First Law and Smarr Formula of Black Hole Mechanics in Nonlinear Electrodynamics (Gauge theories)

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Outlines

1. Introduction
2. Motivation
3. Nonlinear electrodynamics
4. First law of NED black holes
5. Smarr formula for NED black holes
6. Conclusions

1. Introduction

Analogues between thermodynamics and black hole mechanics

Law	Thermodynamics	Black hole mechanics
Zeroth	T is constant	κ is constant
First	$dE = TdS + pdV + \mu dN$	$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \Phi_H dQ$
Second	$\delta S \geq 0$	$\delta A \geq 0$
Third	Impossible to achieve $T = 0$	Impossible to achieve $\kappa = 0$

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A classical black hole is a perfect absorber, not emitting anything. In 1974, Hawking discovered the emission of particles from black holes, which confirms the thermodynamical nature of black holes and the temperature $T = \kappa/2\pi$ is real.

Proof of the first law

A general proof given by Iyer and Wald (Phys. Rev. D 50, 846 1994)

They derived the first law from a general diffeomorphism covariant Lagrangian

$$L = L(g_{ab}, R_{abcd}, \nabla_a R_{bcde}; \psi, \nabla_a \psi)$$

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They showed that the entropy of black hole corresponds to a Noether charge which is constructed from the Lagrangian.
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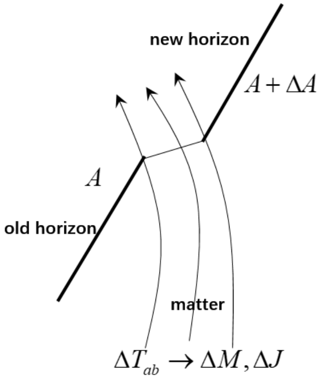
We generalized this formula to Einstein-Maxwell and Einstein-Yang-Mills theories(S.Gao, Phys. Rev. D 68, 044016 2003)

Physcial process version of the first law

[S.Gao and R.Wald, Phys.Rev.D, 64, 084020(2001)]



Let $\xi^a = t^a + \Omega_H \varphi^a$ be a Killing vector field normal to the horizon. Let ΔT_{ab} denote the stress-energy tensor of the matter perturbation. Then the changes in mass and angular momentum are given by

$$\Delta M = \int_H \Delta T_{ab} t^a k^b = \int dV \int d^2S \Delta T_{ab} t^a k^b \quad (2)$$

$$\Delta J = -\int_H \Delta T_{ab} \varphi^a k^b = -\int dV \int d^2S \Delta T_{ab} \varphi^a k^b \quad (3)$$

where k^a is the tangent to the null generators of the horizon H with the affine parameter V. On the horizon, we have the relation

$$k^a = \frac{1}{\kappa V} \xi^a \quad (4)$$

Using the Raychaudhuri's equation

$$\frac{d\theta}{dV} = -8\pi \Delta T_{ab} k^a k^b \quad (5)$$

we have

$$\kappa \int dV \int d^2S \frac{d\theta}{dV} V = -8\pi \int V \int d^2S \Delta T_{ab}(t^a + \Omega_H \varphi^a)k^b$$

which just gives

$$\kappa \Delta A = 8\pi(\Delta M - \Omega_H \Delta J)$$

2. Motivation

In 1968, Bardeen found a regular black hole without central singularity.

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2 \,, \tag{6}$$

where

$$f(r) = 1 - \frac{2Mr^2}{(r^2+q^2)^{3/2}} \,. \tag{7}$$

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It has been noticed that the Bardeen solution does not satisfy the usual first law.

In fact, a general proof for the first law of black hole mechanics in the context of nonlinear electrodynamics has been given by Rasheed (1997). By varying the Komar mass and the NED Lagrangian, he found the following first law that applies to stationary black holes with NED matter sources:

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q + \Psi_H \delta P \,, \tag{8}$$

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The purpose of our work is to generalize Rasheed's treatment and find a general first law for NED black holes.

3. Nonlinear Electrodynamics

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To derive the equations of motion, β_i must be treated as constants. However, to derive the first law, they will be treated as variables. This is crucial in our derivation.

By variation and discarding the boundary terms, we have

$$\delta\mathcal{L} = \left(R_{ab} - \frac{1}{2}Rg_{ab} - 8\pi T_{ab}\right)\delta g^{ab} + 4\sqrt{-g}\nabla_a G^{ab}\delta A_b\,, \qquad (11)$$

where G^{ab} is defined by

$$G^{ab} = -h'(F,\beta_i)F^{ab}\,. \qquad (12)$$

and

$$T_{ab} = \frac{1}{4\pi}\left[-G_a{}^c F_{bc} + \frac{1}{4}h(F,\beta_i)g_{ab}\right] \qquad (13)$$

is the stress-energy tensor of the nonlinear electromagnetic field.

$\delta S = 0$ yields the field equation

$$\nabla_a G^{ab} = 0\,, \quad R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} \qquad (14)$$

In stationary spacetimes, there exists an electric potential Φ and a magnetic potential Ψ

$$E_a = -\nabla_a \Phi \,, \tag{15}$$

$$H_a = -\nabla_a \Psi \,. \tag{16}$$

4. First law of NED black holes

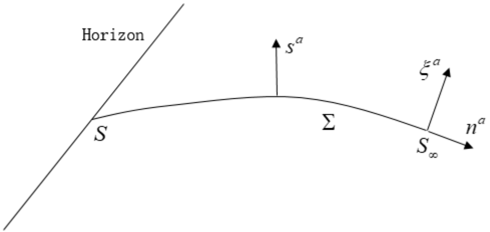

Figure: The three dimensional hypersurface Σ connecting the horizon and infinity.

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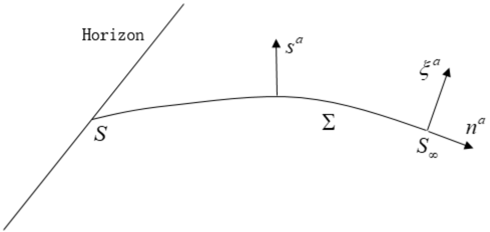

Figure: The three dimensional hypersurface Σ connecting the horizon and infinity.

The Komar mass is defined by

$$M = -\frac{1}{8\pi}\int_{S_\infty} \epsilon_{abcd}\nabla^c\xi^d \,. \tag{17}$$

and the electric charge is given by

$$Q = \frac{1}{8\pi}\int_{S_\infty} \epsilon_{abcd}G^{cd} \,. \tag{18}$$

Applying Stocks's theorem on Σ, Eq. (17) becomes

$$M = -\frac{1}{8\pi}\int_S \epsilon_{abcd}\nabla^c\xi^d - \frac{1}{8\pi}\int_\Sigma d_c(\epsilon_{abcd}\nabla^c\xi^d)\,. \tag{19}$$

By standard calculation, the first integral in Eq. (19) yields [Wald]

$$-\frac{1}{8\pi}\int_S \epsilon_{abcd}\nabla^c\xi^d = \frac{\kappa A}{4\pi}\,, \tag{20}$$

and the second integral yields

$$-\frac{1}{8\pi}\int_\Sigma d_c(\epsilon_{abcd}\nabla^c\xi^d) = \frac{1}{4\pi}\int_\Sigma R_{ab}s^a\xi^b dV\,. \tag{21}$$

Then using Einstein's equation, Eq. (19) can be written as

$$M = \frac{\kappa A}{4\pi} + 2\int_\Sigma \left(T_{ab} - \frac{1}{2}Tg_{ab}\right)s^a\xi^b dV\,. \tag{22}$$

$$\delta M = \frac{1}{4\pi}(\kappa\delta A + A\delta\kappa) + 2\delta \int_\Sigma \left(T_{ab} - \frac{1}{2} T g_{ab} \right) s^a \xi^b dV \,. \qquad (23)$$

Comparing with

$$\int_\Sigma \epsilon_{abcd}\xi^d(\delta R + \gamma^{ef} R_{ef}) = -2A\delta\kappa - 8\pi\delta M \,, \quad \gamma_{ab} = \delta g_{ab} \qquad (24)$$

we have

$$\delta M \quad = \quad \frac{\kappa}{8\pi}\delta A - \frac{1}{2}\int_\Sigma \gamma^{ef}(T_{ef})\epsilon_{abcd}\xi^d + \delta \int_\Sigma T_{cd} s^c \xi^d \epsilon^{(3)} \,. \qquad (25)$$

$$\mathcal{L}_{EM} = \sqrt{-g}\, h(F, \beta_i)\,. \tag{26}$$

$$\delta\mathcal{L}_{EM} = -8\pi\sqrt{-g}\, T_{ab}\delta g^{ab} - 2\sqrt{-g}\, G^{ab}\delta F_{ab} + \sqrt{-g}\delta h\,, \tag{27}$$

where

$$\delta h \equiv \frac{\partial h}{\partial \beta_i}\delta\beta_i\,. \tag{28}$$

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$\Longrightarrow$

$$\delta M = \frac{\kappa}{8\pi}\delta A + \delta I_1 + \delta I_2 + \sum_i \left(\frac{1}{16\pi}\int_\Sigma \frac{\partial h}{\partial \beta_i}\xi^d \epsilon_{abcd} \right) \delta\beta_i\,. \tag{29}$$

$$I_1 = \frac{1}{4\pi}\int_\Sigma G_c{}^e F_{de} s^c \xi^d \epsilon^{(3)} \tag{30}$$

$$\delta I_2 = -\frac{1}{8\pi}\int_\Sigma G^{ab}\delta F_{ab}\xi^d s_d \epsilon^{(3)}\,. \tag{31}$$

Computing δI_1

$$I_1 = \frac{1}{4\pi}\int_\Sigma G_c{}^e \nabla_e \Phi s^c \epsilon^{(3)} \,. \tag{32}$$

$\Longrightarrow$

$$\begin{aligned} I_1 &= \frac{1}{4\pi}\int_\Sigma \nabla_e (G_c{}^e \Phi) s^c \epsilon^{(3)} \\ &= \frac{1}{4\pi}\int_\Sigma \epsilon_{abcd} \nabla_e (G^{de} \Phi) \,. \end{aligned} \tag{33}$$

One can show that if

$$S_{ab} = \epsilon_{abcd} w^{cd} \,, \tag{34}$$

then

$$dS_{cab} = 2\epsilon_{cabd} \nabla_e w^{[ed]} \,. \tag{35}$$

So

$$I_1 = \frac{1}{8\pi} \int_\Sigma dS_{abc} \,, \tag{36}$$

where

$$S_{ab} = \epsilon_{abcd} \Phi G^{cd} \,. \tag{37}$$

By Stokes's theorem and the boundary condition $\Phi \to 0$ at infinity, we have

$$I_1 = \frac{1}{8\pi} \int_S S_{ab} = \frac{1}{8\pi} \int_S \epsilon_{abcd} \Phi G^{cd} = \Phi_H Q \,, \tag{38}$$

where $\Phi_H \equiv \Phi|_H$ and in the last step, we have used the result that Φ is constant on the horizon. Hence

$$\delta I_1 = \Phi_H \delta Q + Q \delta \Phi_H \,. \tag{39}$$

Computing δI_2

$$Y^a \equiv \epsilon^{abcd}(*G_{cd}\delta E_b - H_b \delta F_{cd})\,. \tag{40}$$

$$\begin{aligned} t_1^a &= \epsilon^{abcd} * G_{cd}\delta E_b \\ &= \frac{1}{2}\epsilon^{abcd}\epsilon_{cdef}G^{ef}\delta E_b \\ &= -\frac{1}{2}2!2!\delta_e^{[a}\delta_f^{b]}G^{ef}\delta E_b \\ &= -2G^{ab}\xi^e\delta F_{be}\,. \end{aligned} \tag{41}$$

$$\begin{aligned} t_2^a &= \epsilon^{abcd} * G_{be}\xi^e\delta F_{cd} \\ &= -\frac{1}{2}\epsilon^{bacd}\epsilon_{beij}G^{ij}\xi^e\delta F_{cd} \\ &= \frac{1}{2}3!\delta_e^{[a}\delta_i^c\delta_j^{d]}G^{ij}\xi^e\delta F_{cd} \\ &= 3\xi^{[a}G^{cd]}\delta F_{cd} \\ &= 3\frac{2}{3!}(\xi^a G^{cd} + \xi^c G^{da} + \xi^d G^{ac})\delta F_{cd} \\ &= \xi^a G^{cd}\delta F_{cd} + 2\xi^e G^{ad}\delta F_{de}\,. \end{aligned} \tag{42}$$

Adding Eq. (41) and (42), we have

$$Y^a = \xi^a G^{cd} \delta F_{cd} \,. \tag{43}$$

So Eq. (31) can be written as

$$\begin{aligned}
\delta I_2 &= -\frac{1}{8\pi} \int Y^d s_d \epsilon^{(3)} \\
&= -\frac{1}{8\pi} \int \epsilon_{abcd} Y^d \\
&= -\frac{1}{8\pi} \int \epsilon_{abcd} \epsilon^{dkef} (* G_{ef} \delta E_b - H_k \delta F_{ef}) \\
&= -\frac{1}{8\pi} \int \epsilon_{abcd} (-2) G^{de} \delta E_e + \frac{1}{8\pi} \int \epsilon_{abcd} \epsilon^{dkef} H_k \delta F_{ef} \\
&= -\frac{1}{4\pi} \int \epsilon_{abcd} G^{de} \delta (\nabla_e \Phi) - \frac{1}{8\pi} \int \epsilon_{abcd} \epsilon^{dkef} (\nabla_k \Psi) \delta F_{ef} \\
&= -\frac{1}{4\pi} \int \epsilon_{abcd} \nabla_e (G^{de} \delta \Phi) - \frac{1}{8\pi} \int \epsilon_{abcd} \epsilon^{dkef} \nabla_k (\Psi \delta F_{ef}) \\
&+ \frac{1}{8\pi} \int \epsilon_{abcd} \epsilon^{dkef} \Psi \nabla_k \delta F_{ef} \,.
\end{aligned} \tag{44}$$

The last term vanishes because $\nabla_{[k} F_{ef]} = 0$.

So

$$\delta I_2 = \frac{1}{4\pi}\int_\Sigma \epsilon_{abcd}\nabla_e(G^{ed}\delta\Phi) + \frac{1}{8\pi}\int_\Sigma \epsilon_{abcd}\nabla_k(\epsilon^{kdef}\Psi\delta F_{ef})\,. \quad (45)$$

By applying Eqs. (34),(35) and Stokes's theorem, we obtain

$$\begin{aligned}\delta I_2 &= -\frac{1}{8\pi}\int_S \epsilon_{abcd}G^{ed}\delta\Phi_H - \frac{1}{16\pi}\int_S \epsilon_{abcd}\epsilon^{cdef}\Psi\delta F_{ef} \\ &= -Q\delta\Phi_H + \frac{\Psi_H}{4\pi}\int_S \delta F_{cd} \\ &= -Q\delta\Phi_H + \Psi_H\delta P \end{aligned} \quad (46)$$

By substituting Eqs. (39) and (46) into Eq. (29), we obtain the final form of the first law

$$\boxed{\delta M = \frac{\kappa}{8\pi}\delta A + \Phi_H\delta Q + \Psi_H\delta P + \sum_i K_i\delta\beta_i\,.} \quad (47)$$

where

$$K_i = -\frac{1}{16\pi}\int_\Sigma \frac{\partial h}{\partial\beta_i}\xi^d\epsilon_{dabc} \quad (48)$$

5. Smarr formula for NED black holes

$$S = \int d^4x\sqrt{-g}[R + h(F)]\,, \tag{49}$$

Consider the transformation

$$A_a \to \alpha A_a \tag{50}$$

In Einstein-Maxwell theory, the metric should change

$$g_{ab} \to \alpha^2 g_{ab} \tag{51}$$

to preserve the Lagrangian. Consequently,

$$\sqrt{-g} \to \alpha^4\sqrt{-g}, \quad R \to \alpha^{-2}R, \quad F \to \alpha^{-2}F\,, h(F,\beta_i) \to \alpha^{-2}h(F,\beta_i) \tag{52}$$

and

$$M \to \alpha M,\ \kappa \to \alpha^{-1}\kappa\,, \Phi_H \to \Phi_H, \xi^a \to \alpha^{-1}\xi^a\,, \tag{53}$$

$$A \to \alpha^2 A \quad Q \to \alpha Q, \quad \Psi_H \to \Psi_H, \quad P \to \alpha P\,. \tag{54}$$

We assume

$$\beta_i \to \alpha^{b_i}\beta_i\,. \tag{55}$$

The value of b_i depends on the specific form of $h(F,\beta_i)$ such that $h(F,\beta_i) \to \alpha^{-2}h(F,\beta_i)$. Then

$$K_i \to \alpha^{1-b_i}K_i \tag{56}$$

Since the theory is invariant under the scale transformation, the first law

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Phi_H\delta Q + \Psi_H\delta P + \sum_i K_i\delta\beta_i\,. \tag{57}$$

should hold after the transformation, i.e.,

$$\delta(\alpha M) = \frac{\alpha^{-1}\kappa}{8\pi}\delta(\alpha^2 A) + \Phi_H\delta(\alpha Q) + \Psi_H\delta(\alpha P) + \sum_i \alpha^{1-b_i}K_i\delta(\alpha^{b_i}\beta_i)\,. \tag{58}$$

Variation with respect to all the quantities (including α)yields

$$\boxed{M = \frac{\kappa A}{4\pi} + \Phi_H Q + \Psi_H P + \sum_i b_i K_i\beta_i} \tag{59}$$

Application to Bardeen black holes

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2 \,, \tag{60}$$

where

$$f(r) = 1 - \frac{2Mr^2}{(r^2+q^2)^{3/2}} \,. \tag{61}$$

The horizon is located at $f(r = r_h) = 0$, which gives the relation

$$M = \frac{(r_h^2+q^2)^{3/2}}{2r_h^2} \,, \tag{62}$$

$$\text{surface gravity: } \kappa = \frac{1}{2} f'(r_h) = \frac{M r_h(-2q^2 + r_h^2)}{(q^2+r_h^2)^{5/2}} \,. \tag{63}$$

Ayón-Beato and García first found the following Lagrangian that generates the Bardeen solution

$$h(F,\beta_i) = h(F,M,q) = -\frac{12M}{q^3}\left(\frac{\sqrt{2q^2F}}{1+\sqrt{2q^2F}}\right)^{5/2} . \tag{64}$$

with

$$F_{ab} = q\sin\theta(d\theta_a d\phi_b - d\phi_a d\theta_b) . \tag{65}$$

So

$$F = \frac{q^2}{2r^4} . \tag{66}$$

we find

$$P = \int F_{\theta\phi} d\theta d\phi = q . \tag{67}$$

Thus, q is just the magnetic charge of the black hole.

$$H_a = \frac{15Mqr^4}{2\left(q^2+r^2\right)^{7/2}} dr_a \,. \tag{68}$$

So

$$\Psi(r) = \frac{3M\left(2q^2r^2\sqrt{q^2+r^2}+r^4\sqrt{q^2+r^2}+q^4\sqrt{q^2+r^2}-r^5\right)}{2q\left(q^2+r^2\right)^{5/2}} \,, \tag{69}$$

The first law (47) can be expressed as

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Psi_H \delta q + K_q \delta q + K_M \delta M \,, \tag{70}$$

with

$$K_q = \frac{3M}{2q(q^2+r_h^2)^3}\left[\left(2q^2r_h^3+r_h^5\right)\sqrt{q^2+r_h^2}-(q^2+r_h^2)^3\right] \,, \tag{71}$$

$$K_M = \frac{q^4+2q^2r_h^2+r_h^4-r_h^3\sqrt{q^2+r_h^2}}{(q^2+r_h^2)^2} \,. \tag{72}$$

Since $M \to \alpha M$, $q \to \alpha q$, i.e., $b_i = 1$, application of Eq. (59) yields the Smarr formula

$$M = \frac{\kappa}{4\pi} A + \Psi_H q + K_q q + K_M M \,. \tag{73}$$

Application to Born-Infeld theory

The Lagrangian describing BI theory is

$$h(F,b) = \frac{4}{b^2}\left(1 - \sqrt{1 + \frac{1}{2}b^2F^2}\right) , \tag{74}$$

where b is a constant called the BI vacuum polarization.

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According to Eqs. (12) and (13), G^{ab} and T_{ab} are given by

$$G^{ab} = \frac{F^{ab}}{\sqrt{1+\frac{1}{2}b^2F}}, \tag{75}$$

$$T_{ab} = \frac{b^2 F_a{}^c F_{bc} + \left(\sqrt{1+\frac{1}{2}b^2F} - 1 - \frac{1}{2}b^2F\right) g_{ab}}{b^2\sqrt{1+\frac{1}{2}b^2F}}. \tag{76}$$

Born-Infeld solution is a spherically symmetric solution associated with the Lagrangian (74):

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad (77)$$

where the function $m(r)$ satisfies

$$m'(r) = \frac{1}{b^2}\left(\sqrt{r^4 + b^2 Q^2} - r^2\right)\,, \quad (78)$$

One can verify that Q is the electric charge and

$$M = \lim_{r\to\infty} m(r) \quad (79)$$

is the ADM mass.

By integration, we find

$$
\begin{aligned}
m(r) &= M - \frac{1}{b^2}\int_r^\infty dx\left(\sqrt{x^4+b^2Q^2}-x^2\right) \\
&= M - \frac{r^4 - r^2\sqrt{b^2Q^2+r^4} + 2b^2Q^2\,{}_2F_1(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{b^2Q^2}{r^4})}{3b^2 r}, \qquad (80)
\end{aligned}
$$

where ${}_2F_1$ is the hypergeometric function.
Since on the horizon

$$
m(r_h) = \frac{r_h}{2}\,, \qquad (81)
$$

the mass in Eq. (80) can be written as

$$
M = \frac{r_h}{2} + \frac{r_h^4 - r_h^2\sqrt{b^2Q^2+r_h^4} + 2b^2Q^2\,{}_2F_1(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{b^2Q^2}{r_h^4})}{3b^2 r_h}\,. \qquad (82)
$$

Treating b as a constant, the first law is

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Phi_H \delta Q \,. \tag{83}$$

Note that Eq. (83) does not correspond to an integral form, i.e., the Smarr formula.

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The scale argument requires the variation of b. The coefficient

$$\begin{aligned} K &\equiv -\frac{1}{16\pi}\int_{r_h}^{\infty} 4\pi \frac{\partial h}{\partial b} r^2 dr \\ &= -\frac{1}{3b^3 r_h}\left[2r_h^4 - 2r_h^2\sqrt{b^2Q^2 + r_h^4} + b^2Q^2\,{}_2F_1\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{b^2Q^2}{r_h^4}\right)\right] \end{aligned} \tag{84}$$

Thus, by applying our formula (47), we finally have

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Phi_H \delta Q + K\delta b \,. \tag{85}$$

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$b \to \alpha b$ preserves the action. Therefore

$$M = \frac{\kappa A}{4\pi} + \Phi_H Q + K b\,. \tag{86}$$

This is the desired Smarr formula for BI black holes.

6. Conclusions

We provide a rigorous proof for the first law and Smarr formula that apply to a general nonlinear electrodynamics theory. Compared to Rasheed's result, our first law has a wider application.

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Our work suggests that there are two kinds of variables in Lagrangians. When deriving the equations of motion of the theory, only dynamical fields should be varied and nondynamical variables are held fixed. When deriving the first law and Smarr formula, all variables should be varied.

Thank You!