 136 (2026) 4, 040402

- We generalize the gauging procedure to general dimensions from nonabelian group $D_{2n} = \mathbb{Z}_N \rtimes \mathbb{Z}_2$.
- By gauging the nonnormal \mathbb{Z}_2 , we realize $d\text{-Rep}(\mathbb{Z}_n^{(d-1)} \rtimes \mathbb{Z}_2^{(0)})$ noninvertible symmetry on general lattice with a tensor network representation.
- Our construction inspires relevant studies in high energy phenomenology



Tsutomu Yanagida
IPMU



Qiuyue Liang,
IPMU

Non-invertible symmetry as an axion-less solution to the strong CP problem

Phys.Lett.B 868 (2025) 139706

However, symmetries that lack group structures are described by fusion algebra, which allows for different couplings. One way to realize non-invertible symmetries is by gauging the outer automorphism of a non-Abelian group, whose coset¹ generally exhibits non-invertible symmetry [12,17,32]. Non-invertible symmetries are described by fusion rules, enabling different types of couplings [18]. In this paper, we are

Application 2: find new phases and their lattice realizations

Symmetry protected topological phases

— A gapped phase with symmetry G , that is *uniquely gapped* on a closed chain, but with *boundary edge modes* on a open chain

— Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ cluster state $H = -\sum_j \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z$ $U_e = \prod_{k=1}^{L/2} \sigma_{2k}^x, U_o = \prod_{k=1}^{L/2} \sigma_{2k-1}^x$

$H_{PBC} = -\sum_{j=1}^L \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z$ GS constraints: $\sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z = 1, j = 1, \dots, L \rightarrow GSD = 1$

$H_{OBC} = -\sum_{j=2}^{L-1} \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z$ GS constraints: $\sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z = 1, j = 2, \dots, L-1 \rightarrow \text{degenerate edge modes}$

Acting on the OBC ground state, we find two symmetry operators factorize at two boundaries

$$U_e = \sigma_1^z (\sigma_{L-1}^z \sigma_L^x), U_o = (\sigma_1^x \sigma_2^z) \sigma_{2L}^z$$

At each boundary, they anticommute. \rightarrow The edge modes form a *projective representation* of the symmetry operators at the boundaries

Symmetry protected topological phases

—There is a systematic classification:

SPTs are organized by **bulk topological terms / boundary anomalies**.

—Group cohomology gives a broad class of interacting bosonic SPTs:

G -SPT in $d + 1$ dimension, or the anomaly of G symmetry in d dimension, is classified by **group cohomology** $H^{d+1}(G, U(1))$. 1106.4772, Chen, Gu, Liu & Wen

—SPT admits edge or surface modes protected by the symmetry, and can be used as a **resource** for measurement-based quantum computation

— My focus is *noninvertible symmetry protected topological phases*.

Noninvertible symmetry protected topological phases



Noninvertible Symmetry-Protected Topological Order in a Group-Based Cluster State

*C. Fechisin, N. Tantivasadakarn, V. V. Albert, **Phys.Rev.X** 15 (2025) 1, 011058*

—These novel phases could serve as the **resource states** for measurement based quantum computation.

Quantum information



Cluster State as a Noninvertible Symmetry-Protected Topological Phase

*S. Seifnashri, S. Shao, **Phys.Rev.Lett.** 133 (2024) 11, 116601*

—Realize **all** $\text{Rep}(D_8)$ SPTs in spin chain.

Spin system



Categorical Symmetries in Spin Models with Atom Arrays

*A. Warman, F. Yan, A. Tiwari, S. Schafer-Nameki, **Phys.Rev.Lett.** 135 (2025) 20, 206503*

—Design models to **simulate** phases with noninvertible symmetries including SPTs

Experiment proposal

Systematical understanding?

Lattice model in higher dimensions?

Duality (discrete gauging)

General method

Duality \mathcal{D}

Theories with symmetry \mathcal{C}_1



Theories with symmetry \mathcal{C}_2

— Gapped phases of symmetry \mathcal{C}_1 and that of symmetry \mathcal{C}_2 are ***one-to-one***

Example: If \mathcal{D} is *KW* ($\mathcal{D}: \sigma_i^z \sigma_{i+1}^z \rightarrow \sigma_i^x, \sigma_i^x \rightarrow \sigma_{i-1}^z \sigma_i^z$), and $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{Z}_2$

$$H_{sym} = -\sum_{i=1}^L \sigma_i^x \xleftarrow{\text{KW}} H_{SSB} = -\sum_{j=1}^L \sigma_j^z \sigma_{j+1}^z$$

—Classification (construction) of the gapped phases of \mathcal{C}_2 (a group symmetry)

→ **classification (construction) of the gapped phases of \mathcal{C}_1 (a noninvertible symmetry)**, in

particular for \mathcal{C}_1 **SPT phases**

First example: Rep(D_8) SPT

Gauging \mathbb{Z}_2

Theories with Rep(D_8) symmetry



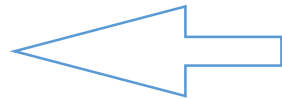
Theories with symmetry D_8

$$\text{Rep}(D_8): W, 1, a, b, ab \leftrightarrow D_8: 1, \eta, \eta^2, \eta^3, U, \eta U, \eta^2 U, \eta^3 U$$

— Rep(D_8) SPT \rightarrow D_8 phases spontaneously breaking $\mathbb{Z}_2 = \{1, U\}$

— *Classification of Rep(D_8) SPTs* \rightarrow Classification of unbroken part of D_8

$$\mathbb{Z}_1 \oplus \mathbb{Z}_2$$



Unbroken subgroups: $\begin{cases} \mathbb{Z}_4^\eta = \{1, \eta, \eta^2, \eta^3\} \\ \mathbb{Z}_2^{\eta^2} \times \mathbb{Z}_2^{\eta U} = \{1, \eta^2\} \times \{1, \eta U\} \end{cases}$

Classification of unbroken group SPT $H^2(\mathbb{Z}_4, U(1)) = \mathbb{Z}_1,$

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$$

More examples of noninvertible SPTs

Gauging \mathbb{Z}_2

Theories with symmetry \mathcal{C}_1



Theories with symmetry \mathcal{C}_2

—This method holds generally if \mathcal{C}_2 contains a 0-form non-normal \mathbb{Z}_2 sub-symmetry

$$(1 + 1)d: \text{Rep}(D_{2n}) \leftrightarrow D_{2n} = \mathbb{Z}_n \rtimes \mathbb{Z}_2$$

$$(2 + 1)d: 2\text{-Rep}(\mathbb{Z}_n^{(1)} \rtimes \mathbb{Z}_2^{(0)}) \leftrightarrow D_{2n}$$

—It *translates the problems of fusion category to that of group*

In our paper, we list the classification with $d\text{-Rep}(\mathbb{Z}_n^{(d-1)} \rtimes \mathbb{Z}_2^{(0)})$ symmetry for $d = 1, 2, 3$

Classification	$d = 1$	$d = 2$	$d = 3$
$n = 1 \pmod 2$	1	n	1
$n = 2 \pmod 4$	$1 + 1$	$n + n$	$1 + 1$
$n = 0 \pmod 4$	$1 + 2$	$n + 3n/2$	$1 + 3$

Construction of $\text{Rep}(D_8)$ SPT phases in $(1+1)d$

—This method allows construction of lattice models with noninvertible symmetry given the knowledge of the **duality** and **dual SSB models**

$\text{Rep}(D_8)$ SPT phases

KW ($D: \sigma_i^z \sigma_{i+1}^z \rightarrow \sigma_i^x, \sigma_{i+1}^x \rightarrow \sigma_{i-1}^z \sigma_i^z$)

D_8 phases with \mathbb{Z}_2^U SSB

$$U = \prod_{j=1}^L \sigma_j^x$$

Trivial phase $H_{trivial} = - \sum_{j=1}^L \sigma_j^x$

$H_{IsingSSB} = - \sum_{j=1}^L \sigma_j^z \sigma_{j+1}^z$

Unbroken symmetry: \mathbb{Z}_4^η

Odd SPT:

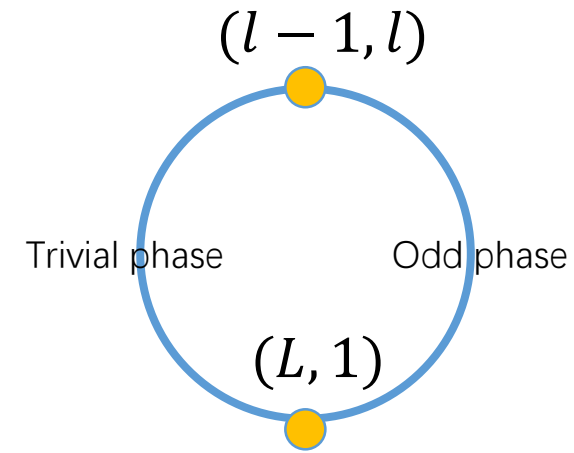
$$H_{odd} = \sum_{k=1}^{L/2} \sigma_{2k}^x + \sigma_{2k-1}^z \sigma_{2k+1}^x \sigma_{2k+3}^z + \sigma_{2k-1}^z \sigma_{2k}^x \sigma_{2k+1}^x \sigma_{2k+2}^z \sigma_{2k+3}^z$$

$$H_{oddSSB} = \sum_{k=1}^{L/2} \sigma_{2k-1}^z \sigma_{2k}^z + \sum_{l=0}^3 \eta^{-l} (\sigma_{2k-1}^y \sigma_{2k}^x \sigma_{2k+1}^x \sigma_{2k+2}^y) \eta^l$$

Unbroken symmetry: $\mathbb{Z}_2^{\eta^2} \times \mathbb{Z}_2^{\eta U}$

The other phase is given by one site lattice translation and is called even phases

Edge modes of non-invertible SPT phases



- Consider the two distinct $\text{Rep}(D_8)$ SPT phases on a closed chain
- Acting on the ground state on a closed chain with two interfaces, symmetry operators **factorizes**

$$U_e = 1, U_o = \sigma_{l+1}^y \sigma_{L-1}^y = U_o^L U_o^R$$

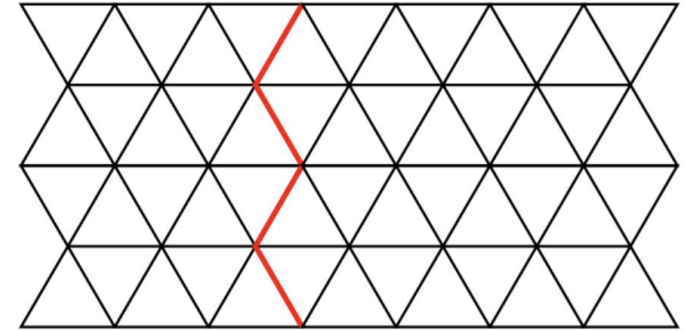
$$W = \sigma_{l+1}^x \sigma_{L-1}^z + \sigma_{l+1}^z \sigma_{L-1}^x = W^{L,1} W^{R,1} + W^{L,2} W^{R,2}$$

- The edge modes come from the projective representation

$$U_o^L W^{L,i} = -W^{L,i} U_o^L, U_o^R W^{R,i} = -W^{R,i} U_o^R, \text{ for } i = 1, 2$$

- It is named odd phase because U_e acts trivially on the ground states
- This analysis can be generalized to $\text{Rep}(D_{2n})$ SPTs in $(1+1)d$

Noninvertible symmetry in $(2 + 1)d$



— D_{2n} symmetry in $(2 + 1)$ dimensions

Consider the model on a general lattice Γ

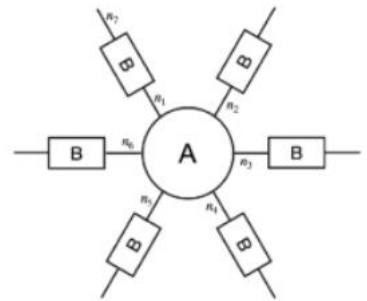
Hilbert space: $\bigotimes_{v \in \Gamma} (\mathbb{C}^2 \otimes \mathbb{C}^n)_v$, operators: $\sigma_v^x, \sigma_v^z, X_v, Z_v$

Symmetry operator: $\eta = \prod_{v \in \Gamma} X_v, U = \prod_{v \in \Gamma} C_v \sigma_v^x, \rightarrow D_{2n} = \{\eta, U | \eta^n = 1, U^2 = 1, \eta U = U \eta^{-1}\}$

— Gauging \mathbb{Z}_2^U symmetry

1. **Coupling** the theory to \mathbb{Z}_2 gauge field μ_l^x, μ_l^z on the link

2. Imposing the Gauss law constraints $(G_v = C_v \sigma_v^x \prod_{l \ni v} \mu_l^z = 1)$ to **project onto the gauge invariant sector**

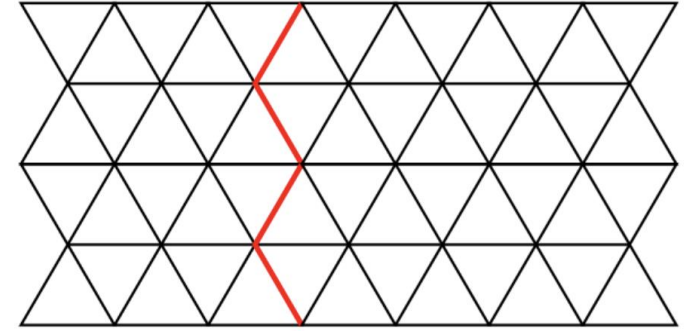


0-form symmetry $U \rightarrow$ 1-form symmetry $W_\gamma = \prod_{l \in \gamma} \mu_l^x$

0-form symmetry $\eta^k \rightarrow$ 0-form noninvertible symmetry W_k w/ fusion algebra $W_{k_1} \times W_{k_2} = W_{k_1+k_2} + W_{k_1-k_2}$

— This construction can be generalized to arbitrary lattices

Examples of noninvertible SPTs in $(2 + 1)d$



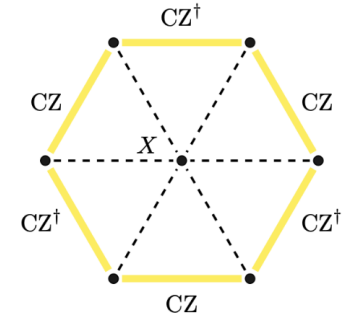
— \mathbb{Z}_2 SSB with symmetry D_8

$$H_m = - \sum_{\langle i,j \rangle \in \Gamma} \sigma_i^z \sigma_j^z - \sum_{i \in \Gamma} X_i \left(\prod_{(i,j,k) \in \Delta} \text{CZ}_{j,k} \prod_{(i,j,k) \in \nabla} \text{CZ}_{j,k}^\dagger \right)^m + (\text{h.c.}), \quad m = 0, 1$$



— SPTs with 2-Rep $(\mathbb{Z}_4^{(1)} \rtimes \mathbb{Z}_2^{(0)})$ symmetry

$$H_m^{\text{SPT}} = - \sum_{\langle i,j \rangle} \mu_{\langle i,j \rangle}^x - \sum_i X_i \left(\prod_{(i,j,k) \in \Delta} \text{CZ}_{i,j,k}^\mu \prod_{(i,j,k) \in \nabla} (\text{CZ}_{i,j,k}^\mu)^\dagger \right)^m + (\text{h.c.}), \quad m = 0, 1$$



Therefore, we give a first systematic construction (2-fusion category) noninvertible SPTs in $(2 + 1)D$.

1. Review of recent developments on generalized symmetries
2. My work on noninvertible symmetries and related phases
3. Conclusion and some outlook

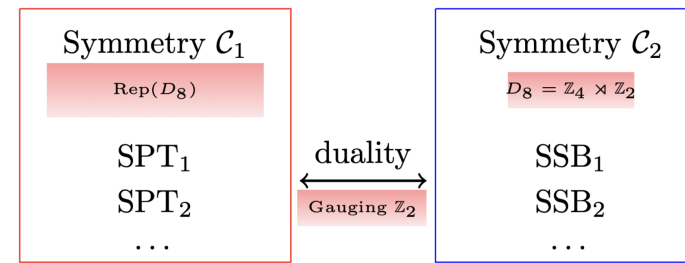
Conclusion

—Although symmetry is one of the most important ideas in physics, only recently we realized that there exist plenty of generalized symmetries *hidden* from our old-fashioned viewpoint of symmetry

—While the whole area is still in the beginning stage, it is important to unlock the power of these symmetries, in particular in topological phases and their applications to quantum information science

—My own research focuses on using **duality** to understand **noninvertible** symmetries, their SPT phases, and beyond-landau transitions.

Summary and outlook



1. Duality transformation can **translate** the problem of noninvertible symmetry to the problem of invertible symmetry while **keeping the phase diagram structure invariant**
2. Via Kramers-Wannier duality (gauging \mathbb{Z}_2 symmetry), a broad class of NISPT can be mapped to \mathbb{Z}_2 SSB with dual group symmetry, from which we can derive the classification and construction of NISPT
3. We give the **complete classification** of NISPT in higher dimensions for the first time.
4. We also systematically construct the SPTs with 2-fusion categories in $(2+1)d$ on a tensor product Hilbert space for the first time.

Outlook: from theory to experiment

Topological phases

Experimental realization

Generalized symmetry



Higher form symmetry

...

Noninvertible symmetry

Abelian topological order

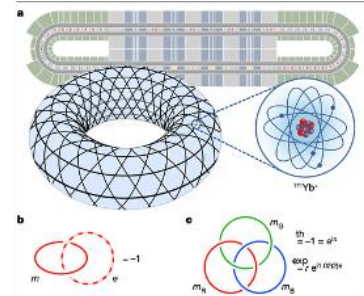
Nonabelian topological order

Noninvertible SPT

Quantum Hall effects

Trapped-ion processor
Nature 626, 505–511 (2024)

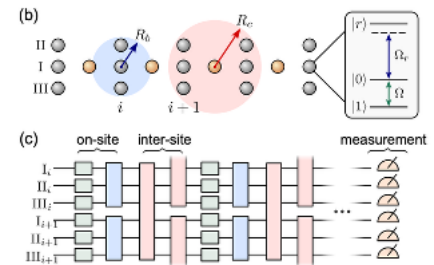
Rydberg atom array
Phys. Rev. Lett. 135, 206503 (2025)



What difficulties might arise in experiments?

In realistic system with noise and decoherence, we need to study mixed state phases

The duality method also works for mixed states!



…it is possible for a theory to have a high degree of symmetry was hidden…

The physicist's task is to find this deeper symmetry.



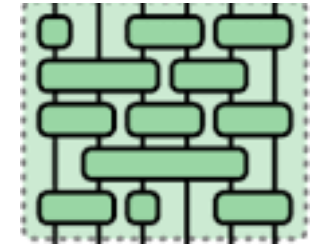
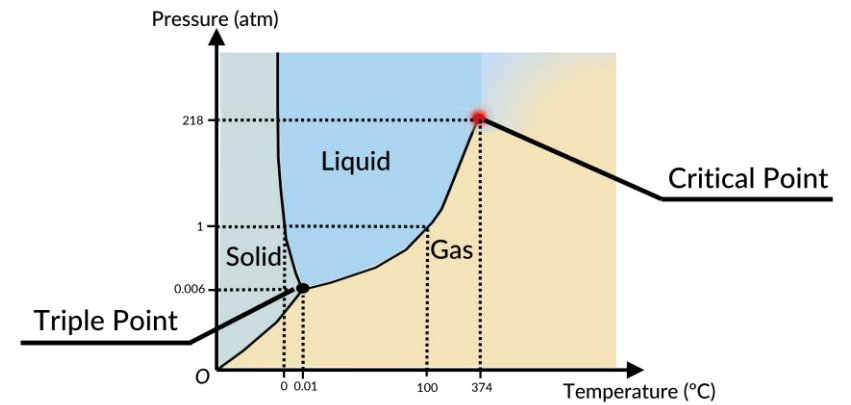
Thank you!

Backup slides

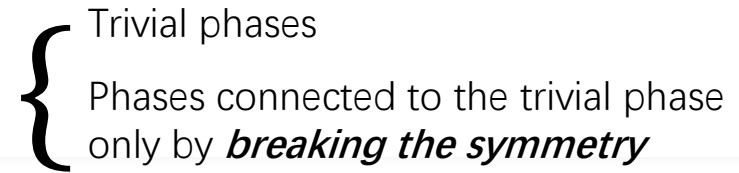
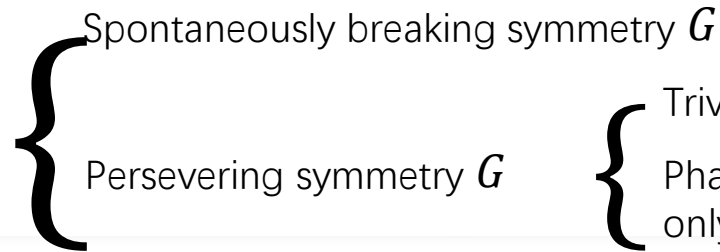
Phases

—Landau’s paradigm: phases represent symmetries

—Gapped phase: equivalence class of local, gapped Hamiltonians that can be interpolated without **closing the gap**



$(1 + 1)d$ system with symmetry G

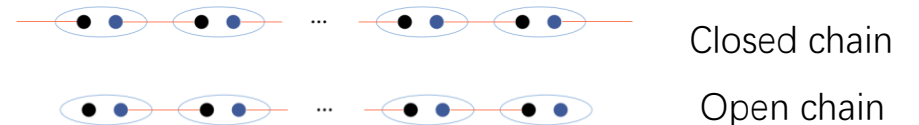


Exotic phases *beyond* Landau’s paradigm: ***Symmetry protected topological (SPT) phases***

—A gapped ***short-range-entangled*** phase with symmetry G

—***Uniquely gapped*** on a closed chain

—With ***boundary edge modes*** on an open chain (Anomaly inflow)



SPT

—There is a systematic classification:

SPTs are organized by **bulk topological terms / boundary anomalies**.

—Group cohomology gives a broad class of interacting bosonic SPTs:

G -SPT in $d + 1$ dimension, or the anomaly of G symmetry in d dimension, is classified by **group cohomology** $H^{d+1}(G, U(1))$. 1106.4772, Chen, Gu, Liu & Wen

—Application in quantum information:

SPT can be used as a **resource** for measurement-based quantum computation

— In recent years many other new exotic phases can be explained with the **generalization** of symmetry

Zoo of noninvertible symmetries

—Tambara-Yamagami fusion category for an abelian group G

Object: $g \in G, D$

$$\text{Fusion: } D \times D = \sum_{g \in G} g, gD = Dg = D$$

Example: Critical Ising model

w/ Kramers-Wannier duality symmetry

1941, Kramers, Wannier

—Rep(G) fusion category for a non-abelian group G

Object: irreducible representations (irreps) of G

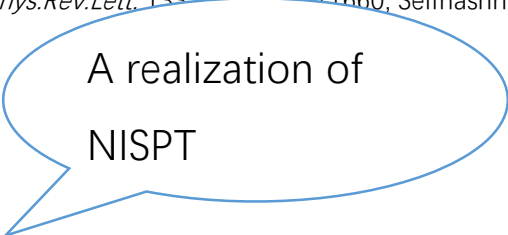
Fusion: Tensor product of irreps

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ cluster state

w/ Rep(D_8) symmetry

Phys.Rev.Lett. 133 (2024) 11, 11660, Seifnashri, Shao

...



A realization of
NISPT

Digression: Application of noninvertible symmetry

— **Flavor model-building**: noninvertible selection rules obtained (for example) by gauging a discrete non-normal subgroup symmetry can realize Yukawa textures that are difficult or impossible to reproduce with ordinary symmetries. 2409.05270 Kobayashi, Otsuka, Tanimoto

— **Testable EFT/BSM signatures**: explicit examples show noninvertible fusion rules imposing distinctive perturbative selection rules (e.g., forcing certain processes to be loop-induced) and can stabilize dark matter when the DM is charged under a noninvertible symmetry. 2503.19964 Suzuki, Xu

— **Strong CP direction**: noninvertible selection rules can enforce quark-mass texture structures used in axion-less strong-CP model building. 2505.05142 Liang, Yanagida

— ...

How to find noninvertible symmetry in lattice models?

— Find a noninvertible operator that commutes with the Hamiltonian

— A noninvertible duality transformation becomes a noninvertible symmetry at the self dual point

E.g. KW duality transformation in $(1 + 1)d$,

Wegner duality transformation in $(3 + 1)d$

*Tambara-Yamagami fusion category

— If a noninvertible duality transformation maps theory A (with group symmetry) to theory B , one can construct noninvertible symmetry in theory B via **sandwich construction**

E.g. $\text{Rep}(D_{2n})$ symmetry in $(1 + 1)d$,

$d\text{-Rep}(\mathbb{Z}_n^{(d-1)} \rtimes \mathbb{Z}_2^{(0)})$ symmetry in $(d + 1)$ -dimension

*Representation fusion category

— There also exist other constructions (like anyon chain) to get lattice models with e.g. Fibonacci fusion category. But they are not realized on a tensor product Hilbert space.

General method: Rep(D_8)



Theories with symmetry \mathcal{C}_1



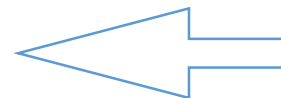
Theories with symmetry \mathcal{C}_2

$$\text{Rep}(D_8): \mathbf{W}, 1, a, b, ab \leftrightarrow D_8: 1, \eta, \eta^2, \eta^3, U, \eta U, \eta^2 U, \eta^3 U$$

— Symmetry \mathcal{C}_1 (with a \mathbb{Z}_2 sub-symmetry) SPT \rightarrow the KW dual theory has symmetry \mathcal{C}_2 but is \mathbb{Z}_2 SSB
Rep(D_8) SPT D_8 phases spontaneously breaking $\mathbb{Z}_2 = \{1, U\}$

— **Classification of the SPT phases for \mathcal{C}_1** \rightarrow Classification of unbroken part of \mathcal{C}_2

$\text{Rep}(D_8) \text{ SPT phases: } \mathbb{Z}_1 \oplus \mathbb{Z}_2$



$$\mathbb{Z}_2^{\eta^2} \times \mathbb{Z}_2^{\eta U} = \{1, \eta^2\} \times \{1, \eta U\}$$

$$H^2(\mathbb{Z}_4, U(1)) = \mathbb{Z}_1, H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$$

General method: Rep(D_{2n})

KW

Theories with symmetry \mathcal{C}_1



Theories with symmetry \mathcal{C}_2

—This method holds generally if \mathcal{C}_2 contains a 0-form non-normal \mathbb{Z}_2 sub-symmetry

$$(1 + 1)d: \text{Rep}(D_{2n}) \leftrightarrow D_{2n} = \mathbb{Z}_n \rtimes \mathbb{Z}_2$$

$$(2 + 1)d: 2\text{-Rep}(\mathbb{Z}_n^{(1)} \rtimes \mathbb{Z}_2^{(0)}) \leftrightarrow D_{2n}$$

—It *translates the problems of fusion category to that of group*

—Rep(D_{2n}) SPTs $\rightarrow \mathbb{Z}_2^U$ SSB with $D_{2n} = \mathbb{Z}_n^\eta \rtimes \mathbb{Z}_2^U$

(1) $n = 1 \text{ mod } 2$, unbroken is \mathbb{Z}_n^η ,

$$\rightarrow H^2(\mathbb{Z}_n, U(1)) = \mathbb{Z}_1$$

(2) $n = 0 \text{ mod } 2$, unbroken is (i) \mathbb{Z}_n^η or (ii) $\mathbb{Z}_{n/2}^{\eta^2} \rtimes \mathbb{Z}_2^{\eta U}$,

Classification	$d = 1$	$d = 2$	$d = 3$
$n = 1 \text{ mod } 2$	1	n	1
$n = 2 \text{ mod } 4$	1 + 1	$n + n$	1 + 1
$n = 0 \text{ mod } 4$	1 + 2	$n + 3n/2$	1 + 3

Example: Rep(D_8) SPT phases in $(1 + 1)d$

—This method allows construction of lattice models with noninvertible symmetry given the knowledge of the **duality** and **dual SSB models**

Rep(D_8) SPT phases: $\mathbb{Z}_1 \oplus \mathbb{Z}_2$

KW (D: $\sigma_i^z \sigma_{i+1}^z \rightarrow \sigma_i^x, \sigma_i^x \rightarrow \sigma_{i-1}^z \sigma_i^z$)

D_8 phases with \mathbb{Z}_2^U SSB

$$U = \prod_{j=1}^L \sigma_j^x$$

Unbroken symmetry:

1. $\mathbb{Z}_4^\eta : H^2(\mathbb{Z}_4, U(1)) = \mathbb{Z}_1$

2. $\mathbb{Z}_2^{\eta^2} \times \mathbb{Z}_2^{\eta U} : H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$

\mathbb{Z}_1 : Trivial phase $H_{trivial} = -\sum_{j=1}^L \sigma_j^x$

$$H_{IsingSSB} = -\sum_{j=1}^L \sigma_j^z \sigma_{j+1}^z$$

Example: Rep(D_8) SPT phases in $(1 + 1)d$

\mathbb{Z}_2^U breaking phase with D_8 symmetry

$$H_{\text{oddSSB}} = \sum_{k=1}^{L/2} \sigma_{2k-1}^z \sigma_{2k}^z + \sum_{l=0}^3 \eta^{-l} (\sigma_{2k-1}^y \sigma_{2k}^x \sigma_{2k+1}^x \sigma_{2k+2}^y) \eta^l$$



Rep(D_8) SPT

Odd SPT: $H_{\text{odd}} = \sum_{k=1}^{L/2} \sigma_{2k}^x + \sigma_{2k-1}^z \sigma_{2k}^x \sigma_{2k+1}^x \sigma_{2k+2}^z \sigma_{2k+3}^z +$

$\sigma_{2k-1}^z \sigma_{2k+1}^x \sigma_{2k+3}^z$ given by one site lattice translation and is called even phases

Unbroken symmetry:

$$1. \mathbb{Z}_4^\eta : H^2(\mathbb{Z}_4, U(1)) = \mathbb{Z}_1$$

$$2. \mathbb{Z}_2^{\eta^2} \times \mathbb{Z}_2^{\eta U} : H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$$

The first term aligns the spins within the same unit cell in the antiferromagnetic configuration.



$$\begin{aligned} &\rightarrow H_{\text{eff}} \\ &= \sum_{k=1}^{L/2} \tilde{\sigma}_k^y \tilde{\sigma}_{k+1}^y \quad \eta^2 \rightarrow I, \eta U \rightarrow \prod_{k=1}^{L/2} \tilde{\sigma}_k^y \\ &\rightarrow H_{\text{oddSSB}} \text{ preserves } \mathbb{Z}_2^{\eta^2} \times \mathbb{Z}_2^{\eta U} = \{1, \eta^2\} \times \{1, \eta U\} \end{aligned}$$

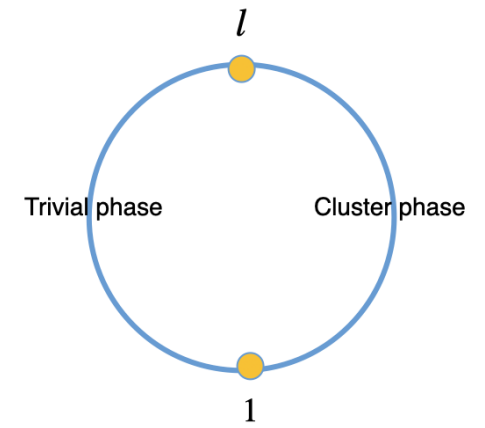
—Different noninvertible SPTs can be detected by edge modes

Edge modes of ordinary SPT phases

— The edge modes are characterized by the *projective representation* of the symmetry operators at the boundaries

E.g. for $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT, where the symmetry are generated by spin flipping on the even and odd sites

$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$: a trivial phase and a nontrivial cluster state



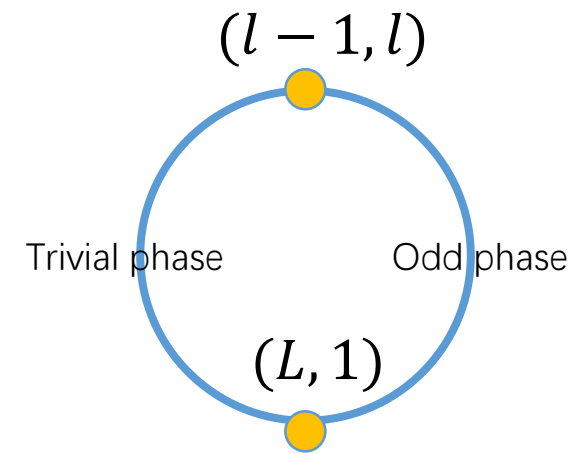
We put the two phases on a closed chain with two interfaces

$$H = -\sum_{j=2}^{l-1} \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z - \sum_{j=l+1}^L \sigma_j^x$$

$$U_e = \sigma_1^z (\sigma_{l-1}^z \sigma_l^x), U_o = (\sigma_1^x \sigma_2^z) \sigma_l^z$$

When acting on the ground state, two \mathbb{Z}_2 doesn't commute at each boundary

Edge modes of non-invertible SPT phases



- Consider the two distinct $\text{Rep}(D_8)$ SPT phases on a closed chain
- Acting on the ground state on a closed chain with two interfaces, symmetry operators *factorizes*

$$U_e = 1, U_o = \sigma_{l+1}^y \sigma_{L-1}^y = U_o^L U_o^R$$

$$W = \sigma_{l+1}^x \sigma_{L-1}^z + \sigma_{l+1}^z \sigma_{L-1}^x = W^{L,1} W^{R,1} + W^{L,2} W^{R,2}$$

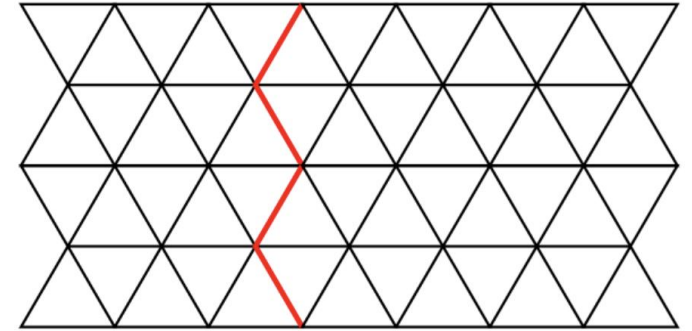
- The edge modes come from the projective representation

$$U_o^L W^{L,i} = -W^{L,i} U_o^L, U_o^R W^{R,i} = -W^{R,i} U_o^R, \text{ for } i = 1, 2$$

- It is named odd phase because U_e acts trivially on the ground states

- This construction & analysis can be generalized to $\text{Rep}(D_{2n})$ SPTs in $(1+1)d$

Noninvertible symmetry in $(2 + 1)d$



— D_{2n} symmetry in $(2 + 1)$ dimensions

Consider the model on a general lattice Γ

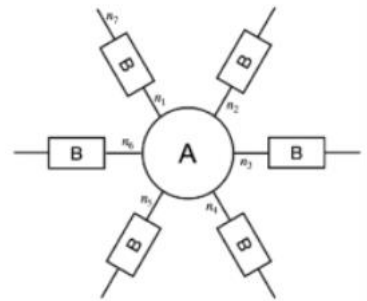
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Symmetry operator: $\eta = \prod_{v \in \Gamma} X_v, U = \prod_{v \in \Gamma} C_v \sigma_v^x, \rightarrow D_{2n} = \{\eta, U | \eta^n = 1, U^2 = 1, \eta U = U \eta^{-1}\}$

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0-form symmetry $U \rightarrow$ 1-form symmetry $W_\gamma = \prod_{l \in \gamma} \mu_l^x$

0-form symmetry $\eta^k \rightarrow$ 0-form noninvertible symmetry W_k w/ fusion algebra $W_{k_1} \times W_{k_2} = W_{k_1+k_2} + W_{k_1-k_2}$

— This construction can be generalized to arbitrary lattices

Phase transitions Beyond Landau

—Landau's paradigm characterize phase transitions between phases breaking a larger symmetry to phases breaking a smaller symmetry

—However, recent studies unveil interesting (continuous) phase transitions between phases breaking **two seemingly unrelated symmetries**, which is termed as deconfined quantum critical point (DQCP).

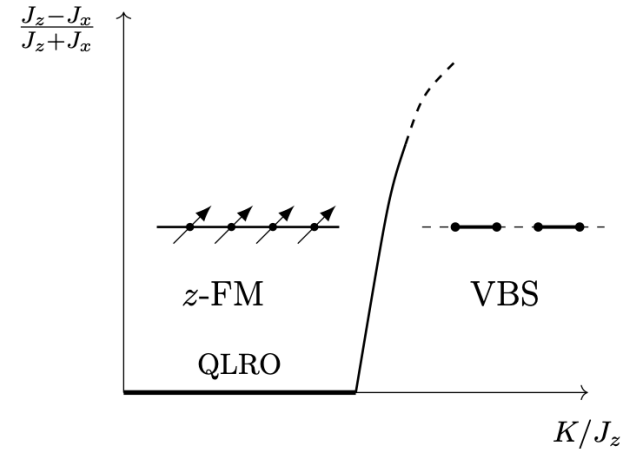
—Prof. Xie Chen proposed that such transitions can be understood as **Generalized Landau Paradigm** using a duality transformation.

PRL Forward-Looking Essay: Generalized Landau Paradigm for Quantum Phases and Phase Transitions

Phase transitions Beyond Landau Ongoing work, with Hiromi Ebisu (Riken) & Bo Han (UCologne)

—Model with DQCP S. Jiang, O. I. Motrunich *Phys.Rev.B* 99 (2019) 7, 075103

$$H = \sum_j (-J_x \sigma_j^x \sigma_{j+1}^x - J_z \sigma_j^z \sigma_{j+1}^z) + \sum_j (K_{2x} \sigma_j^x \sigma_{j+2}^x + K_{2z} \sigma_j^z \sigma_{j+2}^z) + \dots$$



With symmetry $\mathbb{Z}_2^x \times \mathbb{Z}_2^z \times \mathbb{Z}_2^T$

z-FM: \mathbb{Z}_2^z SSB, VBS: \mathbb{Z}_2^T SSB

By gauging $\mathbb{Z}_2^x \times \mathbb{Z}_2^z$, we found that the lattice translation becomes noninvertible,

and the total symmetry is $\text{Rep}(D_8)$

The z-FM phase is dual to a \mathbb{Z}_2^x SSB phase,

Example of categorical Landau Paradigm