Relativistic hydrodynamics for heavy–ion collisions – can a macroscopic approach be applied to a microscopic system?

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thanks to:

Ioannis Bouras, Gabriel S. Denicol, Kai Gallmeister, Carsten Greiner, Pasi Huovinen, Etele Molnár, Harri Niemi, Jorge Noronha, Zhe Xu

Fundamentals of fluid dynamics (I)

Fluid dynamics is a theory that describes the motion of macroscopic fluids

Fluids are liquids (e.g. water $\Longrightarrow$ "hydrodynamics") or gases (e.g. air)

Equations of motion:

– non-relativistic, without dissipation:

Mass conservation: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0\,,$ ρ **: mass density,** $\vec{v}$**: fluid velocity**

Euler's equation: $\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla p\,,$ p **: pressure**

– relativistic, including dissipation:

Net-charge conservation: $\partial_\mu N^\mu = 0\,,$ N^μ**: net-charge 4-current**

Energy-momentum conservation: $\partial_\mu T^{\mu\nu} = 0\,,$ $T^{\mu\nu}$**: energy-momentum tensor**

Fundamentals of fluid dynamics (II)

Range of validity of fluid dynamics:

A fluid "element" contains typically $\sim 10^{23}$ particles per gram

$\Longrightarrow$ interparticle distance $\lambda \ll L$,

L: characteristic macroscopic length scale

(length scale of variation of fluid fields, $L^{-1} \sim |\nabla \vec{v}|/c\,, \; |\nabla p|/p$)

Mean-free path of particle interactions: $\ell \sim \lambda$

$\Longrightarrow$ Knudsen number $\boxed{\mathrm{Kn} \equiv \ell/L}$

$\Longrightarrow$ fluid dynamics is valid if $\mathrm{Kn} \ll 1$

$\Longrightarrow$ fluid dynamics can be derived from an underlying microscopic theory as a power series in Kn (I)

Where does microscopic information enter the fluid-dynamical eqs. of motion?

$\Longrightarrow$ equation of state $p(\epsilon, n)$, transport coefficients $\zeta\,,\; \eta\,,\; \kappa\,, \ldots$

$\Longrightarrow$ "low-energy constants" (II)

$\stackrel{\text{(I,II)}}{\Longrightarrow}$ Fluid dynamics is long-distance, large-time effective theory of a given underlying microscopic theory

Why heavy-ion collisions?

Fundamental theory of strong interactions:

$\Longrightarrow$ Quantum Chromodynamics (QCD)

$\Longrightarrow$ QCD phase diagram:

Hadronic phase:

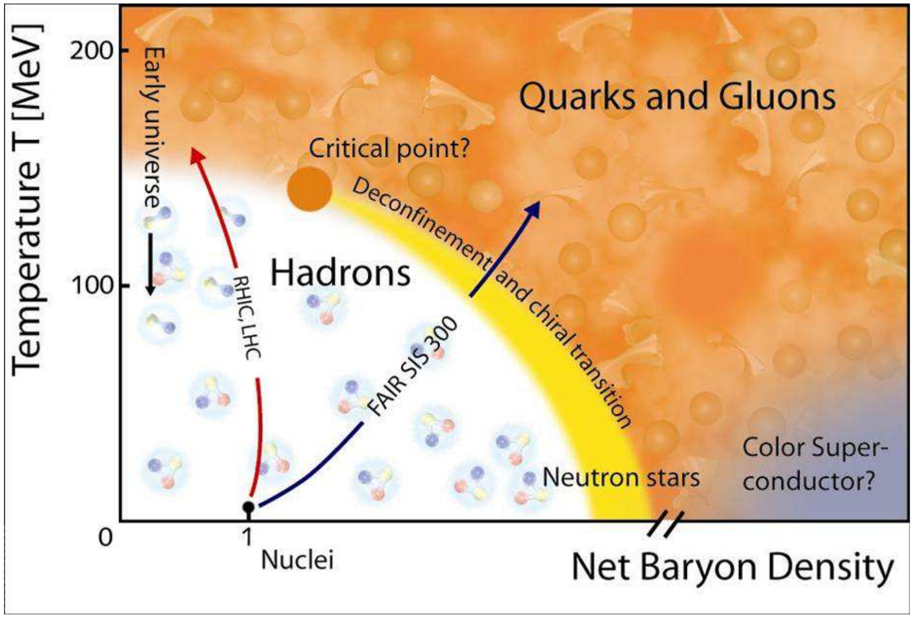
Confinement of quarks and gluons

(Chiral symmetry broken $\langle \bar{q}q \rangle \neq 0$)

Quark-Gluon Plasma (QGP):

Deconfined quarks and gluons

(Chiral symmetry restored $\langle \bar{q}q \rangle \simeq 0$)

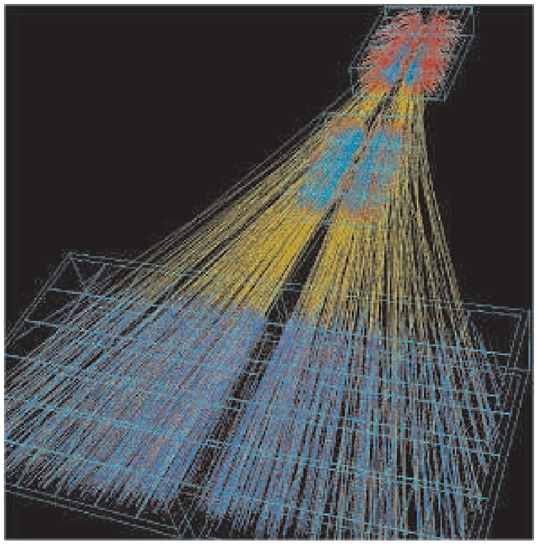


Heating and compressing QCD matter $\Longrightarrow$ heavy-ion collisions!

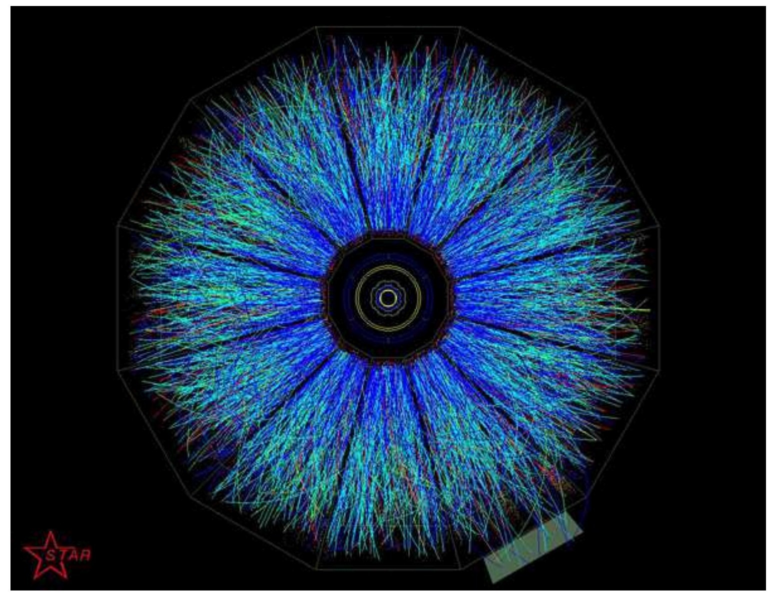
$\Longrightarrow$ Study phase transitions (in particular, deconfinement and chiral transitions) in fundamental theory of nature (QCD) in the laboratory!

$\Longrightarrow$ Study early-universe matter, as it existed $\sim 10^{-6}$ sec after the big bang!

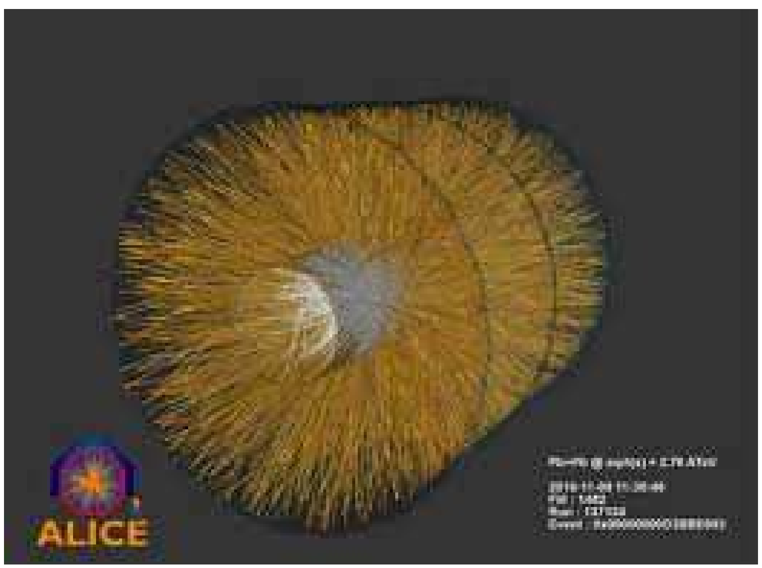
Heavy-ion collisions: the experimentalist's view



Pb+Pb collision at $\sqrt{s_{\mathrm{NN}}} = 17.3$ GeV
NA49 experiment @ CERN-SPS

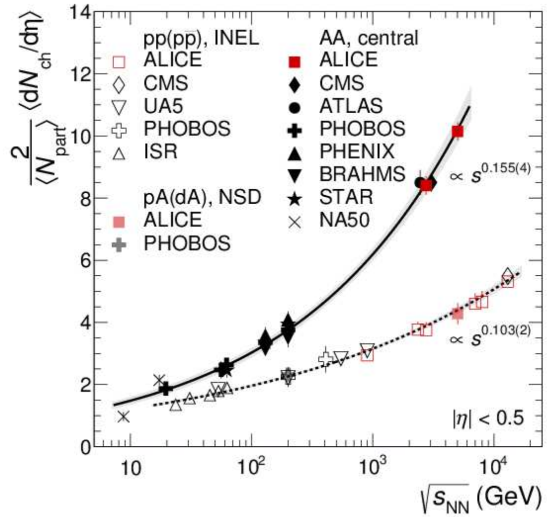


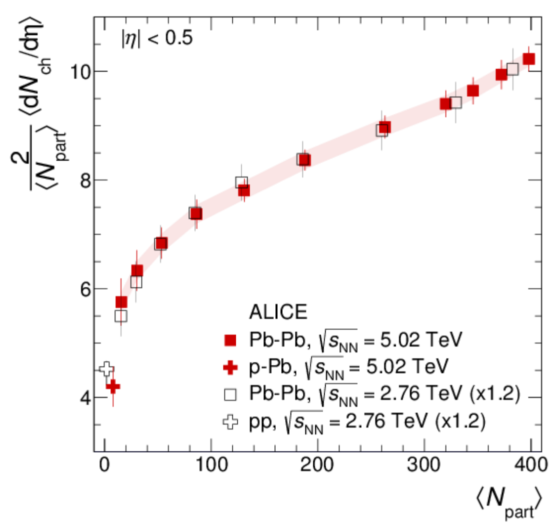
Au+Au collision at $\sqrt{s_{\mathrm{NN}}} = 200$ GeV
STAR experiment @ BNL-RHIC



Pb+Pb collision at $\sqrt{s_{\mathrm{NN}}} = 2.76$ TeV
ALICE experiment @ CERN-LHC

Heavy-ion collisions: creating early-universe matter





Central PbPb collisions at $\sqrt{s_{NN}} = 5.02$ TeV:

– How many particles are created?

Within an angle of $\simeq 40^o$ transverse to beam axis:

$$\Longrightarrow \quad N = \frac{3}{2} N_{ch} \sim \frac{3}{2} \times 10 \times 200 = 3000 \text{ particles!}$$

Average energy: $\frac{dE_T}{d\eta} / \frac{dN_{ch}}{d\eta} \sim 1$ GeV!

– What is the energy density at time τ_0?

J.D. Bjorken, PRD 27 (1983) 140

$$\Longrightarrow \quad \epsilon \sim \frac{1}{A_\perp \tau_0} \frac{dE_T}{d\eta} \sim \frac{3000\,\text{GeV}}{100\,\text{fm}^2 \tau_0} \sim 30 \frac{\text{GeV}}{\text{fm}^3} \text{ at } \tau_0 \sim 1 \frac{\text{fm}}{c}!$$

$\Longrightarrow$ high-(energy-)density environment!

$\Longrightarrow$ in QGP phase of QCD matter!

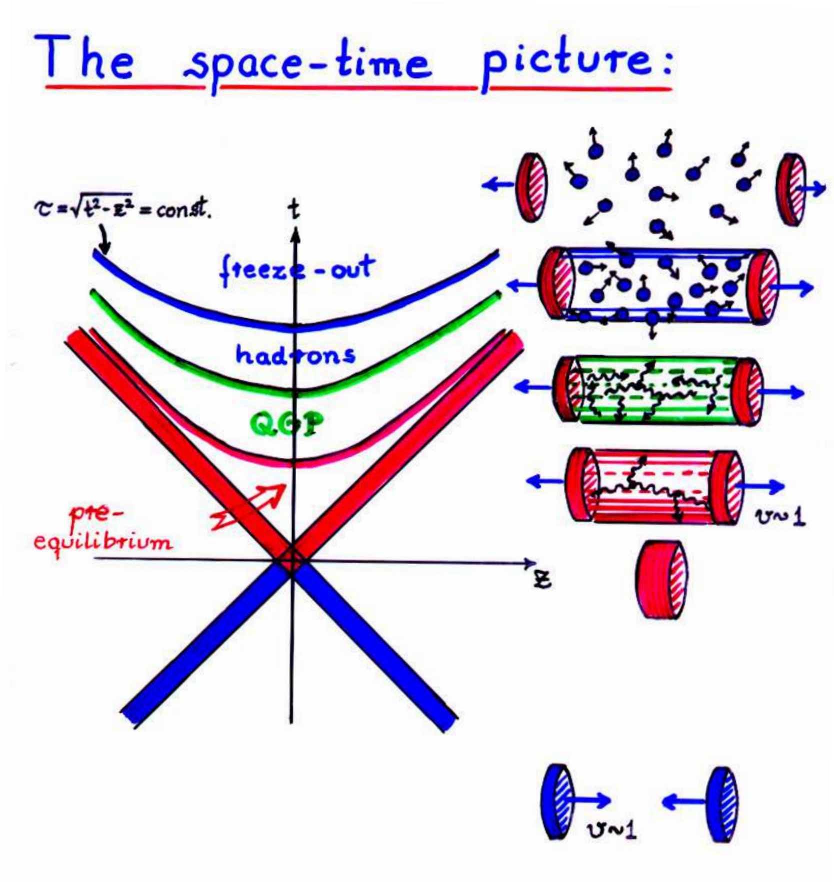
$\Longrightarrow$ interparticle distance $\lambda \sim 1/T \lesssim 0.5$ fm,
while system size $L \sim 10$ fm

$\Longrightarrow$ Kn $\sim \lambda/L \sim 0.05 \ll 1$!

$\Longrightarrow$ fluid dynamics may be applicable!

Heavy-ion collisions: the theorist's view

Bjorken's picture: J.D. Bjorken, PRD 27 (1983) 140



Below certain temperature T_f hadrons freeze-out

⇑

Below certain temperature T_c QGP hadronizes

⇑

. . . QGP

⇑

Leave in their wake highly excited quark-gluon matter which thermalizes ($\mathrm{Kn} \sim \lambda/L \ll 1$!) and becomes a . . .

⇑

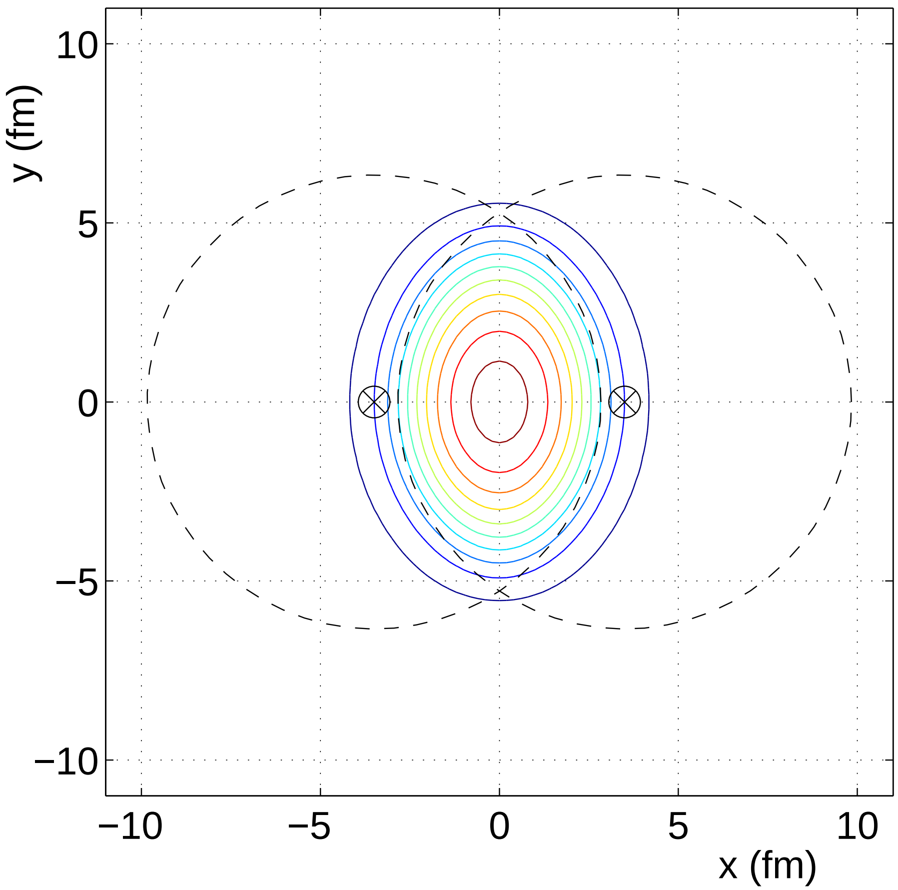
Highly Lorentz-contracted nuclei pass through each other

Note: Bjorken assumed that evolution of QGP and hadronic phase prior to freeze-out can be described by ideal fluid dynamics (1+1-d boost-invariant scaling solution)
$\Longrightarrow$ physics constant on proper-time hypersurfaces $\tau = \sqrt{t^2 - z^2} = const.$

How can we decide whether fluid dynamics applies?

Fluid dynamics implies collective flow:

non-central heavy-ion collision
in transverse $(x - y)$ plane $(z = 0)$:



If particles do not interact with each other, they stream freely towards the detector

$\Longrightarrow$ single-inclusive particle spectrum:

$$E\,\frac{dN}{d^3\vec{p}} \equiv E\,\frac{dN}{dp_z\,d^2\vec{p}_\perp}\,, \quad p_\perp = \sqrt{p_x^2 + p_y^2}$$

transverse momentum

$$\equiv E\,\frac{dN}{dp_z\,p_\perp dp_\perp\,d\varphi} \equiv \frac{dN}{dy\,p_\perp dp_\perp d\varphi}\,,$$

$\tanh y \equiv \frac{p_z}{E}$, y : longitudinal rapidity

is independent of azimuthal angle φ

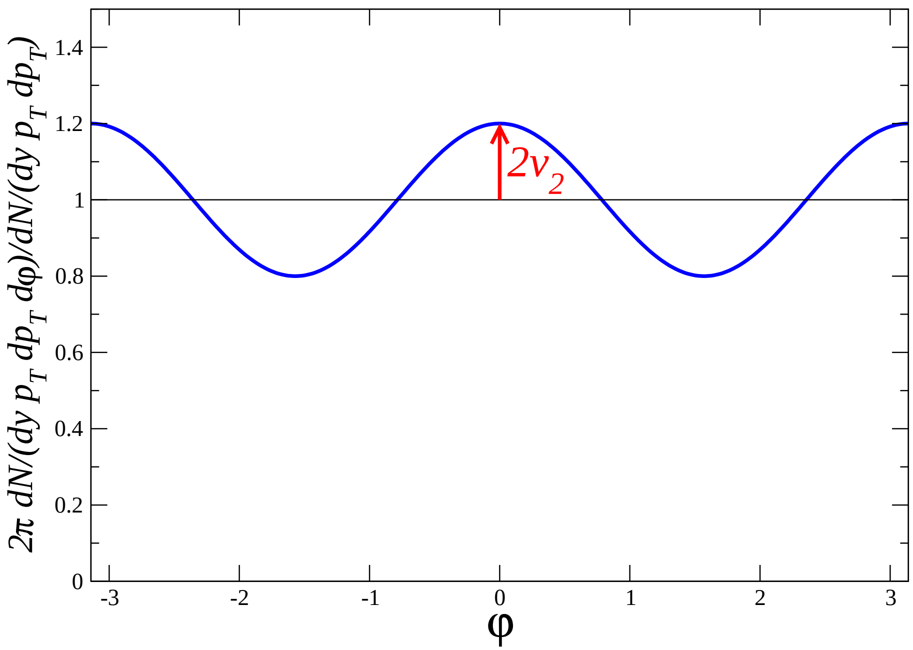
$\Longrightarrow$ information on initial geometry is lost

But: If particles interact strongly (like in a fluid), collective flow develops

$\Longrightarrow$ initial spatial asymmetry is, by difference in pressure gradients, converted to final momentum anisotropy

Characterization of collective flow

Event-averaged single-inclusive particle spectrum at $y = 0$ as function of φ:



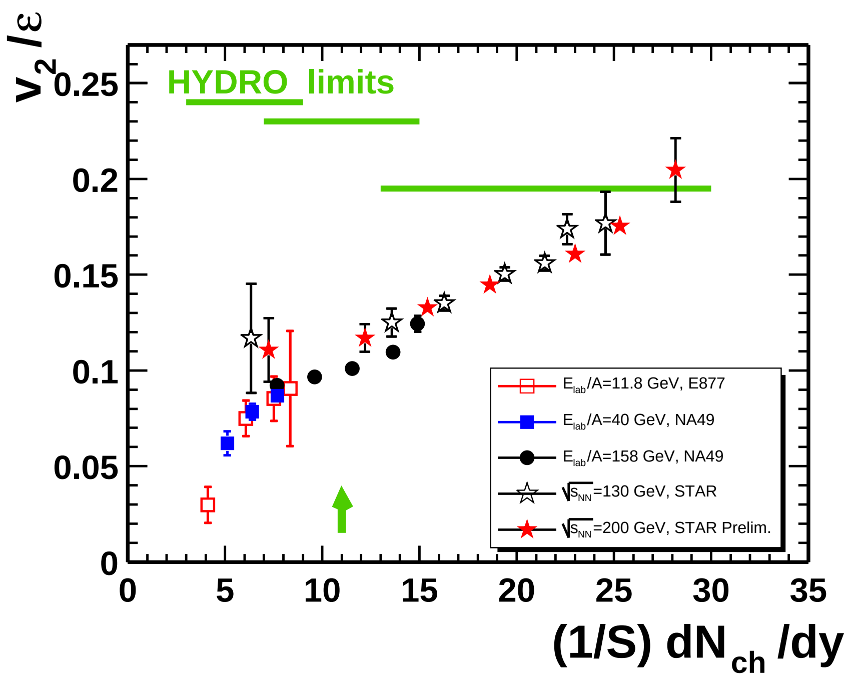
$\Longrightarrow$ **preferential emission of particles in the reaction $(x - z)$ plane**

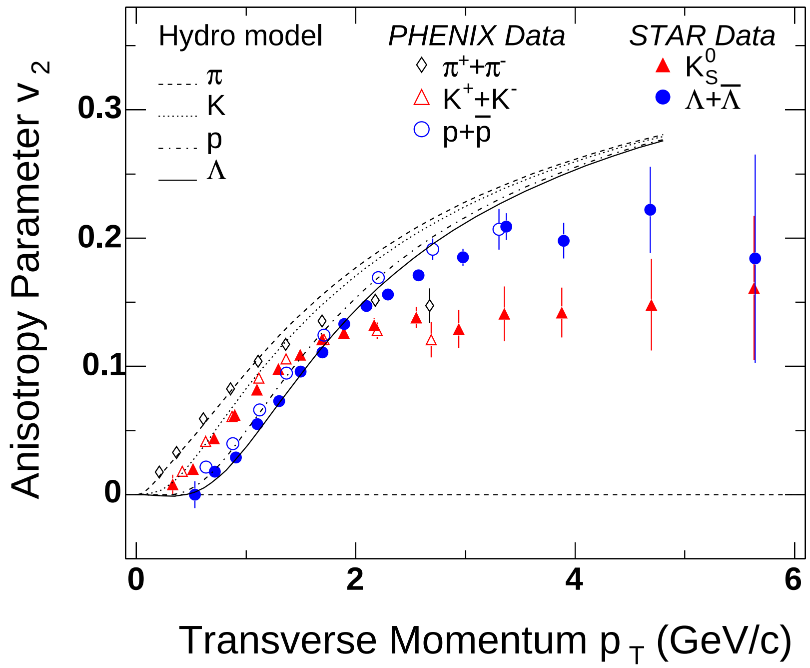
$\Longrightarrow$ **Fourier decomposition of single-inclusive particle spectrum:**

$$E\frac{dN}{d^3\vec{p}} \equiv \frac{dN}{dy\, p_\perp dp_\perp d\varphi} \equiv \frac{1}{2\pi}\frac{dN}{dy\, p_\perp dp_\perp}\left[1 + 2\sum_{n=1}^{\infty} v_n(y, p_\perp)\, \cos(n\varphi)\right]$$

v_1 : **directed flow**, v_2 : **elliptic flow** v_3 : **triangular flow** , **etc.**

Data confronts theory





⟹ approach to fluid-dynamics with increasing centrality and beam energy!

⟹ prediction of fluid dynamics: mass ordering of $v_2(p_T)$!

Success story no. 1: quantitative description of elliptic flow at RHIC within ideal fluid dynamics

⟹ no dissipative effects! ⟹ "RHIC scientists serve up the perfect fluid"

Two problems (I)

1. There is no real ideal fluid!

shear viscosity $\eta \sim \frac{T}{\langle\sigma\rangle} \to 0 \iff$ average scattering cross section $\langle\sigma\rangle \to \infty$

minimal value for shear viscosity to entropy density ratio:

(i) from uncertainty principle ("quantum limit"): $\frac{\eta}{s} \simeq \frac{1}{12}$

P. Danielewicz, M. Gyulassy, PRD 31 (1985) 53

(ii) from AdS/CFT correspondence: conjectured lower bound $\frac{\eta}{s} = \frac{1}{4\pi}$

P. Kovtun, D.T. Son, A. Starinets, PRL 94 (2005) 111601

$\Longrightarrow$ What is $\frac{\eta}{s}$ of hot and dense hadronic matter?

If $\frac{\eta}{s} \ll 1 \Longrightarrow$ matter is strongly interacting!

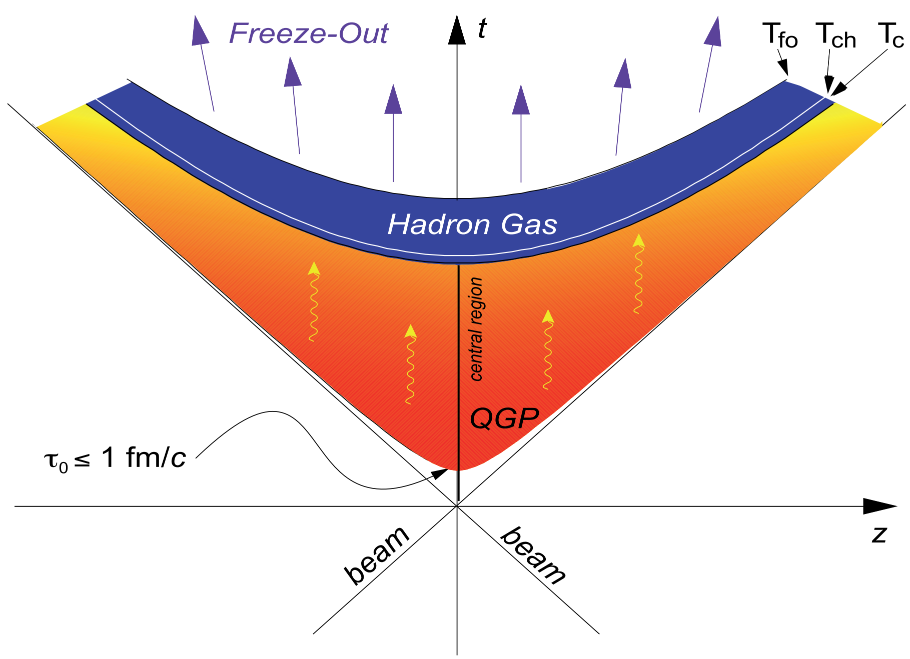
$\Longrightarrow$ "strongly coupled quark-gluon plasma" (sQGP)

Two problems (II)

2. Fluid-dynamical equations of motion: $\partial_\mu T^{\mu\nu} = 0$

$\Longrightarrow$ partial differential equations

$\Longrightarrow$ require initial conditions on a space-time hypersurface



energy-momentum tensor $T^{\mu\nu}(\tau_0, \vec{x})$
on initial space-time hypersurface
$\tau \equiv \sqrt{t^2 - z^2} \equiv \tau_0 = const.$

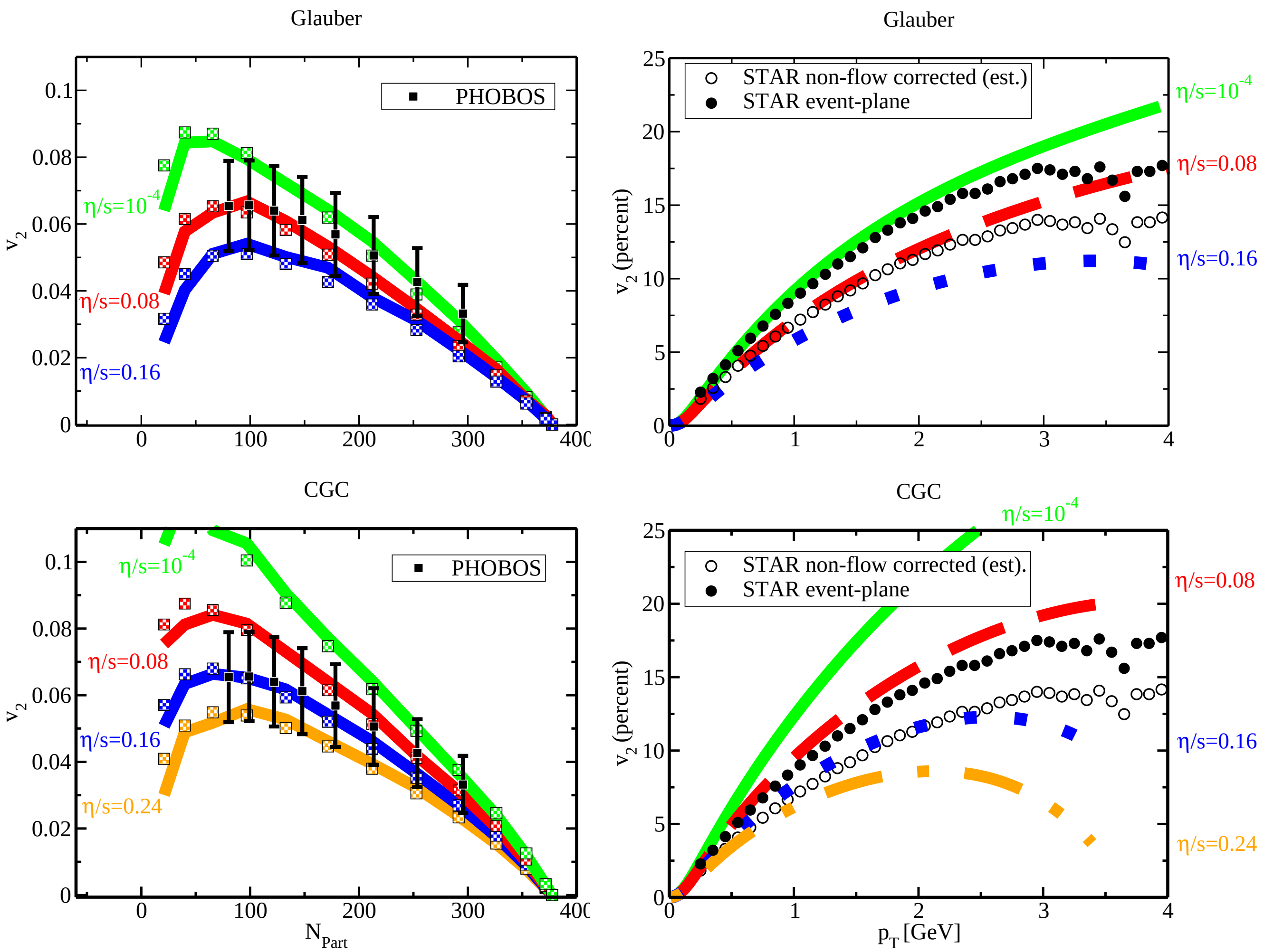
$\Longrightarrow$ continuum of parameters to fit to experimental data

$\Longrightarrow$ experimental data may allow for non-zero viscosity!

$\Longrightarrow$ need calculations within dissipative fluid dynamics and with realistic initial conditions!

Interplay between dissipation and initial conditions

M. Luzum, P. Romatschke, PRC 78 (2008) 034915; Erratum C 79 (2009) 039903



Glauber initial conditions:

$$\eta/s \simeq 0.08$$

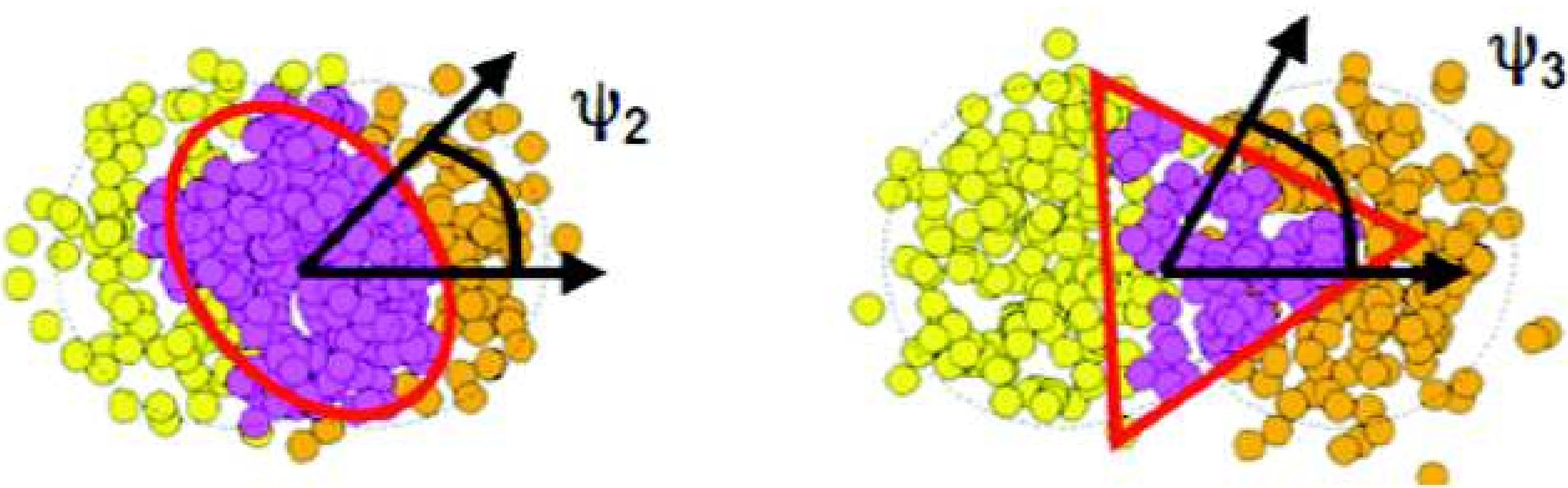
Color-Glass Condensate (CGC) initial conditions:

$$\eta/s \simeq 0.16$$

Event-by-event fluctuations

event by event: fluctuations of initial geometry

$\Longrightarrow$ **rotate participant plane vs. reaction plane $\psi_2 \neq 0$**



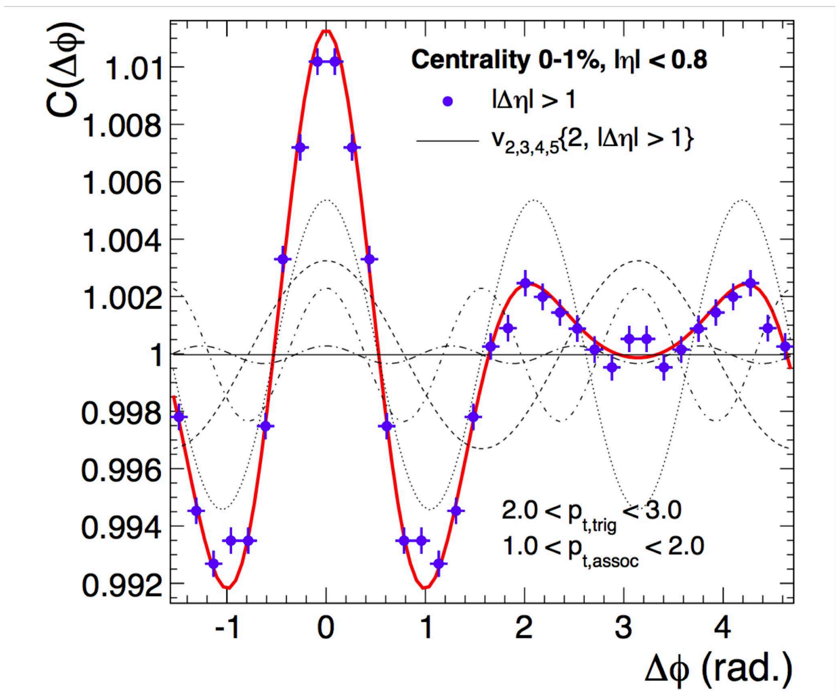
$\Longrightarrow$ **(i) induce higher flow harmonics! $v_n \neq 0\,,\; n = 3, 4, \ldots$**

$\Longrightarrow$ **(ii) provide additional constraint on η/s and initial conditions!**

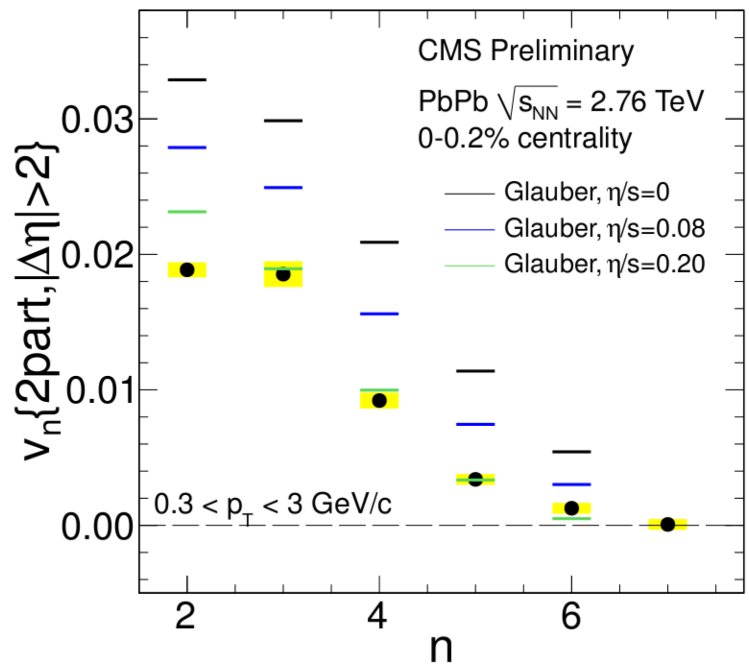
Event-by-event higher flow harmonics

⟹ **two-particle correlation functions as superposition of higher flow harmonics**

B. Alver, G. Roland, PRC 81 (2010) 054905



ALICE Collaboration, PRL 107 (2011) 032301

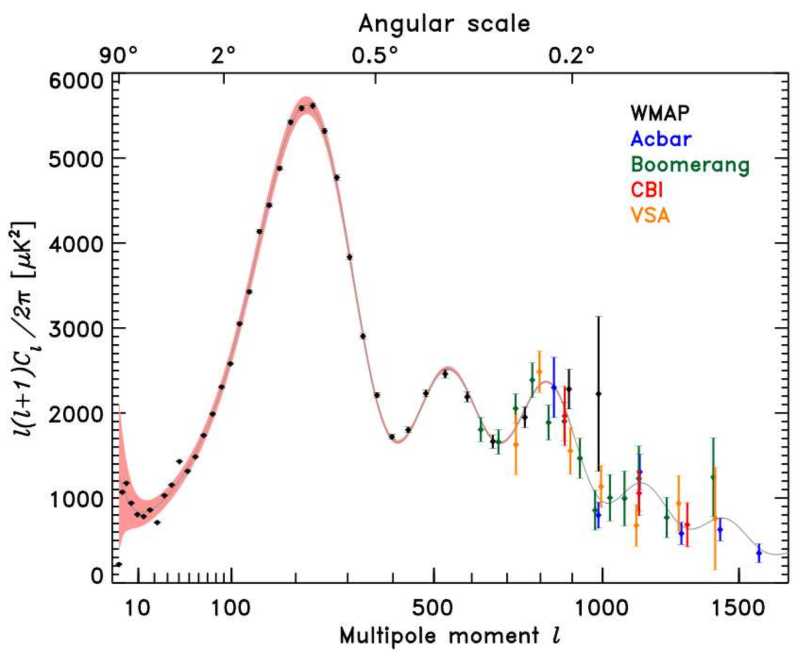


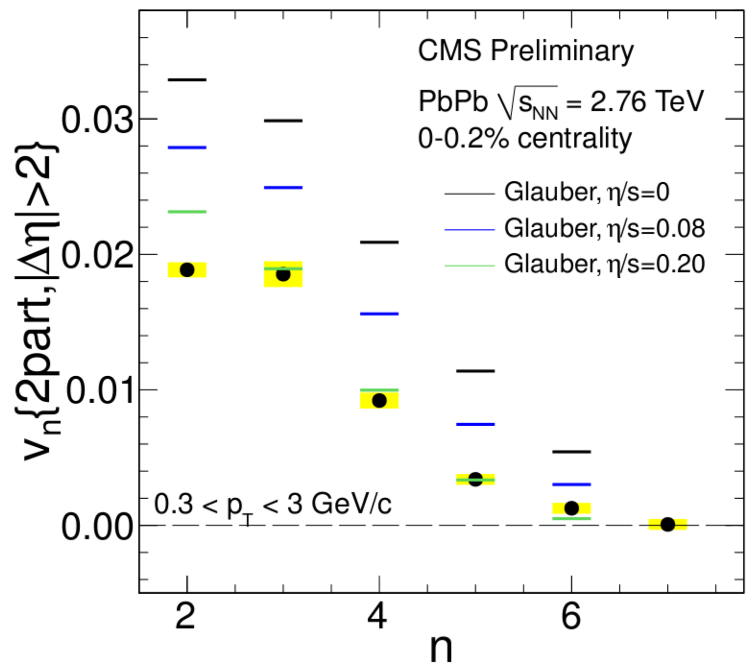
G. Roland for the CMS collaboration, NPA 904 – 905 (2013) 43c

Event-by-event higher flow harmonics

⟹ **two-particle correlation functions as superposition of higher flow harmonics**

B. Alver, G. Roland, PRC 81 (2010) 054905





WMAP multipole power spectrum

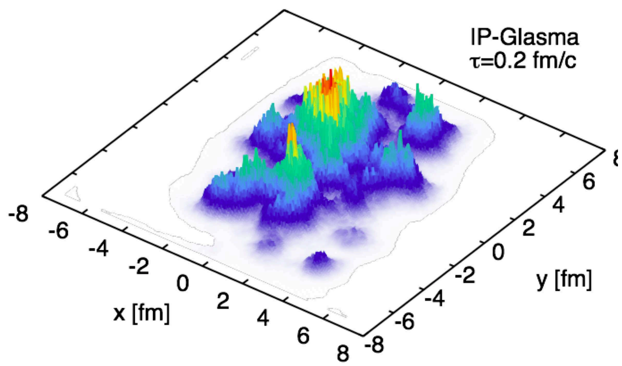
⟹ **Big Bang vs. "little bang"**

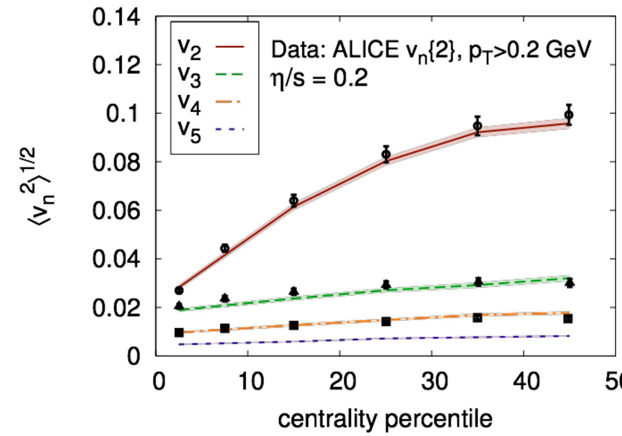
G. Roland for the CMS collaboration, NPA 904 – 905 (2013) 43c

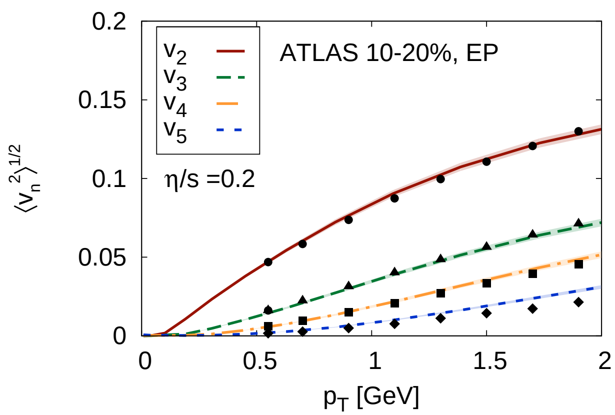
Additional constraint on η/s and initial conditions

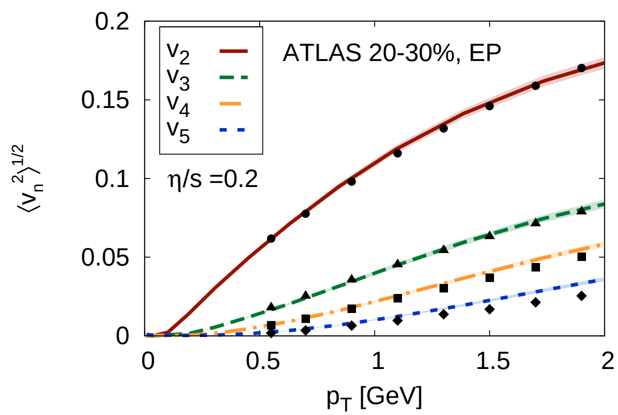
initial conditions:
IP glasma

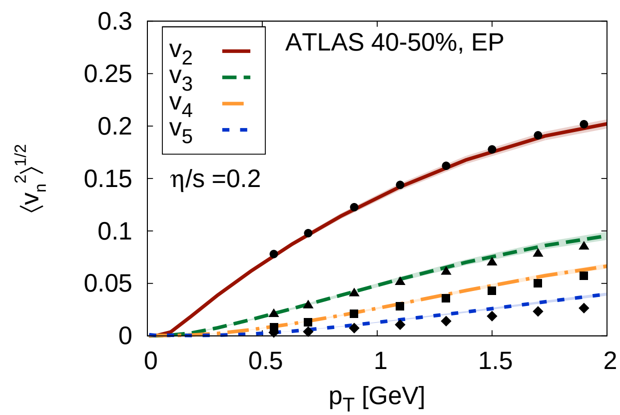
$\eta/s = 0.2 = const.$











C. Gale, S. Jeon, B. Schenke, P. Tribedy, R. Venugopalan, PRL 110 (2013) 1, 012302

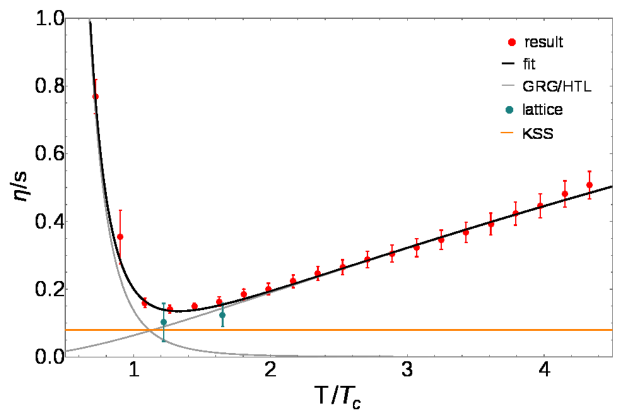
$\Longrightarrow$ **Success story no. 2: Quantitative description of collective flow by dissipative fluid dynamics, for all centralities and event by event!**
(with IP glasma initial conditions, $\eta/s = const.$)

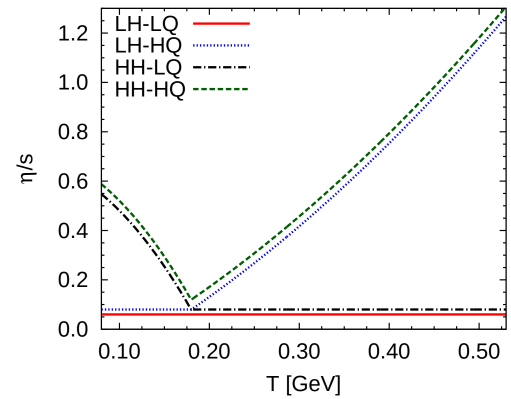
$\eta/s = const.$ vs. $\eta/s(T)$ (I)

η/s is not constant but a function of T (and μ)!

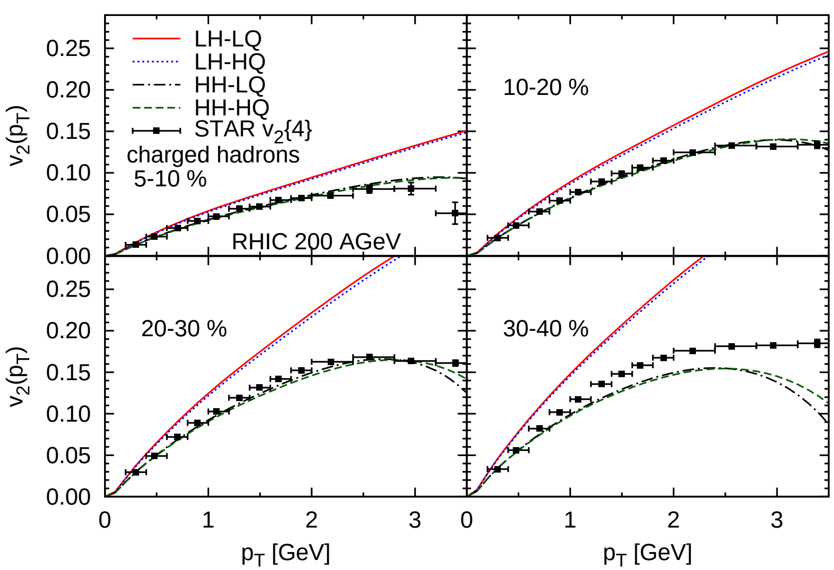
N. Christiansen, M. Haas, J.M. Pawlowski, N. Strodthoff, PRL 115 (2015) 11, 112002

H. Niemi, G.S. Denicol, P. Huovinen, E. Molnár, DHR, PRL 106 (2011) 212302; PRC 86 (2012) 014909

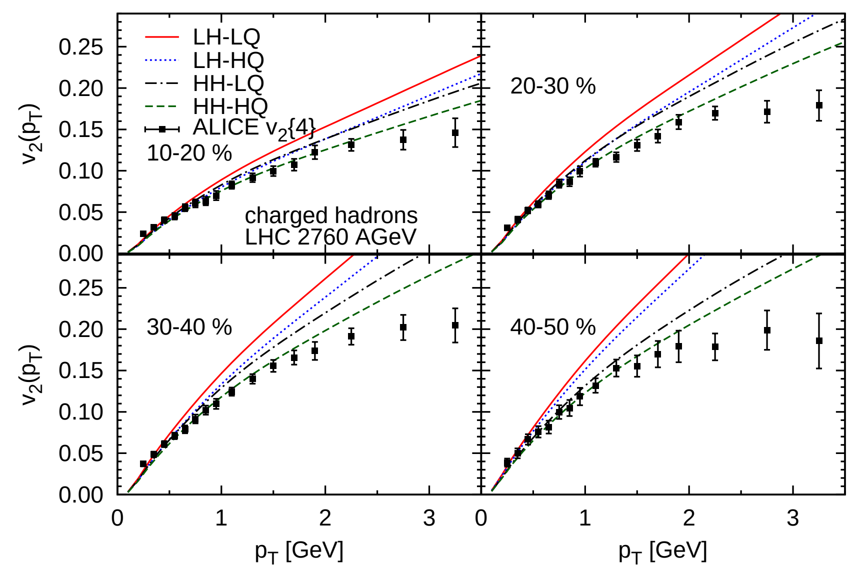




RHIC: $v_2(p_T)$ not sensitive to η/s in QGP!

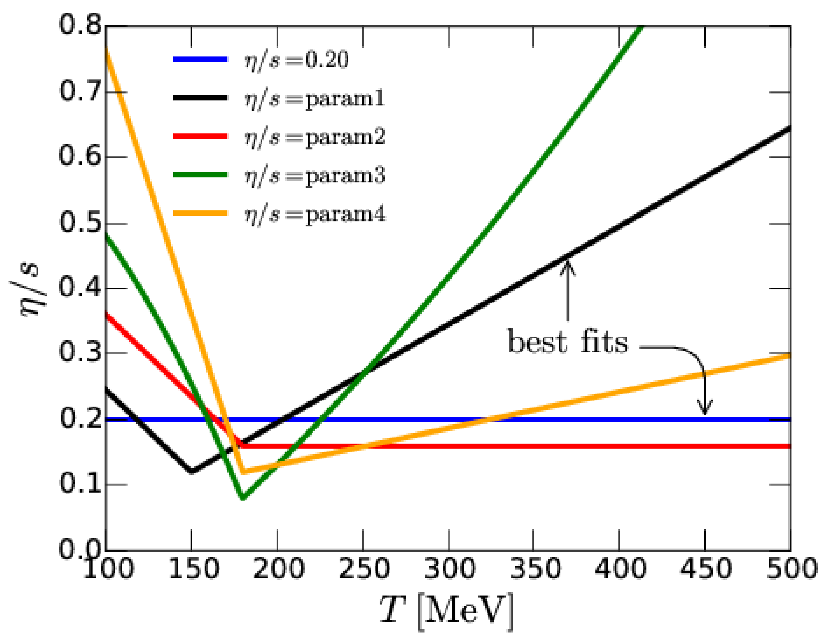


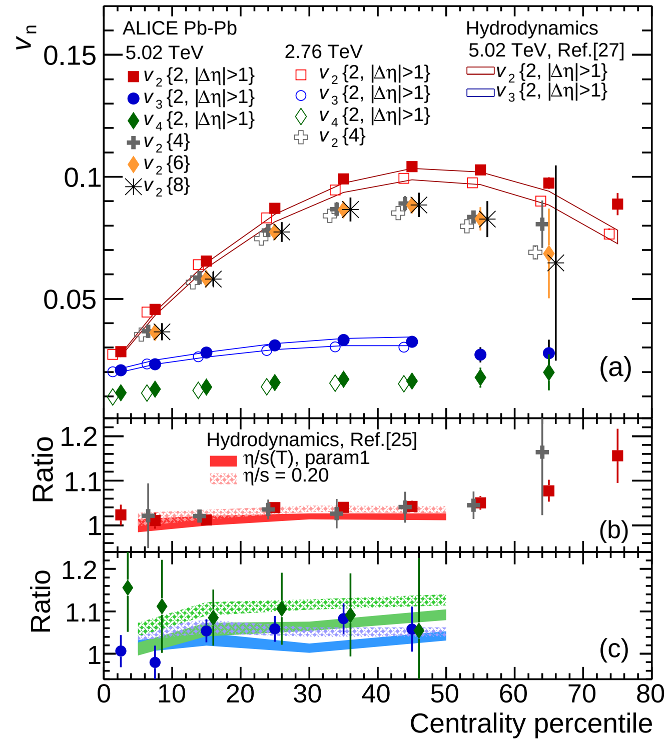
LHC: sensitivity increases!



$\eta/s = const.$ vs. $\eta/s(T)$ (II)

Prediction for highest LHC energies:
EbyE, NLO-pQCD, initial-state saturation + dissipative fluid dynamics (EKRT) model H. Niemi, K.J. Eskola, R. Paatelainen, K. Tuominen, PRC 93 (2016) 014912





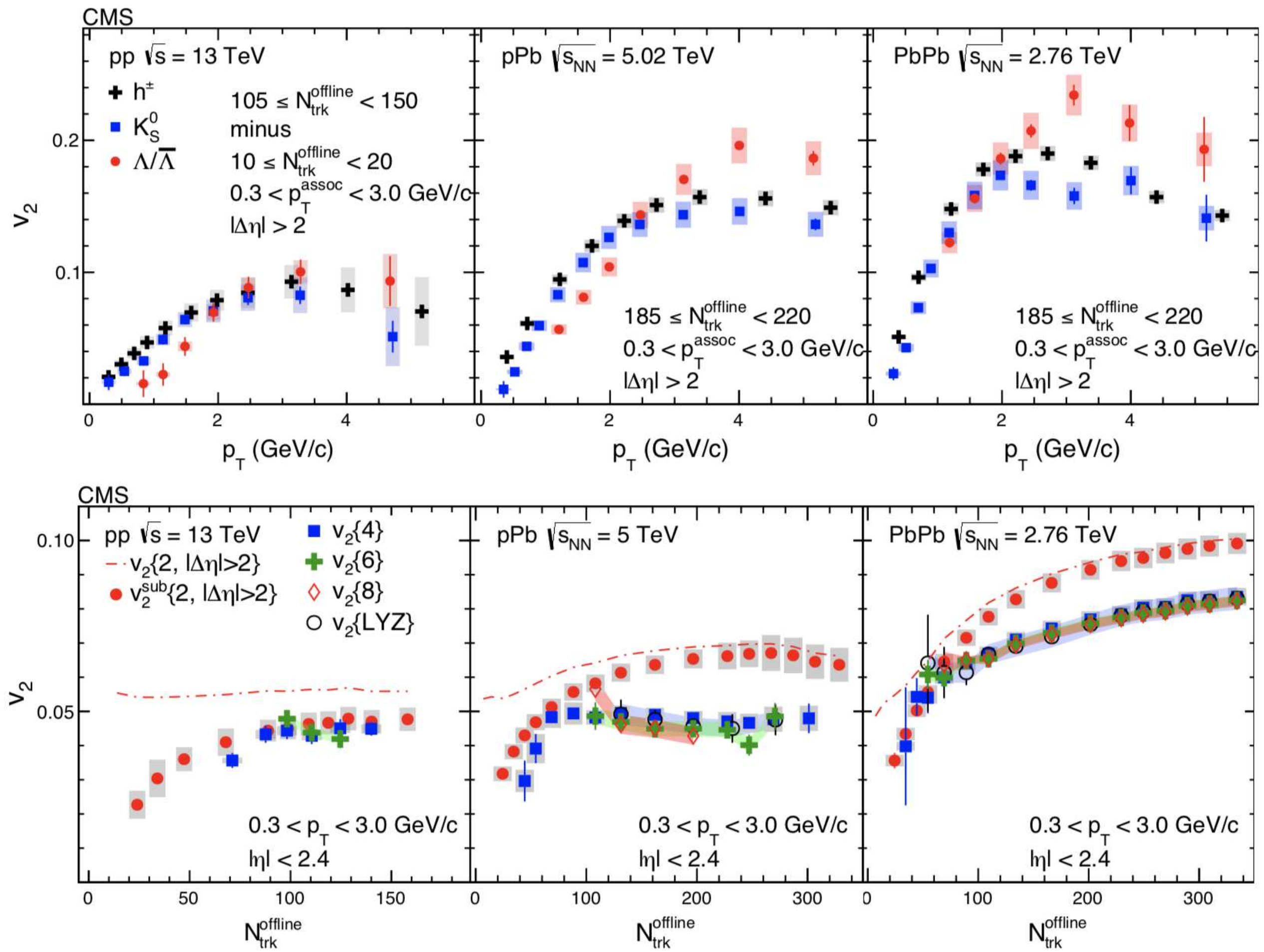
ALICE coll., PRL 116 (2016) 132302

Success story no. 3: initial state and subsequent evolution sufficiently well understood to make quantitative predictions!

But: available range of collision energies not yet sufficient to determine $\eta/s(T)$!

Collective flow in small systems: how small is too small?

Considerable flow seen in pp- and pA-collisions!

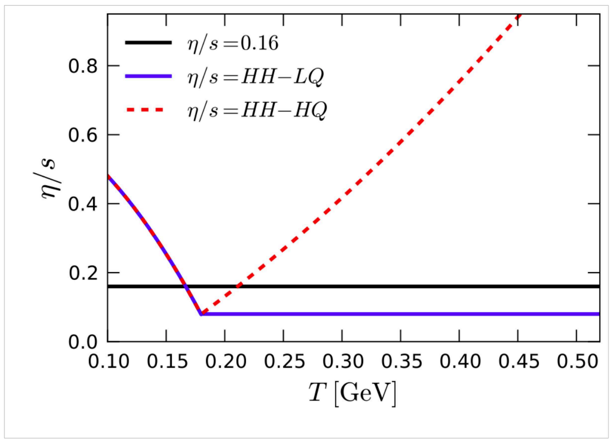


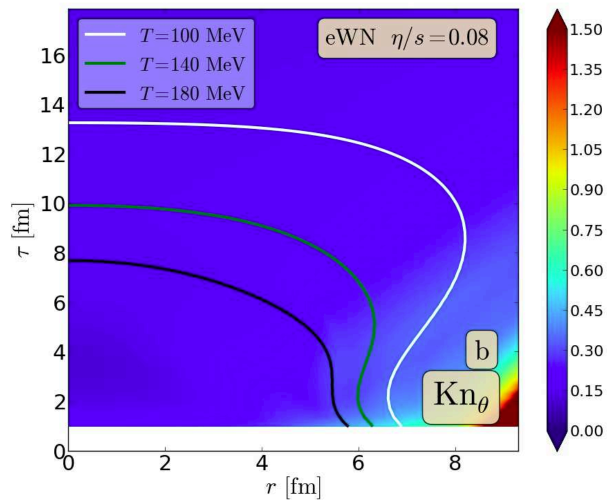
W. Li for the CMS coll. (proc. of QM2017), arXiv:1704.03576 [nucl-ex]

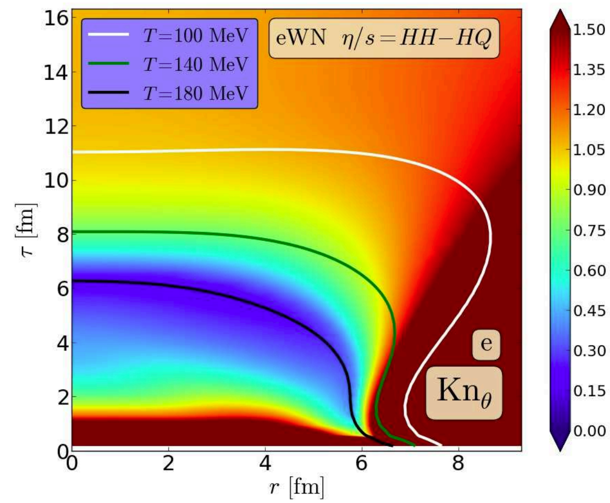
But: is fluid dynamics applicable to describe such small systems???

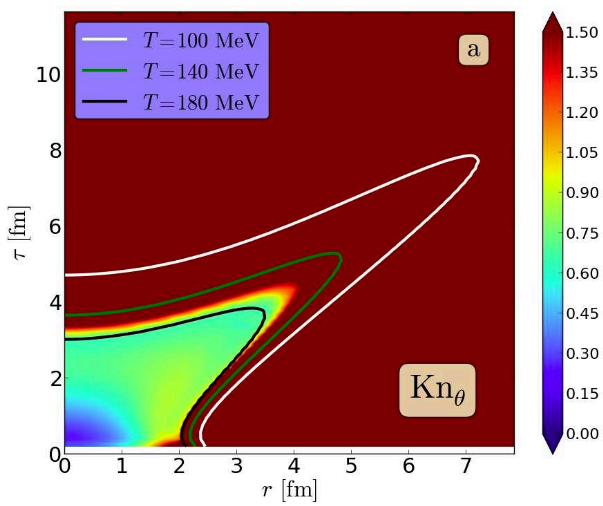
Consistency check

Compute $\mathrm{Kn} \equiv \frac{\ell}{L} \sim \frac{|\nabla \vec{v}|}{\langle\sigma\rangle n} \sim \frac{\eta}{s}\frac{|\nabla \vec{v}|}{T}$ **H. Niemi, G.S. Denicol, arXiv:1404.7327 [nucl-th]**









- **AA collisions, event-averaged initial conditions:**
  If $\eta/s \ll 1$, fluid dynamics applicable
- **pA collisions:**
  Even for $\eta/s \ll 1$, fluid dynamics barely applicable

$\Longrightarrow$ **Small systems behave collectively, but cannot be reliably described by (standard) fluid dynamics!**

$\Longrightarrow$ **Improve theory of fluid dynamics by including higher-order corrections in Kn!**

Fluid dynamics: degrees of freedom

1. Net charge (e.g., baryon number, strangeness, etc.) current: $N^\mu = n\, u^\mu + n^\mu$

u^μ — fluid 4-velocity, $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

$g_{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$ (West coast!!) metric tensor

$n \equiv u^\mu N_\mu$ — net-charge density in fluid rest frame

$n^\mu \equiv \Delta^{\mu\nu} N_\nu \equiv N^{<\mu>}$ — diffusion current (flow of net charge relative to u^μ), $n^\mu u_\mu = 0$

$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ projector onto 3-space orthogonal to u^μ, $\Delta^{\mu\nu} u_\nu = 0$

2. Energy-momentum tensor: $T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p+\Pi)\,\Delta^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu}$

$\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ — energy density in fluid rest frame

p — pressure in fluid rest frame

Π — bulk viscous pressure, $p + \Pi \equiv -\frac{1}{3}\,\Delta^{\mu\nu} T_{\mu\nu}$

$q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ — heat-flux current (flow of energy relative to u^μ), $q^\mu u_\mu = 0$

$\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ — shear-stress tensor, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0\,,\ \pi^\mu{}_\mu = 0$

$a^{(\mu\nu)} \equiv \frac{1}{2}\,(a^{\mu\nu} + a^{\nu\mu})$ symmetrized tensor

$a^{<\mu\nu>} \equiv \left(\Delta_\alpha{}^{(\mu}\Delta^{\nu)}{}_\beta - \frac{1}{3}\,\Delta^{\mu\nu}\Delta_{\alpha\beta}\right) a^{\alpha\beta}$ symmetrized, traceless spatial projection

Fluid dynamics: equations of motion

1. **Net-charge conservation:**

$$\partial_\mu N^\mu = 0 \iff \dot{n} + n\,\theta + \partial\cdot n = 0$$

$\dot{n} \equiv u^\mu \partial_\mu n$ convective (comoving) derivative
(time derivative in fluid rest frame, $\dot{n}_{\mathrm{RF}} \equiv \partial_t n$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. **Energy-momentum conservation:**

$\partial_\mu T^{\mu\nu} = 0 \iff$ energy conservation:

$$u_\nu\,\partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\,\theta + \partial\cdot q - q\cdot\dot{u} - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0$$

acceleration equation:

$$\Delta^{\mu\nu}\,\partial^\lambda T_{\nu\lambda} = 0 \iff$$
$$(\epsilon+p)\dot{u}^\mu = \nabla^\mu(p+\Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q\cdot\partial u^\mu - \Delta^{\mu\nu}\,\partial^\lambda \pi_{\nu\lambda}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient (spatial gradient in fluid rest frame)

Solvability

Problem:

5 equations, but 15 unknowns (for given u^μ): ϵ, p, n, Π, n^μ (3), q^μ (3), $\pi^{\mu\nu}$ (5)

Solution:

1. **clever choice of frame (Eckart, Landau,...): eliminate n^μ or q^μ**

   $\Longrightarrow$ **does not help! Promotes u^μ to dynamical variable!**

2. **ideal fluid limit: all dissipative terms vanish, $\Pi = n^\mu = q^\mu = \pi^{\mu\nu} = 0$**

   $\Longrightarrow$ **6 unknowns: ϵ, p, n, u^μ (3) (not quite there yet...)**

   $\Longrightarrow$ **fluid is in local thermodynamical equilibrium**

   $\Longrightarrow$ **provide equation of state (EOS) $p(\epsilon, n)$ to close system of equations**

3. **provide additional equations for dissipative quantities**

   $\Longrightarrow$ **relativistic dissipative fluid dynamics**

   (a) **First-order theories: e.g. generalization of Navier-Stokes (NS) equations to the relativistic case (Landau, Lifshitz)**

   (b) **Second-order theories: e.g. Israel-Stewart (IS) equations**

Navier-Stokes equations

Navier-Stokes (NS) equations: first-order relativistic dissipative fluid dynamics

1. **bulk viscous pressure:** $\boxed{\Pi_{\rm NS} = -\zeta\,\theta}$

   ζ **bulk viscosity**

2. **diffusion current:** $\boxed{n^\mu_{\rm NS} = \kappa_n\,\nabla^\mu\alpha}$

   $\beta \equiv 1/T$ **inverse temperature,**

   $\alpha \equiv \beta\,\mu,$ μ **chemical potential,**

   κ_n **net-charge diffusion coefficient**

3. **shear stress tensor:** $\boxed{\pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}}$

   η **shear viscosity,**

   $\sigma^{\mu\nu} = \nabla^{<\mu}u^{\nu>}$ **shear tensor**

$\Longrightarrow$ **algebraic expressions in terms of thermodynamic and fluid variables**

$\Longrightarrow$ **simple... but: unstable and acausal equations of motion!!**

W.A. Hiscock, L. Lindblom, PRD 31 (1985) 725

Israel-Stewart equations

Israel-Stewart (IS) equations: second-order relativistic dissipative fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

"Simplified" version:

$$\tau_\Pi \, \dot{\Pi} + \Pi = \Pi_{\rm NS}$$
$$\tau_n \, \dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS}$$
$$\tau_\pi \, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS}$$

cf. also T. Koide, G.S. Denicol, Ph. Mota, T. Kodama, PRC 75 (2007) 034909

$\Longrightarrow$ dynamical (instead of algebraic) equations for dissipative terms!

solution: e.g. bulk viscous pressure $\Pi(t) = \Pi_{\rm NS}\left(1 - e^{-t/\tau_\Pi}\right) + \Pi(0)\, e^{-t/\tau_\Pi}$

$\Longrightarrow$ dissipative quantities $\Pi\,,\ n^\mu\,,\ \pi^{\mu\nu}$ relax to their respective NS values $\Pi_{\rm NS}\,,\ n^\mu_{\rm NS}\,,\ \pi^{\mu\nu}_{\rm NS}$ on time scales $\tau_\Pi\,,\ \tau_n\,,\ \tau_\pi$

$\Longrightarrow$ stable and causal fluid-dynamical equations of motion!

see, e.g., S. Pu, T. Koide, DHR, PRD 81 (2010) 114039

However: Simplified IS equations do not contain all possible second-order terms!

Power counting (I)

3 length scales: 2 microscopic, 1 macroscopic

- **interparticle distance (thermal wavelength)** $\lambda \sim \beta \equiv 1/T$
- **mean-free path** $\ell \sim (\langle\sigma\rangle n)^{-1}$
- **length scale over which macroscopic fluid fields vary** $L\,,\quad \partial_\mu \sim L^{-1}$

$$n,\, s \sim T^3 = \beta^{-3} \sim \lambda^{-3}\,, \qquad \eta \sim T/\langle\sigma\rangle = (\langle\sigma\rangle\lambda)^{-1} \quad \Longrightarrow \quad \boxed{\frac{\ell}{\lambda} \sim \frac{1}{\langle\sigma\rangle n}\,\frac{1}{\lambda} \sim \frac{1}{\langle\sigma\rangle\lambda}\,\frac{1}{n} \sim \frac{\eta}{s}}$$

$\Longrightarrow \dfrac{\eta}{s}$ **solely determined by 2 microscopic length scales! (similarly:** $\dfrac{\zeta}{s}\,,\ \dfrac{\kappa_n}{\beta\, s}$ **)**

3 regimes:

- **dilute-gas limit** $\dfrac{\ell}{\lambda} \sim \dfrac{\eta}{s} \gg 1 \iff \langle\sigma\rangle \ll \lambda^2 \Longrightarrow$ **weak-coupling limit**
- **viscous fluids** $\dfrac{\ell}{\lambda} \sim \dfrac{\eta}{s} \sim 1 \iff \langle\sigma\rangle \sim \lambda^2$

  interactions happen on the scale $\lambda \Longrightarrow$ **moderate coupling**
- **ideal-fluid limit** $\dfrac{\ell}{\lambda} \sim \dfrac{\eta}{s} \ll 1 \iff \langle\sigma\rangle \gg \lambda^2 \Longrightarrow$ **strong-coupling limit**

Power counting (II)

Knudsen number: $\mathrm{Kn} \equiv \frac{\ell}{L} \sim \ell\, \partial_\mu$

$\Longrightarrow$ expansion in Knudsen number equivalent to gradient (derivative) expansion

$\Longrightarrow$ if microscopic particle dynamics (small scale $\sim \ell$) is well separated from macroscopic fluid dynamics (large scale $\sim L$), expansion in powers of $\mathrm{Kn} \ll 1$ expected to converge!

$\Longrightarrow$ Estimate Navier-Stokes terms: use: $\epsilon + p = Ts + \mu n \;\Longrightarrow\; \beta\,\epsilon \sim s$!

$\Longrightarrow$ $\frac{\Pi_{\mathrm{NS}}}{\epsilon} = -\frac{\zeta}{\beta\,\epsilon}\beta\,\theta \sim -\frac{\zeta}{s}\frac{\lambda}{\ell}\ell\,\theta \sim \ell\,\partial_\mu u^\mu \sim \mathrm{Kn}$ (similarly $\frac{n^\mu_{\mathrm{NS}}}{s}, \frac{\pi^{\mu\nu}_{\mathrm{NS}}}{\epsilon} \sim \mathrm{Kn}$)

But: in IS theory $\Pi,\, n^\mu,\, \pi^{\mu\nu}$ independent dynamical quantities!

$\Longrightarrow$ (inverse) Reynolds number(s): $\mathrm{Re}^{-1} \sim \frac{\Pi}{\epsilon}, \frac{n^\mu}{s}, \frac{\pi^{\mu\nu}}{\epsilon}$

$\Longrightarrow$ asymptotically, terms $\sim \mathrm{Re}^{-1} \sim \mathrm{Kn}$

$\Longrightarrow$ additional relaxation term in IS equation is of second order:

$$\frac{1}{\epsilon}\tau_\Pi\,\dot{\Pi} \sim \frac{1}{\epsilon}u^\mu\,\ell\,\partial_\mu\Pi \sim \mathrm{Re}^{-1}\mathrm{Kn} \sim \mathrm{Kn}^2$$

$\Longrightarrow$ to be consistent, have to include other second-order terms as well!

Israel-Stewart equations revisited

$$\tau_\Pi \dot{\Pi} + \Pi = \Pi_{\rm NS} + \mathcal{K} + \mathcal{J} + \mathcal{R}$$

$$\tau_n \dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS} + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu$$

$$\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu}$$

Kn^2:

$$\mathcal{K} = \zeta_1 \omega_{\mu\nu}\omega^{\mu\nu} + \zeta_2 \sigma^{\mu\nu}\sigma_{\mu\nu} + \zeta_3 \theta^2 + \zeta_4 (\nabla\alpha)^2 + \zeta_5 (\nabla p)^2 + \zeta_6 \nabla\alpha\cdot\nabla p + \zeta_7 \nabla^2\alpha + \zeta_8 \nabla^2 p\,,$$

$$\mathcal{K}^\mu = \kappa_1 \sigma^{\mu\nu}\nabla_\nu\alpha + \kappa_2 \sigma^{\mu\nu}\nabla_\nu p + \kappa_3 \theta\nabla^\mu\alpha + \kappa_4 \theta\nabla^\mu p + \kappa_5 \omega^{\mu\nu}\nabla_\nu\alpha + \kappa_6 \Delta^{\mu\lambda}\partial^\nu\sigma_{\lambda\nu} + \kappa_7 \nabla^\mu\theta\,,$$

$$\mathcal{K}^{\mu\nu} = \eta_1 \omega_\lambda{}^{<\mu}\omega^{\nu>\lambda} + \eta_2 \theta\sigma^{\mu\nu} + \eta_3 \sigma_\lambda{}^{<\mu}\sigma^{\nu>\lambda} + \eta_4 \sigma_\lambda{}^{<\mu}\omega^{\nu>\lambda} + \eta_5 \nabla^{<\mu}\alpha\nabla^{\nu>}\alpha + \eta_6 \nabla^{<\mu}p\nabla^{\nu>}p + \eta_7 \nabla^{<\mu}\alpha\nabla^{\nu>}p + \eta_8 \nabla^{<\mu}\nabla^{\nu>}\alpha + \eta_9 \nabla^{<\mu}\nabla^{\nu>}p$$

Re^{-1}Kn:

$$\mathcal{J} = -\ell_{\Pi n}\nabla\cdot n - \tau_{\Pi n} n\cdot\nabla p - \delta_{\Pi\Pi}\theta\Pi - \lambda_{\Pi n} n\cdot\nabla\alpha + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}$$

$$\mathcal{J}^\mu = \omega^{\mu\nu}n_\nu - \delta_{nn}\theta n^\mu - \ell_{n\Pi}\nabla^\mu\Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla^\lambda\pi_{\nu\lambda} + \tau_{n\Pi}\Pi\nabla^\mu p - \tau_{n\pi}\pi^{\mu\nu}\nabla_\nu p - \lambda_{nn}\sigma^{\mu\nu}n_\nu + \lambda_{n\Pi}\Pi\nabla^\mu\alpha - \lambda_{n\pi}\pi^{\mu\nu}\nabla_\nu\alpha$$

$$\mathcal{J}^{\mu\nu} = 2\pi_\lambda{}^{<\mu}\omega^{\nu>\lambda} - \delta_{\pi\pi}\theta\pi^{\mu\nu} - \tau_{\pi\pi}\pi_\lambda{}^{<\mu}\sigma^{\nu>\lambda} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n} n^{<\mu}\nabla^{\nu>}p + \ell_{\pi n}\nabla^{<\mu}n^{\nu>} + \lambda_{\pi n} n^{<\mu}\nabla^{\nu>}\alpha$$

where $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ vorticity

Re^{-2}:

$$\mathcal{R} = \varphi_1 \Pi^2 + \varphi_2 n\cdot n + \varphi_3 \pi^{\mu\nu}\pi_{\mu\nu}$$

$$\mathcal{R}^\mu = \varphi_4 \pi^{\mu\nu}n_\nu + \varphi_5 \Pi n^\mu$$

$$\mathcal{R}^{\mu\nu} = \varphi_6 \Pi\pi^{\mu\nu} + \varphi_7 \pi_\lambda{}^{<\mu}\pi^{\nu>\lambda} + \varphi_8 n^{<\mu}n^{\nu>}$$

$\Longrightarrow$ **transport coefficients can be determined by matching to underlying theory, e.g. kinetic theory** G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD 85 (2012) 114047

Fluid dynamics for small systems

How well does fluid dynamics approximate solution of Boltzmann equation?

K. Gallmeister, H. Niemi, C. Greiner, DHR, PRC 98 (2018) 024912

Fluid dynamics:

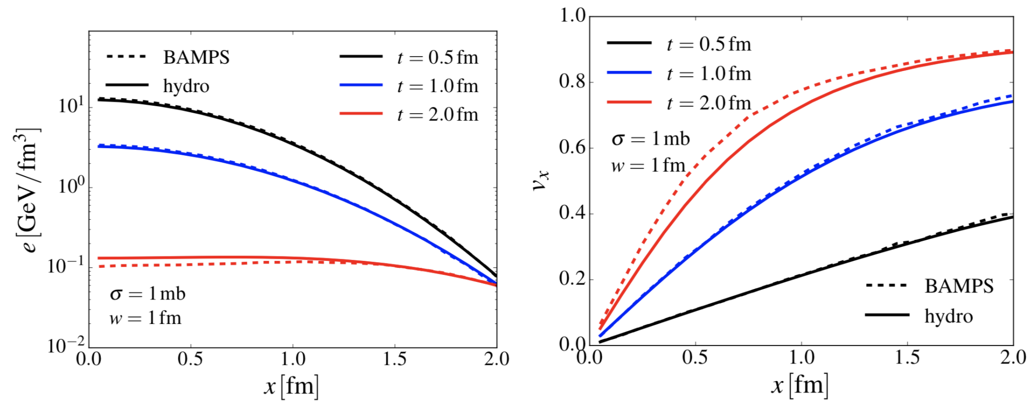
$\Longrightarrow$ **improved IS theory**

Boltzmann equation:

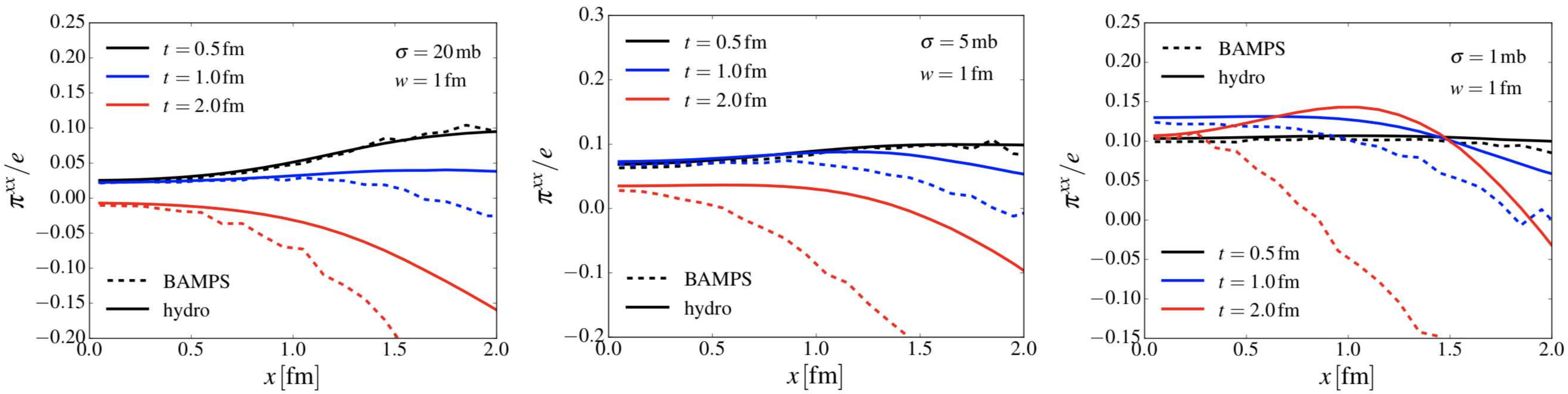
$\Longrightarrow$ **BAMPS**

Initial ϵ profile:

$\Longrightarrow$ **Gaussian, width $w = 1$ fm**



$\Longrightarrow$ **conserved quantities reasonably well described, even for small σ (large η/s)**



$\Longrightarrow$ **larger deviations for dissipative quantities, increasing with σ and time**

$\Longrightarrow$ **fluid dynamics applicable to describe collective flow in small systems!**

But: are other explanations possible?

Collective flow in dilute systems

How much collective flow is produced by kinetic theory?

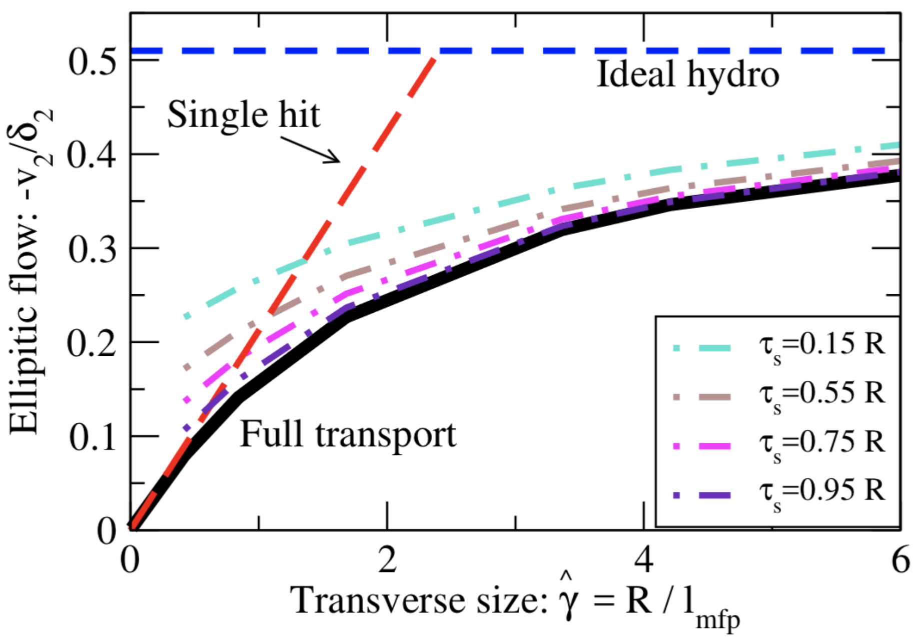
A. Kurkela, U.A. Wiedemann, B. Wu, PLB 783 (2018) 274; arXiv:1805.04081 [hep-ph]

Perturbative solution of Boltzmann equation in powers of Kn^{-1}

$\Longrightarrow$ **first order $O(\mathrm{Kn}^{-1})$ $\Longleftrightarrow$ "single-hit" dynamics**

$\Longrightarrow$ **"collective-flow" response to initial spatial anisotropies δ_n:**

$$\frac{v_2}{\delta_2} = 0.212\,\hat{\gamma} + O(\hat{\gamma}^2) \quad , \quad \frac{v_3}{\delta_3} = 0.140\,\hat{\gamma} + O(\hat{\gamma}^2) \text{ , where } \hat{\gamma} \equiv \frac{R}{\ell} \sim \mathrm{Kn}^{-1}$$



$\Longrightarrow$ **sizable "flow" coefficients even for small collision rates!**

Conclusions

1. **Phase transitions in a fundamental theory of nature (QCD) can be studied in the laboratory via heavy-ion collisions**
2. **Success stories nos. 1 – 3: fluid dynamics can quantitatively describe and predict collective-flow phenomena in AA-collisions**
3. **Determining transport coefficients by comparison to experimental data requires thorough understanding of initial conditions**
4. **System created in pA- and pp-collisions exhibits signs of collective flow, but may be too small to apply (standard) fluid dynamics**
5. **Second-order relativistic dissipative fluid dynamics has been systematically derived as long-distance, large-time limit of kinetic theory**
6. **Second-order relativistic dissipative fluid dynamics provides good description of conserved quantities (and thus of collective flow), even for small systems**
7. **Experimentally observed collective flow in small systems may also be explained by transport theory in the "few-hit" limit**

$\Longrightarrow$ **Can a macroscopic approach be applied to a microscopic system?**
Yes, if one carefully monitors its range of applicability!