Entanglement structure from modified IR geometries

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Based on arXiv 2404.02737 with Xin-Xiang Ju, Teng-Zhou Lai, Bo-Hao Liu , &Wen-Bin Pan

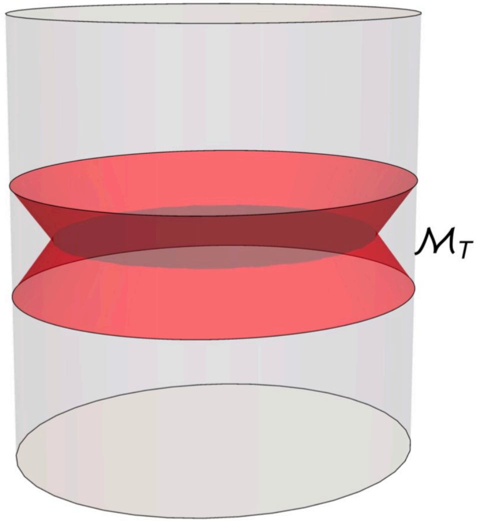
Motivation

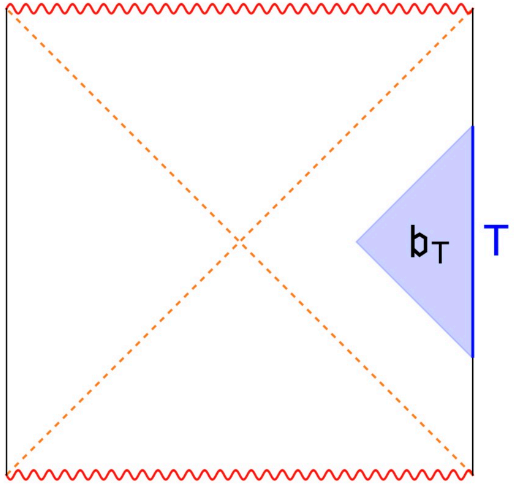
- Quantum entanglement closely connected to geometry
- A new proposal on the relation between quantum entanglement and geometry: radial scale/boundary real space scale correspondence; in contrast to the UV/IR relation emphasizing the scale in momentum space;
- Hints for this proposal
  - Hole-ography, differential entropy (Balasubramanian, Chowdhury, Czech, de Boer, 2013&2014), generalized Rindler wedge (Ju, Pan, Sun, Wang, 2023)
  - Subregion-subregion duality
  - AdS/MERA (Swingle,2009)

Motivation

✦ Hint from hole-ography and the generalized Rindler wedge (GRW)

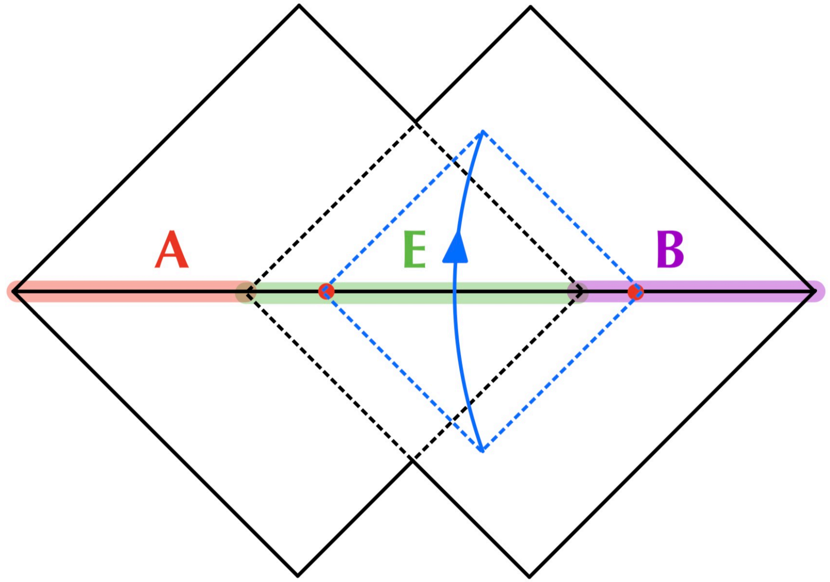
- Defining a bulk spacetime subregion from accelerating observers: subregion enclosed by the horizon of accelerating observers
- In the simplest case: a hole in AdS spacetime; the subregion outside the hole corresponds to a time band at the boundary, hole-orgraphy;
- In the most general case: Rindler convex subregions; the subregion outside corresponds to a spcetime subregion at the boundary, GRW



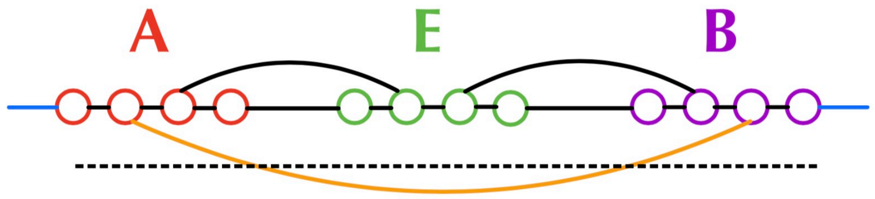


Motivation

- ✦ Hint from hole-ography and the generalized Rindler wedge (GRW)
- The boundary dual of the area of the surface is the differential entropy, which could be viewed as the cost in a state merging protocol eliminating long scale entanglement in the real space;
- The quantum state: a Markov chain, defined by conditional mutual information I(A:B|E) being zero;
- In the GRW framework, this is explained as a fine-grained entropy of a state with long scale entanglement removed: entanglement between A and B eliminated;
- Further supported by subregion-subalgebra correspondence (Leutheusser, Liu, 2022) and entanglement entropy equivalence
- Therefore: eliminating a bulk IR region corresponds to eliminating long scale entanglement in real space of the boundary theory



time band: an infinite set of causal diamonds aligned together



Motivation

The proposal:

- IR geometry --- long distance scale entanglement structure;
- UV geometry --- short distance scale entanglement structure;

- Modifying IR geometry: changing the long scale entanglement structure, revealed from the entanglement measures, especially the conditional mutual information;

- Results further confirming the proposal.

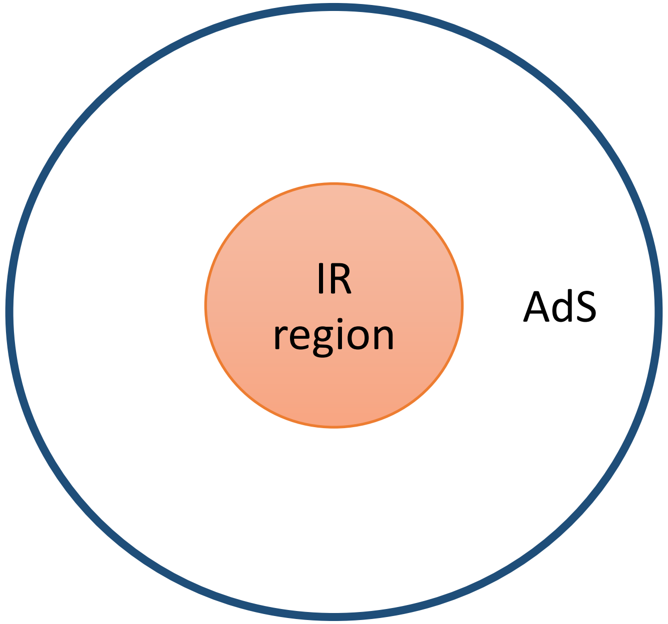
Outline

- Modifying IR geometries on AdS spacetime
  - ➢Geometry and RT surfaces
- Quantum entanglement properties on the modified IR geometries:
  - ➢Mutual information
  - ➢Conditional mutual information and PEE analysis: the different long scale entanglement structures
- Summary and open questions

I Modifying IR geometries on AdS

- Various types of IR geometries have been studied in the context of AdS/CMT: Lifshitz, Hyperscaling violating geometry, etc..
- Geometries flowing from IR to UV under IR irrelevant deformations
- Concerning entanglement structures at different scales, to make the entanglement structure analysis clearly and analytically reflecting the geometry at the IR: IR geometry connected to UV geometry at an interface brane in 2+1d, with connection conditions and matter localized on the brane
- Replace the IR region with a different IR geometry

I Modifying IR geometries on AdS



- To focus on interesting spatial part: time-dependent in the most general case, Cauchy slice extrinsic curvature zero
- Geodesic length reflecting entanglement behavior: two types of “opposite” IR geometries,
- **Spherical IR geometry:** replace the IR region with a 2d sphere, geodesic length becomes larger, more long scale entanglement expected
- **Hyperbolic IR geometry:** replace the IR region with a 2d hyperbolic surface, geodesic length becomes shorter , like an AdS with AdS radius shorter than the outside, less long scale entanglement expected

I Modifying IR geometries on AdS: geometry and RT surface

- **Spherical case:**

$$ds^2 = -f(r)g(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\theta^2$$

- Geometry: IR region replaced by a 2d sphere, outside: AdS spacetime

$$g(r) = \begin{cases} (l^2 - r^2)/l^2, & \text{for } r < r_{IR}, \\ (l_{AdS}^2 + r^2)/(l_{AdS}^2), & \text{for } r > r_{IR}, \end{cases}$$

- r_{IR}: edge of the IR region, l larger than $r_{IR;}$
- An extremal limit: $l \rightarrow \infty$, geodesic length very large
- Make sure extrinsic curvature to be zero at t=0 and let it evolve in time

I Modifying IR geometries on AdS: geometry and RT surface

- Matter at interface of the IR and UV region: common feature of brane world scenario, μ matter energy and j_ν matter momentum density

$$R_c - K^{\mu\nu}K_{\mu\nu} + K^2 = 16\pi G\mu,$$
$$\mathcal{D}^\mu K_{\mu\nu} - \mathcal{D}_\nu K = 8\pi G j_\nu,$$

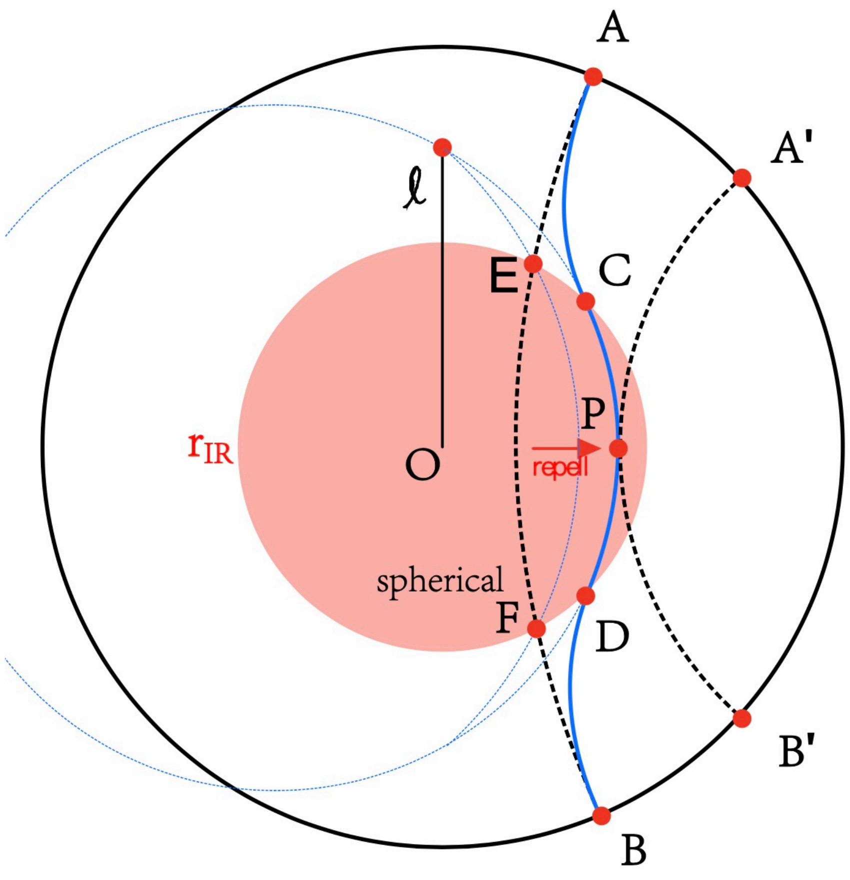
- Connection conditions: the same induced metric on the interface and matter energy momentum tensor supporting the discontinuity in the Ricci tensor

$$g_{\theta\theta} = r^2$$

- Null energy condition to be satisfied;

I Modifying IR geometries on AdS: geometry and RT surface

- RT surface in the spherical IR geometry:



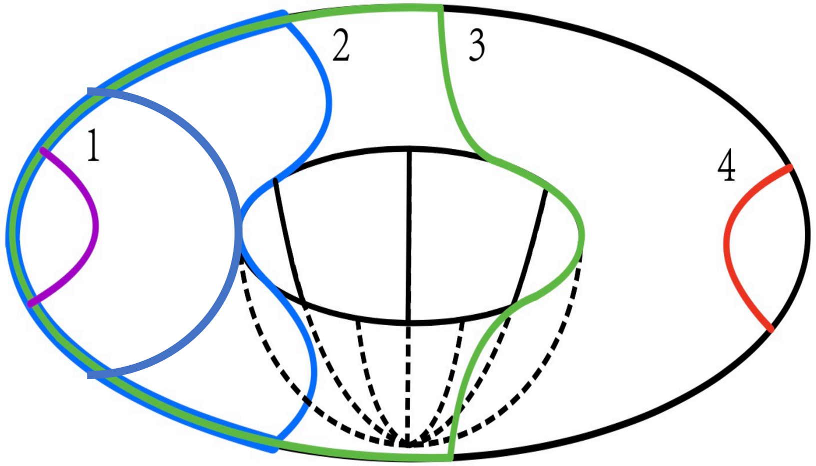
The geodesic length inside is larger than that outside the IR region

For small boundary subregions, RT surfaces are the same as the unmodified AdS case;

For the same large enough boundary interval, the RT surface is repelled to the boundary

I Modifying IR geometries on AdS:
geometry and RT surface

- Focus on the spherical extremal limit: infinite geodesic length
- IR region becomes an entanglement shadow where no RT surfaces can penetrate into
- Four phases of RT surfaces depending on the size of the boundary interval, a critical size $L_c = 2l_{S^1} \arctan \frac{l_{AdS}}{r_{IR}}$



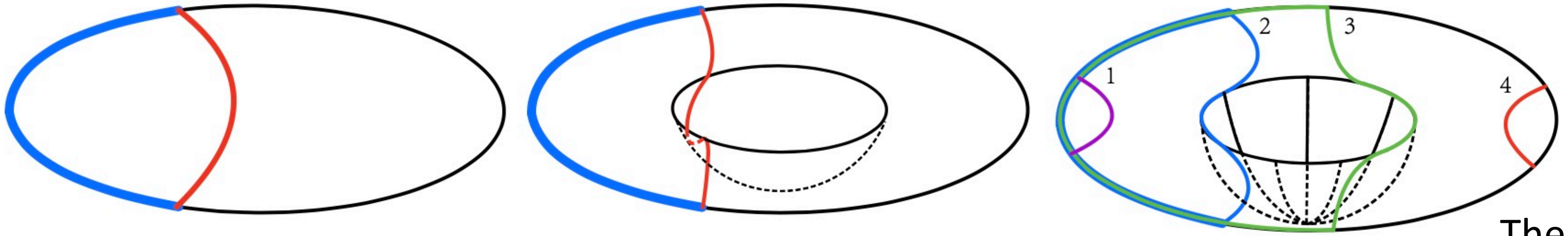
- I, IV: $L<L_c$ or $L_{sys}-L<L_c$, the same as in AdS
- II, III: $L_c<L<L_{sys}/2$, or $L_c< L_{sys}-L < L_{sys}/2$,

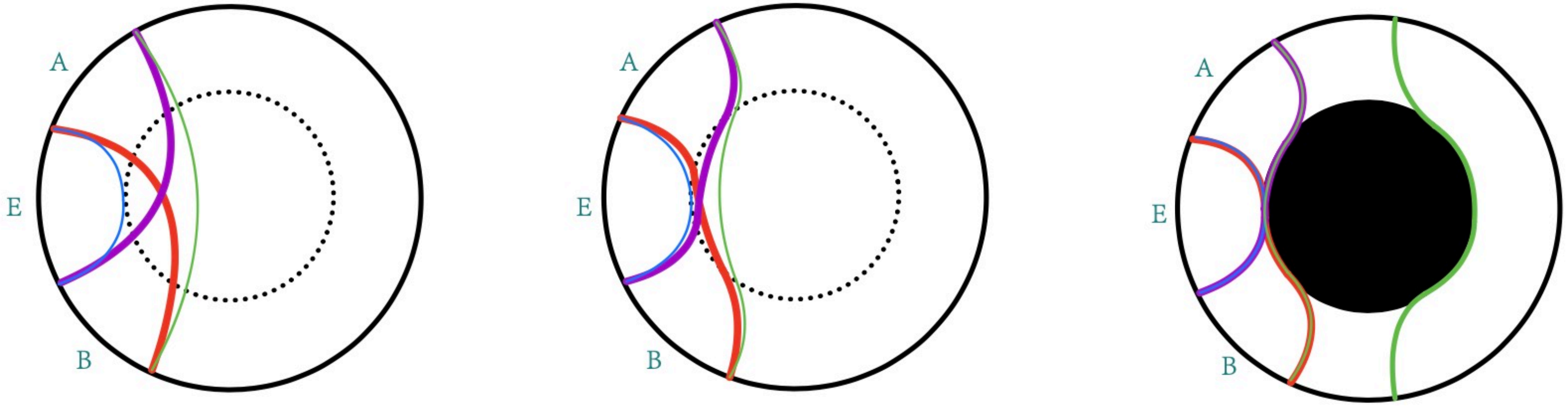
  the RT surface does not penetrate into the

  IR region and instead wraps around the edge

I Modifying IR geometries on AdS:
geometry and RT surface

- A summary of RT surface configurations in the geometry with a spherical IR region





The extremal limit, closely related to the differential entropy and GRW

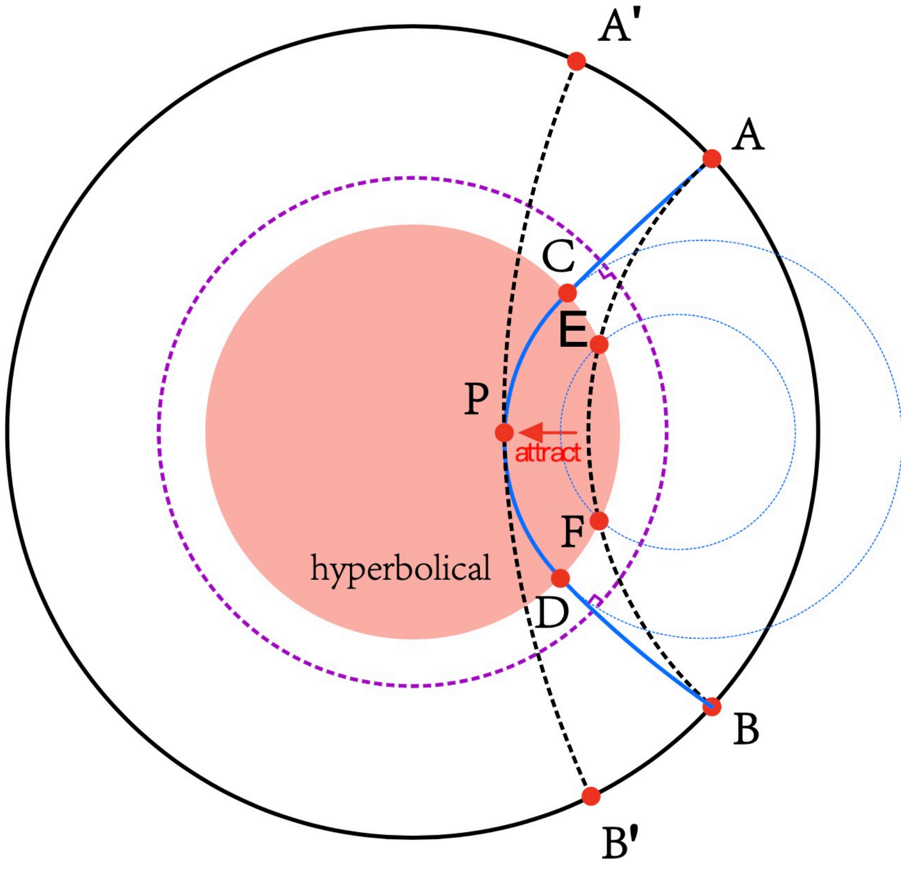
I Modifying IR geometries on AdS:
geometry and RT surface

- **Hyperbolic IR:**
- Geometry

$$g(r) = \begin{cases} (l^2 + r^2)/l^2, & \text{for } r < r_{IR}, \\ (l_{AdS}^2 + r^2)/l_{AdS}^2, & \text{for } r > r_{IR}, \end{cases}$$

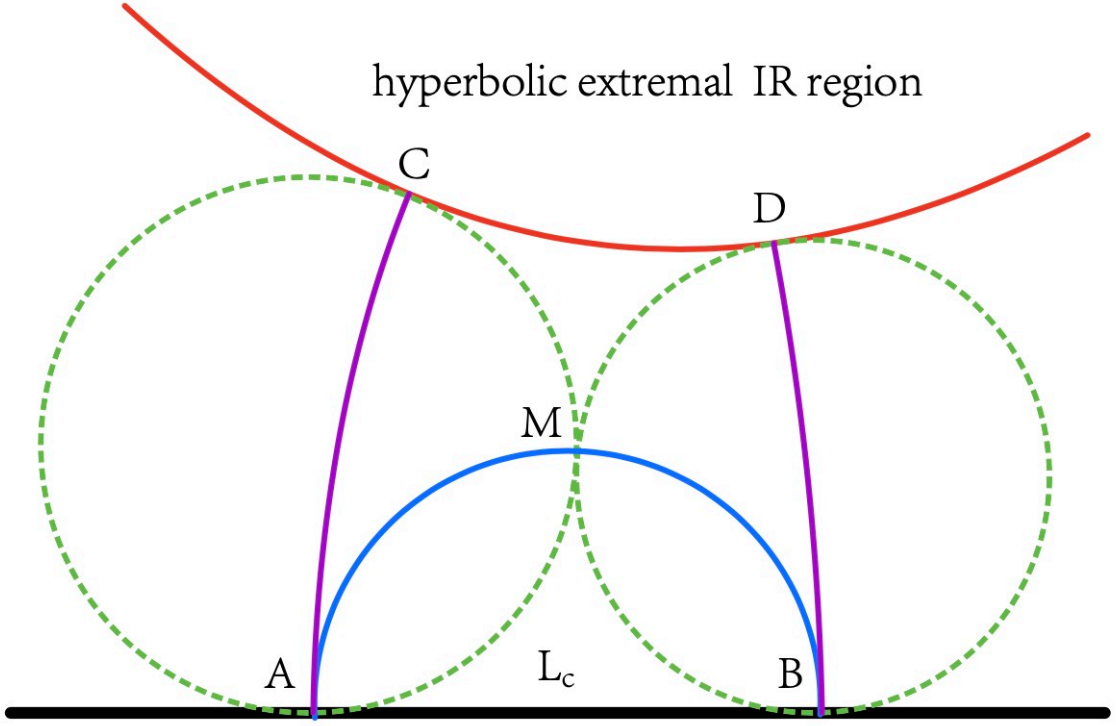
$$0 < l < l_{AdS}$$

- RT surface attracted to the IR region



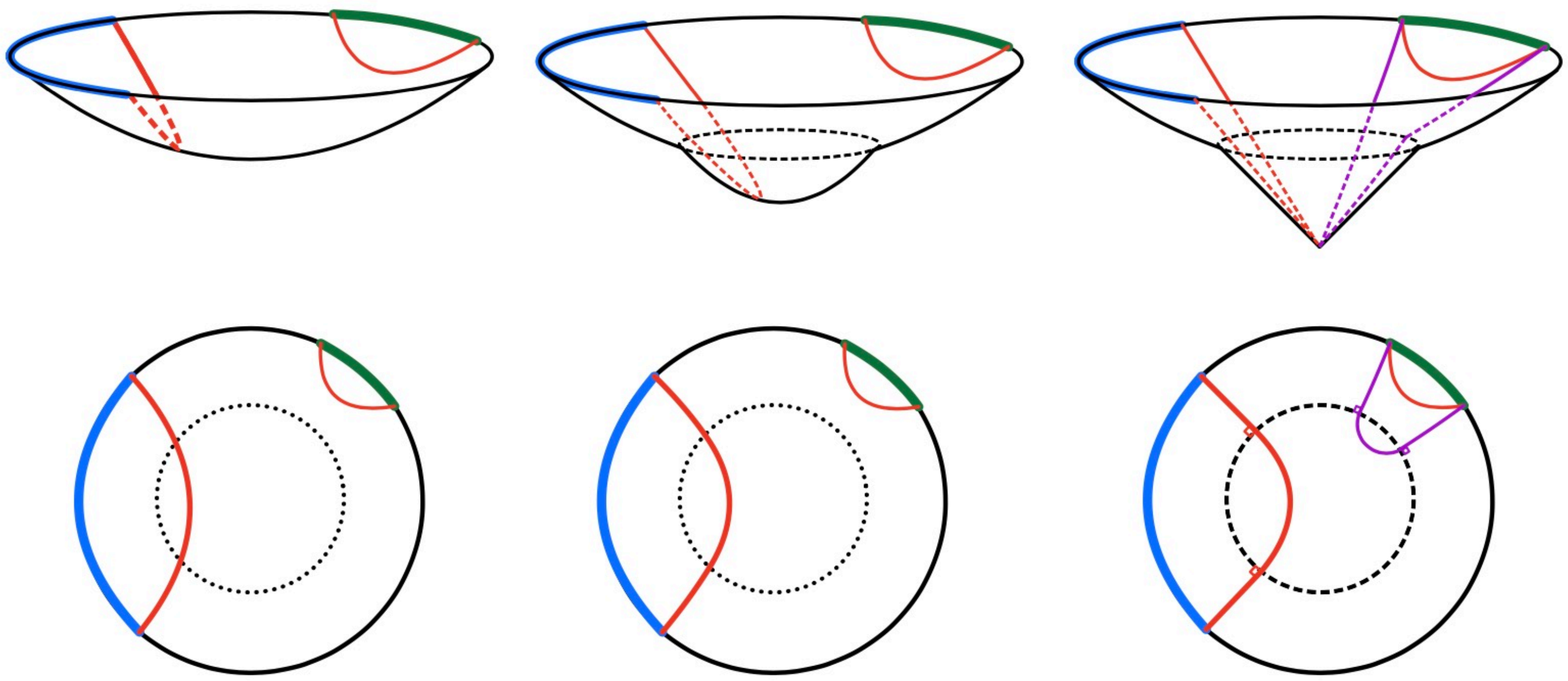
I Modifying IR geometries on AdS:
geometry and RT surface

- extremal limit: $l \to 0$, geodesic length 0, trivial spacetime, related to EoW brane, AdS/BCFT, AdS/ICFT;
- RT surface in the hyperbolic extremal case
- A critical length L_c at which the configuration of the RT surface has a first order phase transition: the RT surface will penetrate into the IR region at $L_c<L<L_{sys}-L_c$



I Modifying IR geometries on AdS:
geometry and RT surface

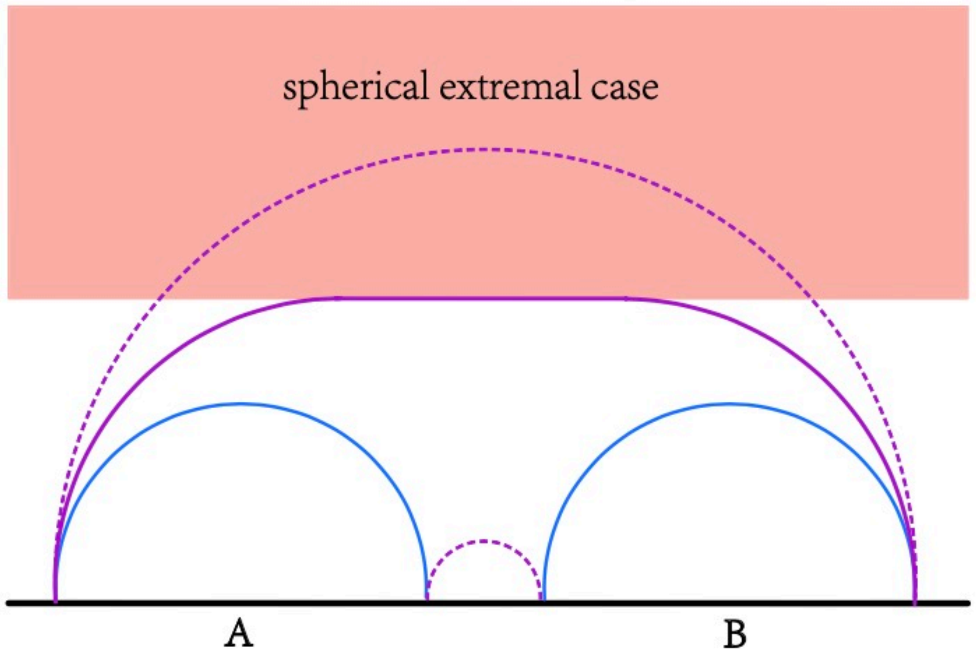
- A summary of the RT configuration



AdS hyperbolic IR extremal hyperbolic IR

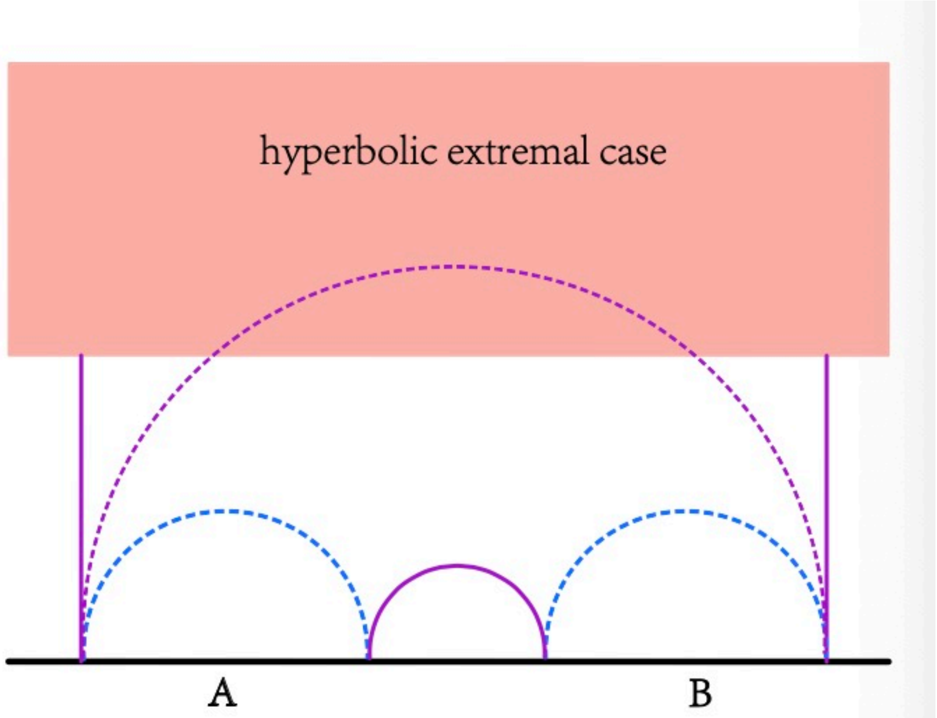
II Entanglement properties

- Entanglement at long scales, two subregions at long scales
- **mutual information**



tend to disconnect, mutual information decreases

no mutual information when distance between AB is larger than L_c



tend to connect, mutual information increases

II Entanglement properties

- To consider entanglement for long scale degrees of freedom:
- Entanglement entropy includes contribution from all scales
- Mutual information alone does not reflect the full entanglement structure of subregions at long scales: depending on the size of the subregions, does not reflect how ρ_{AB} is embedded in the full system

- More entanglement measures: conditional mutual information, partial entanglement entropy

II Entanglement properties: CMI

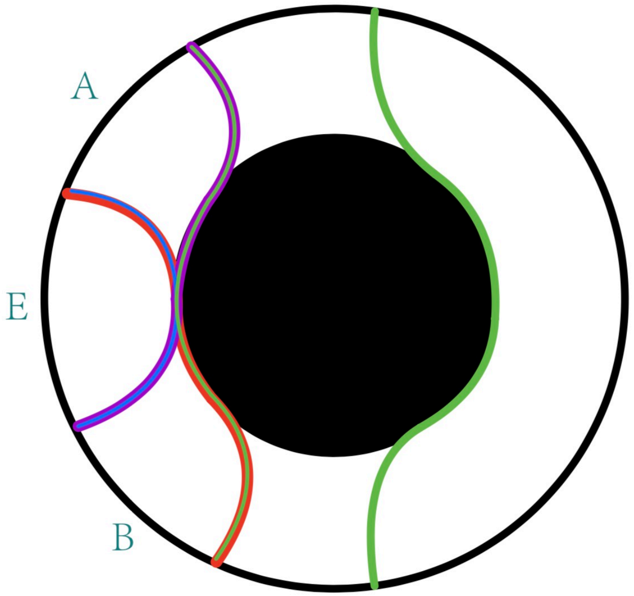
- **Conditional mutual information (CMI)**

$$I(A:B|E) = S_{AE} + S_{BE} - S_{ABE} - S_E$$

- Signifying the mutual information between AB under the condition E
- Quantifies the correlation between A and BE that is not just due to the correlation between A and E
- Strong subadditivity ensures the non-negativity of CMI
- CMI≥MI
- Extensively studied in the context of differential entropy and partial entanglement entropy
- E chosen to be the interval between A and B
- CMI: distinguish entanglement at different distance scales, while the entanglement entropy includes entanglement at all scales

II Entanglement properties: CMI

- CMI properties
- Extremal spherical IR:



CMI vanishes when the length of E is larger than L_c and the whole size of ABE is smaller than half the system size

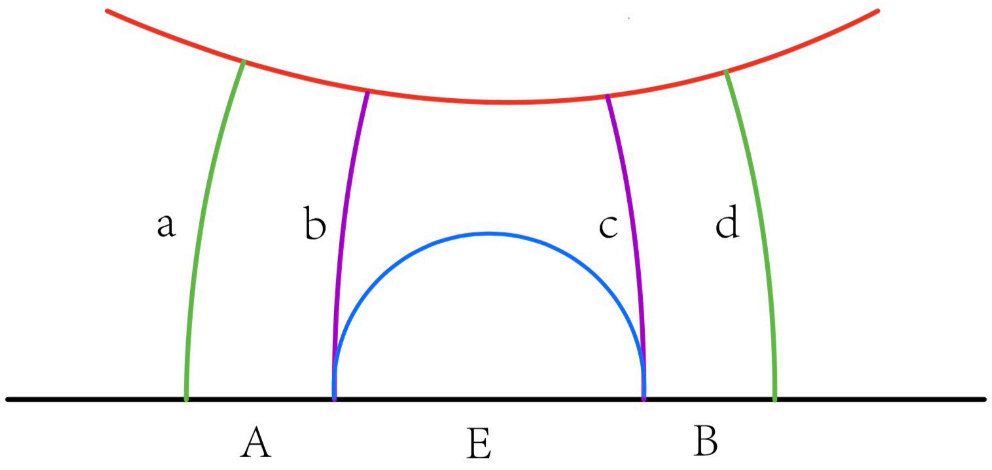
When ABE longer than half the system size, nonzero CMI: correlations at longer distances

In contrast to CMI always being nonzero for pure AdS

More quantitative comparison later

II Entanglement properties: CMI

- Extremal hyperbolic IR:



CMI vanishes when the length of E is larger than L_c

Compared to the extremal spherical case: the whole size of ABE does not need to be smaller than half the system size

No correlations at long distances

In contrast to CMI always being nonzero for pure AdS

More quantitative comparison later

II Entanglement properties: PEE

- Fine entanglement structure: partial entanglement entropy
- Quantify the amount of entanglement of a small subregion
- Definition: the contribution of a subregion A_i to the entanglement entropy of A

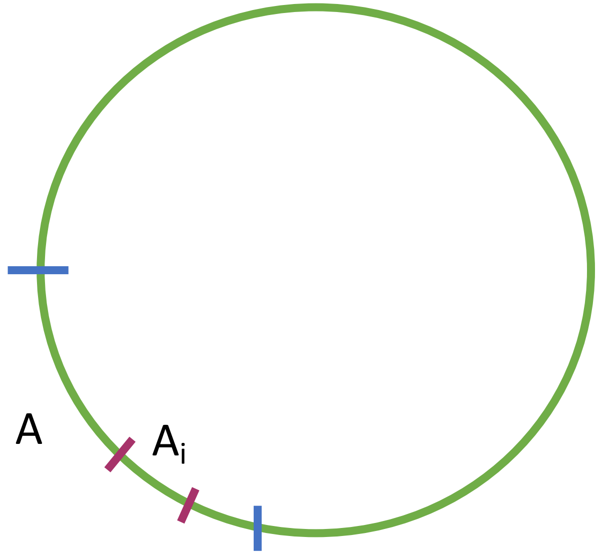
$$s_{\mathcal{A}}(A_i) = \int_{A_i} f_{\mathcal{A}}(\mathbf{x}) d\sigma_{\mathbf{x}}$$

- Two point PEE

$$s_{\bar{B}}(A) = \int_A \mathrm{d}\sigma_{\mathbf{x}} \int_B \mathrm{d}\sigma_{\mathbf{y}} \mathcal{J}(\mathbf{x}, \mathbf{y})$$

- PEE=CMI proposal: $L = |\mathbf{x} - \mathbf{y}|$

$$\mathcal{J}(L) d\sigma_x d\sigma_y = \frac{1}{2} I(d\sigma_x : d\sigma_y | E) = -\frac{1}{2} \frac{d^2 S(L)}{dL^2} d\sigma_x d\sigma_y$$



II Entanglement properties: PEE

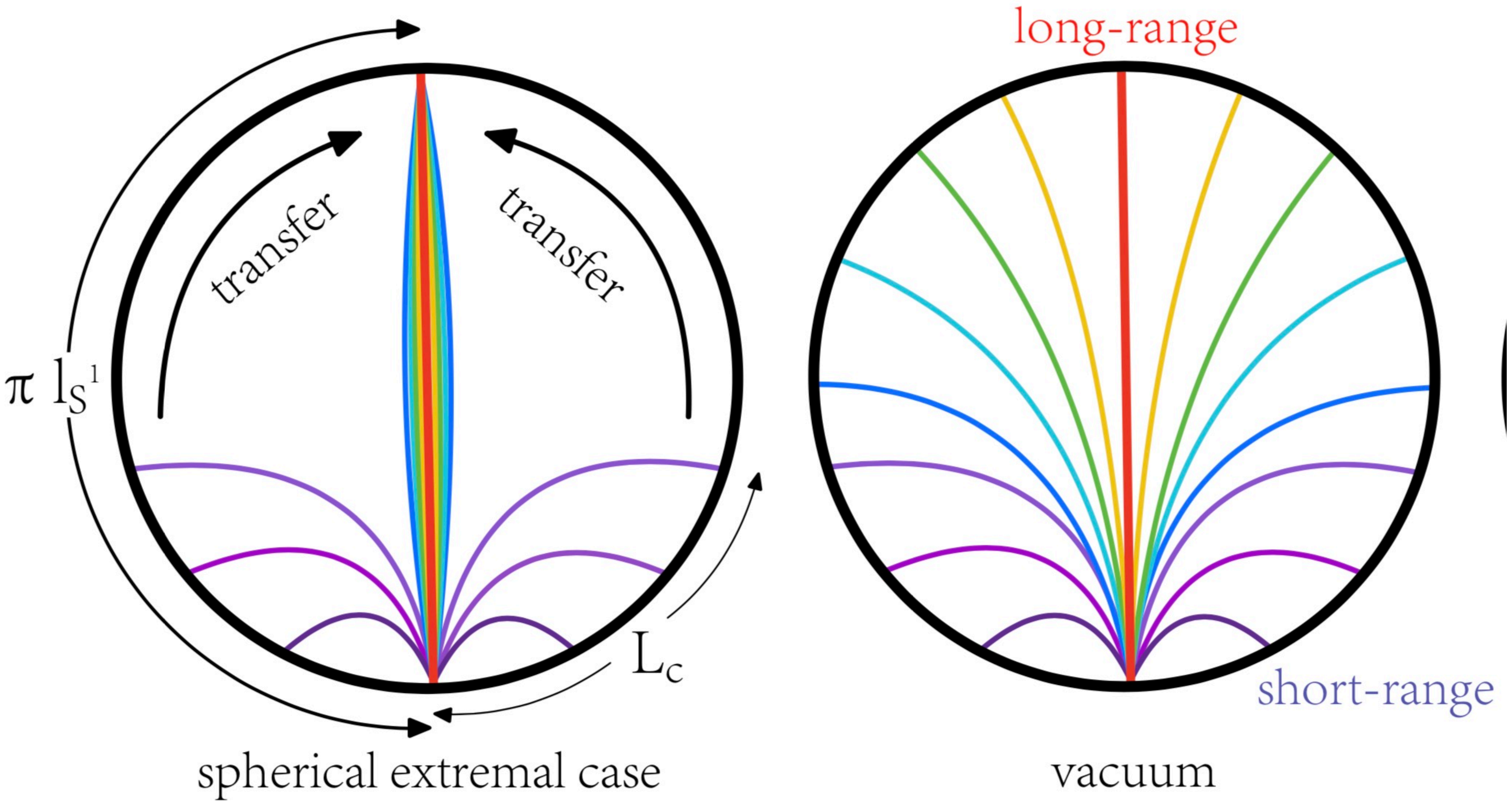
- Behavior of the two point PEE function
- Extremal spherical IR:

$$\mathcal{J}_{sph}(L) = \begin{cases} \mathcal{J}_{vac} = \dfrac{c}{24}\dfrac{\csc(L/2l_{S^1})^2}{l_{S^1}^2}, & (L < L_c \text{ or } L > 2\pi l_{S^1} - L_c) \\ 0, & (L_c \leq L < \pi l_{S^1} \text{ or } \pi l_{S^1} < L \leq 2\pi l_{S^1} - L_c) \\ \dfrac{c\cot(L_c/2l_{S^1})}{6l_{S^1}}\delta(L - \pi l_{S^1}) = \delta(L - \pi l_{S^1})\displaystyle\int_{L_c}^{2\pi l_{S^1} - L_c} \mathcal{J}_{vac}(L)dL. & (L \to \pi l_{S^1}) \end{cases}$$

- Entanglement redistribution: the entanglement entropy of $L<L_c$ does not change, redistribution of entanglement at $L>L_c$

II Entanglement properties: entanglement structures

Entanglement at $L>L_c$ transfers to $L=1/2L_{sys}$



II Entanglement properties: entanglement structures

- The amount of long range entanglement at L=1/2L_{sys}

$$\pi l_{S^1}\int_{\pi l_{S^1}-\epsilon}^{\pi l_{S^1}+\epsilon}\mathcal{I}_{sph}(L)dL=\pi\frac{c}{6}\cot(L_c/2l_{S^1})$$

- Equal to the area of the IR region

$$2\int \text{diverge two-point PEE}=\frac{1}{4G}\text{Area}(\text{IR region})$$

- The area of IR region: the total amount of long scale entanglement

II Entanglement properties: entanglement structures

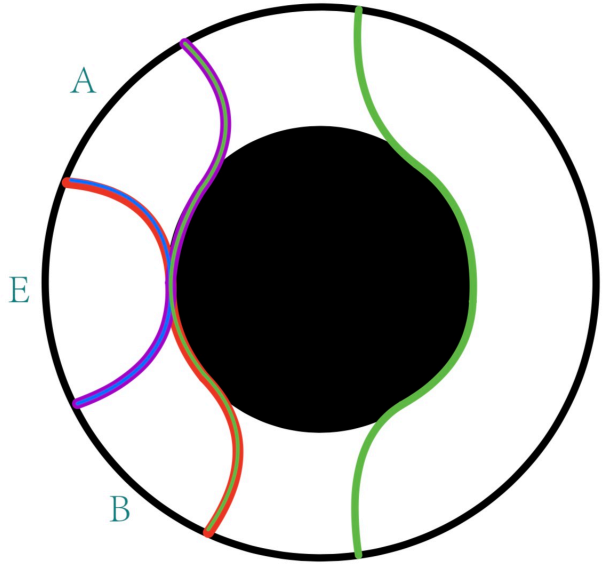
- Remember that the entanglement entropy of the extremal spherical IR

$$\frac{c}{3}\log(\frac{l_{S^1}}{\pi\epsilon}\sin(\frac{L_c}{2l_{S^1}})) + \frac{c\cot(L_c/2l_{S^1})}{6l_{S^1}}(L-L_c), \qquad (L_c \leq L \leq \pi l_{S^1})$$

at $L=1/2L_{sys}$, long range entanglement

$$\pi\frac{c}{6}\cot(L_c/2l_{S^1})$$

- Other terms: shorter range entanglement



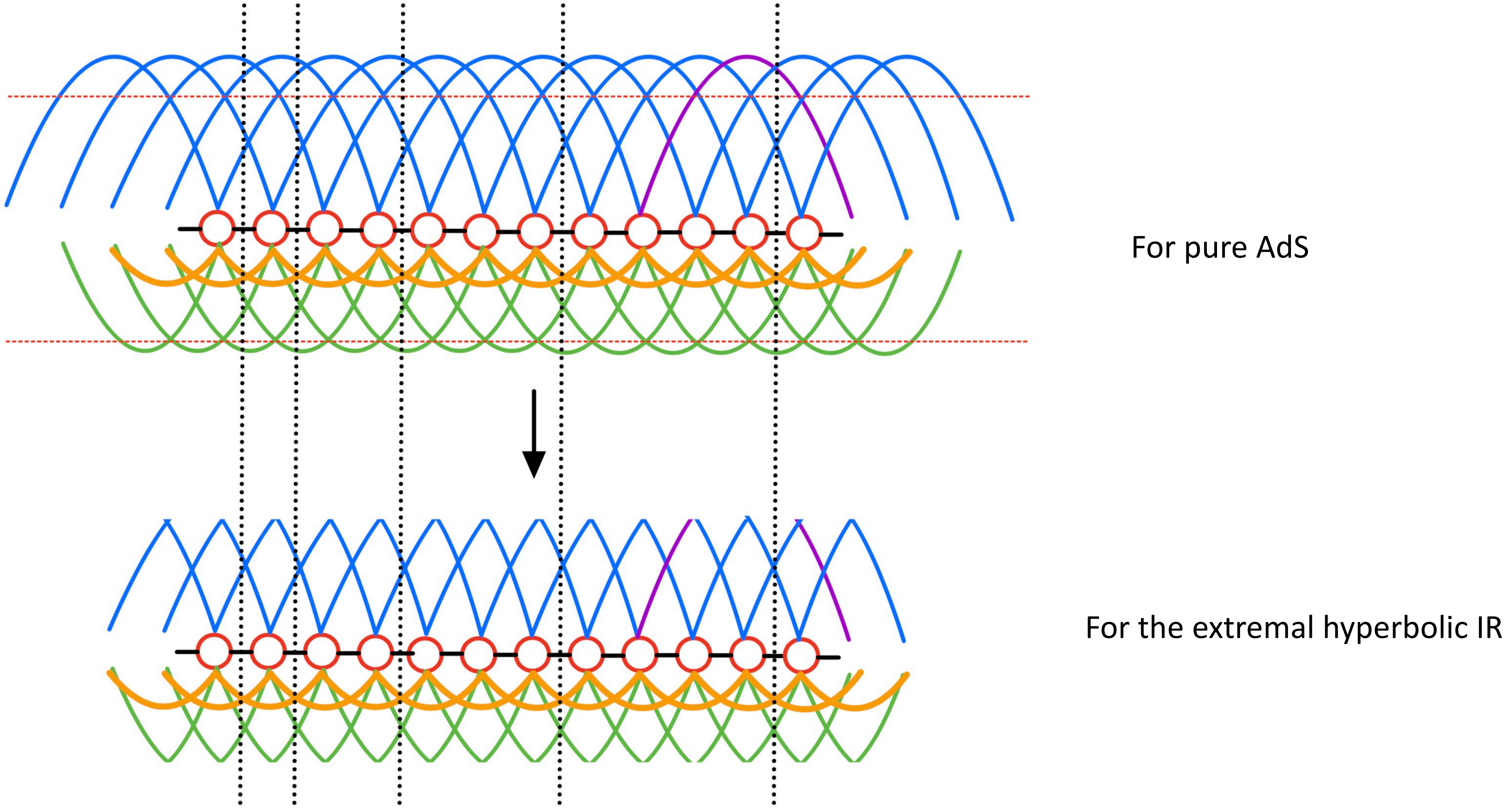
II Entanglement properties: entanglement structures

- Extremal hyperbolic IR:

$$\mathcal{I}_{hyp}(L) = \begin{cases} \mathcal{I}_{vac} = \dfrac{c}{24}\dfrac{\csc(L/l_{S^1})^2}{l_{S^1}^2} & (L < L_c \,\text{or}\, L > 2\pi l_{S^1} - L_c) \\ \dfrac{c\cot(L_c/2l_{S^1})}{12 l_{S^1}}\delta(L-L_c) = \displaystyle\int_{L_c}^{\pi l_{S^1}} -\frac{1}{2}S''_{vac}(L)dL\delta(L-L_c), & (L \to L_c) \\ 0 & (L_c \le L \le 2\pi l_{S^1} - L_c), \end{cases}$$

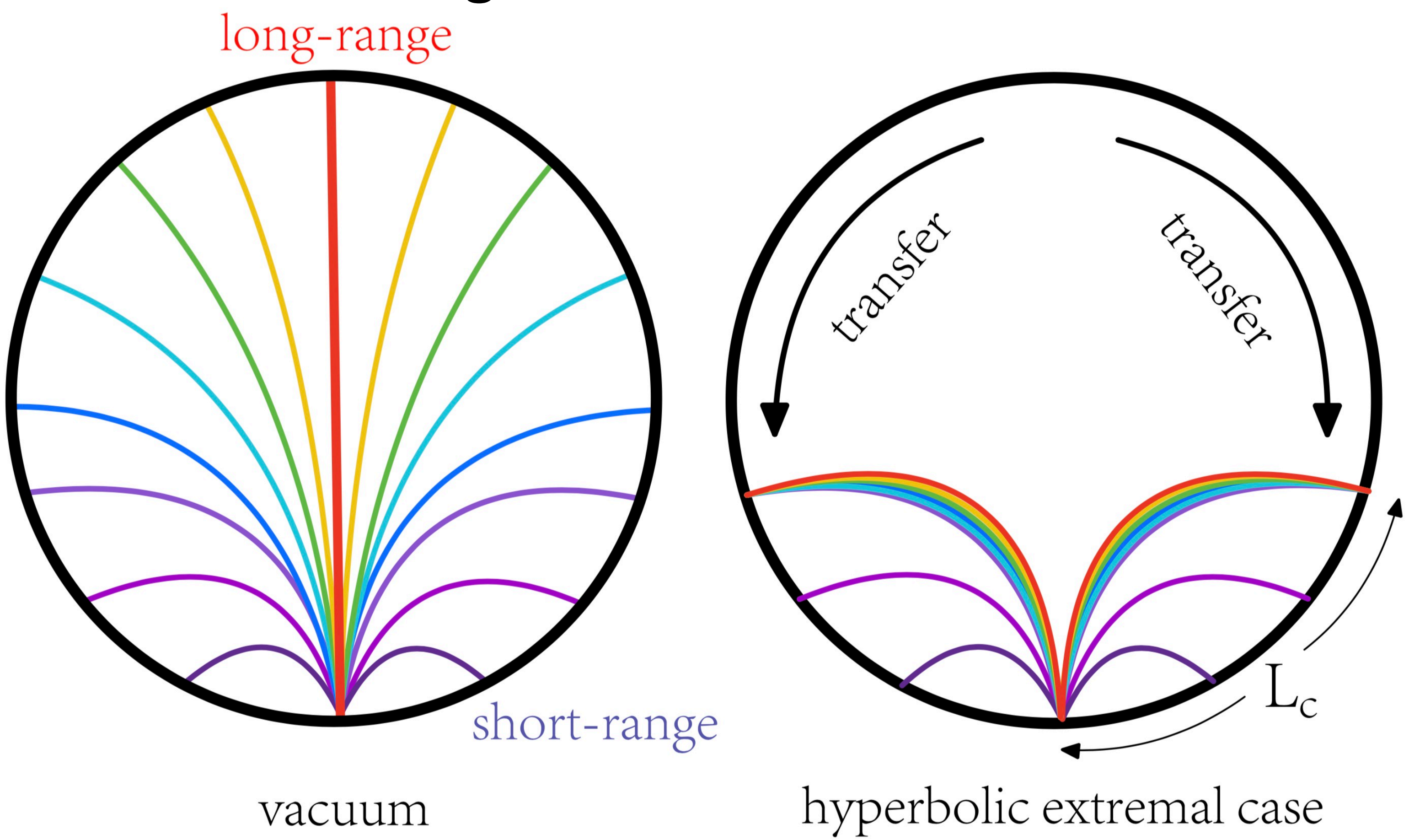
- Entanglement redistribution: the entanglement entropy of $L<L_c$ does not change, redistribution of entanglement at $L \geq L_c$
- entanglement at $L>L_c$ is eliminated and is transferred to the critical length entanglement at $L = L_c$

Illustration for the entanglement structure



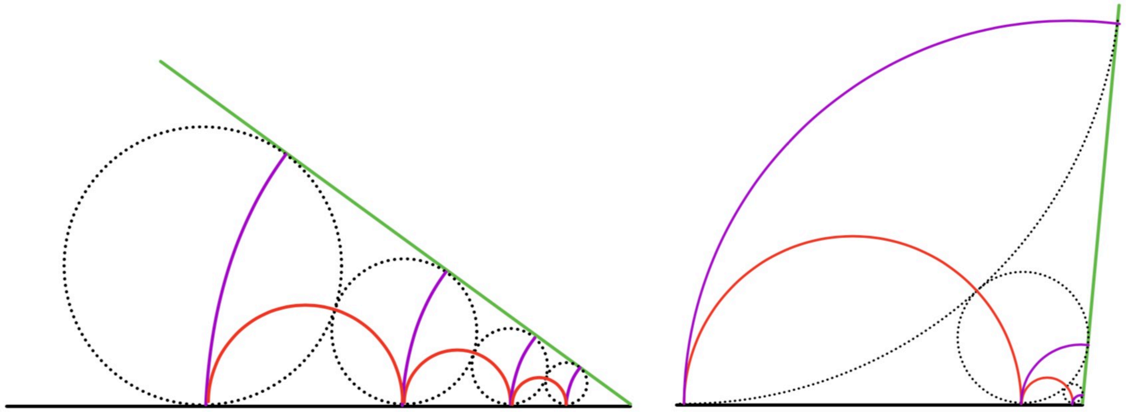
II Entanglement properties: entanglement structures

Illustration for the entanglement structure



II Entanglement properties: entanglement structures

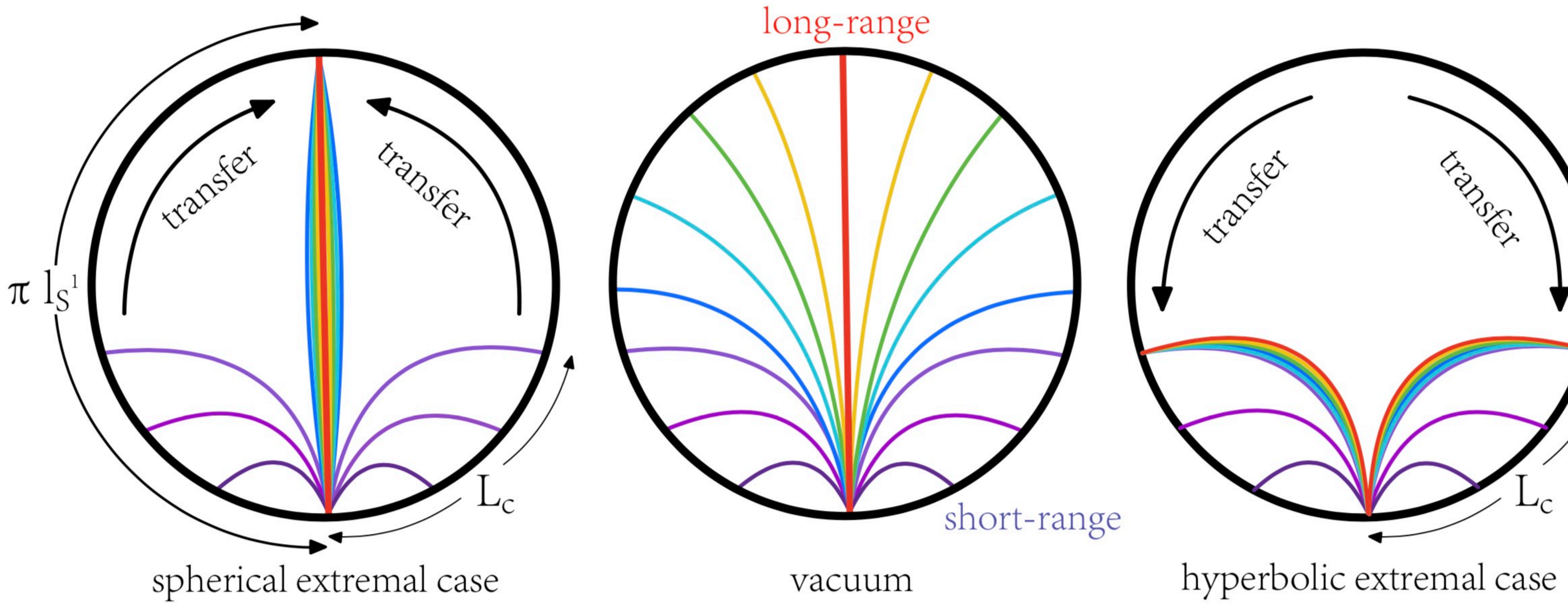
- Relation of the extremal hyperbolic IR with EoW, AdS/BCFT, AdS/ICFT
- The brane dividing the IR and UV region could be in a general shape: it could extend to the boundary
- The same as EoW concerning holographic entanglement entropy, trivial spacetime at the other side of the brane



Summary

- A new proposal to relate the IR geometry to the real space long scale entanglement structure of the dual field theory
- Changing the IR geometry on an AdS spacetime to observes the change in the long scale entanglement structure
- Two opposite types of IR geometries: spherical and hyperbolic IR
- Detect the long scale entanglement structure: besides RT surface, mutual information, specially utilizing conditional mutual information and PEE
- Redistribution of long scale entanglement structure found.

- Summary of the entanglement structure



- Open questions:
- Other IR to UV flows, change in scaling behavior
- Other measures for the entanglement structure at different scales
- Deeper relations with AdS/BCFT and AdS/ICFT?
- A puzzle regarding the critical length in the extremal hyperbolic IR geometry: L_c is one half of the IR scale while not the IR scale

Thank you!