Uniform Asymptotic Approximation and Non-adiabatic evolution of Primordial perturbations

均匀渐近近似与原初量子场的非绝热演化

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Non–adiabatic effects on the field modes

- Consider a field mode $\mathrm{u_k}$ obeys the sODE

$$\frac{\mathrm{d}^2\mathrm{u_k}(\eta)}{\mathrm{d}\eta^2} + \left(\mathrm{k}^2 + \mathrm{m}^2(\eta)\right)\mathrm{u_k}(\eta) = 0. \tag{1}$$

- If $\omega_\mathrm{k}^2(\eta) = \mathrm{k}^2 + \mathrm{m}^2(\eta)$ varies non-adiabtically, i..e,

$$\left|\frac{3\omega_\mathrm{k}'^2}{4\omega^4} - \frac{\omega_\mathrm{k}''}{2\omega_\mathrm{k}^3}\right| \ll 1 \ \text{ is voilated}, \tag{2}$$

- a window of modes can experience exponential growth,

$$\mathrm{u_k}(\eta) \sim \mathrm{e}^{\mp\int|\omega|\mathrm{d}\eta} \tag{3}$$

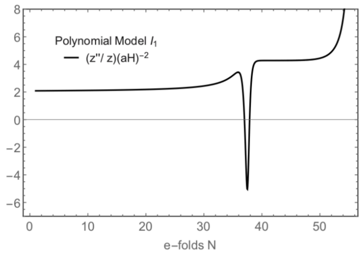
- which can also produce excited states with $\beta_\mathrm{k} \neq 0$ after the non-adiabatic regime,

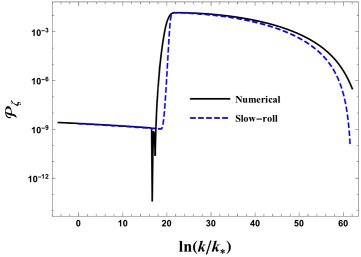
$$\mathrm{u_k}(\eta) \sim \frac{\alpha_\mathrm{k}}{\sqrt{2\mathrm{k}}}\mathrm{e}^{-\mathrm{i}\mathrm{k}\eta} + \frac{\beta_\mathrm{k}}{\sqrt{2\mathrm{k}}}\mathrm{e}^{\mathrm{i}\mathrm{k}\eta} \tag{4}$$

Primordial modes in ultra slow-roll inflation

[H. Di and Y. Gong, arXiv: 1707.09578; I. Dalianis, A. Kehagias, and G. Tringas, arXiv: 1805.09483; C.Fu, P.Wu, H. Yu, arXiv: 1907.05042......]

$$\frac{d^2 u_k(\eta)}{d\eta^2} + \left(k^2 - \frac{z''}{z}\right) u_k(\eta) = 0 \tag{5}$$



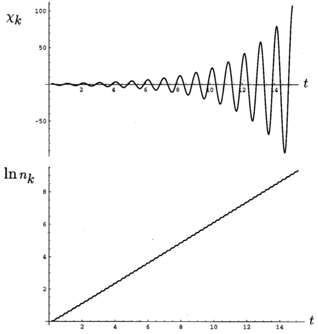


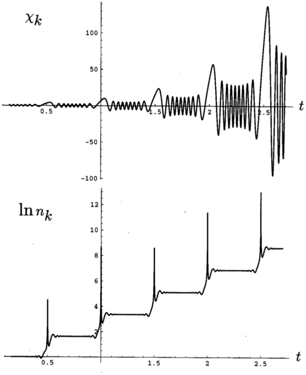
Parametric resonance during reheating

[L. Kofman, A. Linde, and AA. Starobinsky, PRD56(1997)3258]

- Modes coupled to oscillating inflaton field

$$\ddot{\chi}_k + \left(k^2 + g^2\sigma^2 + 2g^2\sigma^2\Phi\sin^2(mt)\right)\chi_k = 0 \qquad (6)$$



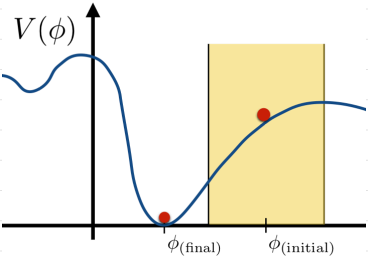


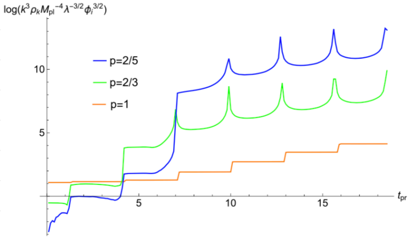
Generation of Oscillons

[J. Liu, ZK. Guo, RG. Cai, and G.Shiu, PRL120(2018)031301; C. Fu, P. Wu, and H. Yu, Phys. Rev. D 97, (2018); S. Antusch, F. Cefala, et. al., JHEP01(2018)083.]

- Inflaton field perturbation with self-resonance potential

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} + V''(\phi)\right)\delta\phi_k = 0 \qquad (7)$$





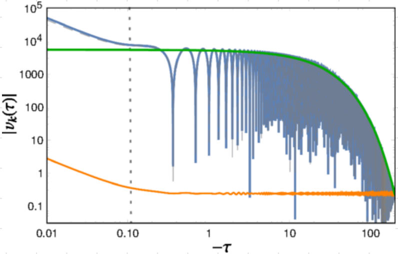
Primordial black holes by sound resonance

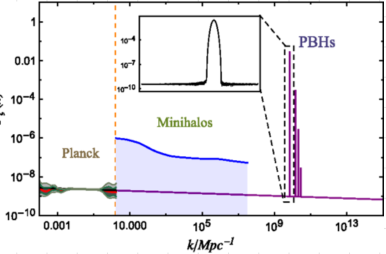
[YF. Cai, X. Tong, DG. Wang, and SF. Yan, PRL121(2018)081306]

- Modes with an oscillating sound speed

$$\frac{d^2 v_k(\eta)}{d\eta^2} + \left(c_s^2 k^2 + m^2(\eta)\right) v_k(\eta) = 0 \qquad (8)$$

$$c_s(\tau) = 1 - 2\xi[1 - \cos(2k_* \tau)] \qquad (9)$$





Quantum Bounce in Loop Quantum Cosmology

[Phys.Rev. D98 (2018) 103528]

- Perturbation modes near quantum bounce

$$\frac{d^2 v_k(\eta)}{d\eta^2} + \left(k^2 - \frac{a''}{a} + \tilde{U}(\eta)\right) v_k(\eta) = 0 \qquad (10)$$

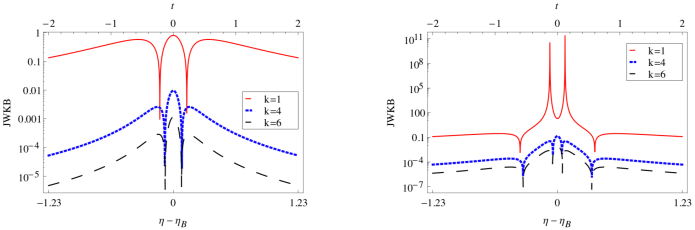


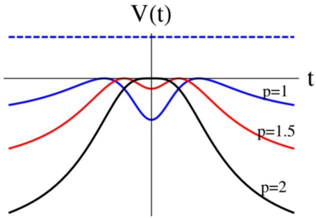
FIG. 3. JWKB criterion is violated near the time of bounce at $t = 0$. The left panel shows the result for the hybrid approach and the right panel shows the result for the dressed metric approach. Note that we used unit $m_{\mathrm{Pl}} = 1$ and set $a_{\mathrm{B}} = 1$ in these figures.

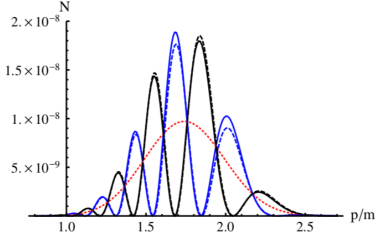
Schwinger pair productions by laser pulses

[C. K. Dumlu and G. V. Dunne, Phys. Rev. Lett. 104, 250402 (2010).]

- Schwinger effects: electron-positron pairs can be produced from vacuum with electric field E $\sim 10^{18}$V/m. Such phenomenon can be achieved by periodic laser pulses.

$$\ddot{\phi}_k + \left(m^2 + p_\perp^2 + [p - A(t)]^2\right)\phi_k = 0 \qquad (11)$$





The sODEs in modern physics

- Schrödinger-like wave equations in quantum mechanics

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\Big(E - V(x)\Big)\psi(x) = 0. \tag{12}$$

- Field perturbations in black hole spacetimes, for example, for Schwarzchild black hole,

$$\frac{d^2R(r)}{dr^2} + \Big(\omega^2 - V(r,\omega)\Big)R(r) = 0. \tag{13}$$

- Mukhanov-Sasski like equations for cosmological perturbations in the early universe

$$\frac{d^2\mu_k^{(s,t)}(\eta)}{d\eta^2} + \left(\omega_k^2(\eta) - \frac{z''_{s,t}}{z_{s,t}}\right)\mu_k^{(s,t)}(\eta) = 0. \tag{14}$$

- ....

The JWKB approximation

JWKB method is of wide application with great success for solving the sODEs, which plays essential roles in the development of many branches of theoretical physics

$$\frac{d^2\mu_k(y)}{dy^2} + \Omega^2(y)\mu_k(y) = 0. \tag{15}$$

- However, it is important to emphasize here that the validity of the JWKB method has to be restricted to the region where the JWKB condition is fulfilled:

$$\left|\frac{3\Omega'^2}{4\Omega^4} - \frac{\Omega''}{2\Omega^3}\right| \ll 1. \tag{16}$$

The JWKB approximate solutions

When the JWKB condition is fulfilled, the approximate solution of

$$\frac{\mathrm{d}^2\mu_{\mathrm{k}}(\mathrm{y})}{\mathrm{dy}^2} + \Omega^2(\mathrm{y})\mu_{\mathrm{k}}(\mathrm{y}) = 0. \tag{17}$$

can be expressed as

$$\mu_{\mathrm{k}}^{\mathrm{JWKB}}(\mathrm{y}) \simeq \frac{1}{\sqrt{2\Omega(\mathrm{y})}} \mathrm{e}^{\pm\mathrm{i}\int\Omega(\mathrm{y})\mathrm{dy}} \tag{18}$$

- **What is the solution looks like if the JWKB condition is violated?**

Violations of the JWKB approximation

$$\left|\frac{3\Omega'^2}{4\Omega^4} - \frac{\Omega''}{2\Omega^3}\right| \ll 1.$$

- Turning points problem.

$$\Omega^2(\mathrm{y}) = 0.$$

- The second-order finite pole ($\mathrm{y} \to 0,\ \alpha = 2$)

$$\Omega^2(\mathrm{y}) \sim \frac{\mathrm{c}}{\mathrm{y}^\alpha} \longrightarrow \left|\frac{3\Omega'^2}{4\Omega^4} - \frac{\Omega''}{2\Omega^3}\right| \sim \frac{\alpha^2 + 4\alpha}{16\mathrm{c}}\mathrm{y}^{\alpha-2}.$$

- Extreme points. About extreme point, $\Omega'_{\mathrm{m}} \sim 0$.

$$\left|\frac{\Omega''_{\mathrm{m}}}{2\Omega^3_{\mathrm{m}}}\right| > 1.$$

An Example for the Second-Order Pole

Radial Schrödinger equation for hydrogen atom

$$\frac{d^2\psi(r)}{dr^2} + \left[\frac{2m}{\hbar^2}\left(E + \frac{Ze^2}{r}\right) - \frac{l(l+1)}{r^2}\right]\psi(r) = 0. \tag{19}$$

The exact energy levels are given by

$$E_n = -\frac{mZ^2e^4}{2\hbar^2 n^2} = -\frac{mZ^2e^4}{2\hbar^2(n_r + 1/2 + l + 1/2)^2} \tag{20}$$

1930, Young and Uhlenbeck found the JWKB method gives
(Young and Uhlenbeck, Phys. Rev. 36, 1154 (1930).)

$$E_n^{JWKB} = -\frac{mZ^2e^4}{2\hbar^2(n_r + 1/2 + \sqrt{l(l+1)})^2}. \tag{21}$$

Exact result can be obtained by replacing $l(l+1) \to (l + \frac{1}{2})^2$.

Langer modification for radial problem

In 1937, Langer suggested a replacement to the radial Schrödinger equation (R. E. Langer, Phys. Rev. 51, 669 (1937)).

$$\frac{d^2\psi(r)}{dr^2} + \left[\frac{2m}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2}\right]\psi(r) = 0. \tag{22}$$

The **Langer modification** is defined as

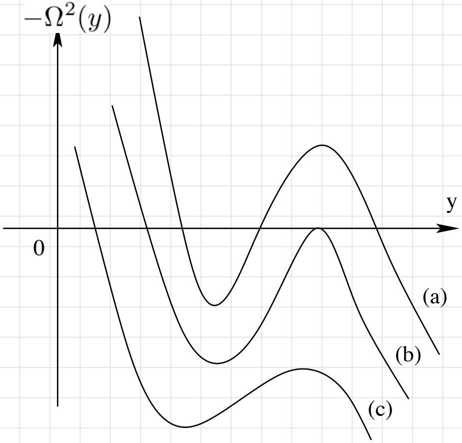
$$l(l+1) \longrightarrow (l + 1/2)^2 \tag{23}$$

Then the JWKB method gives correct results for the radial problem.

This is a standard procedure in quantum mechanics.

Three cases for turning points

turning points **poles** **extreme point**



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The uniform asymptotic approximation

- Treatments for one single turning point [Langer (1931, 1932, 1935, . . . 1949); Olver, Philos. Trans. R. Soc. A 247, 307 (1954); Asymptotics and Special Functions, (AKP Classics, Wellesley, MA 1997)]
  - Applications to inflationary cosmology [Habib et. al., PRL89,281301(2002); PRD70,083507(2004); PRD71,043518(2005); Martin, Ringeval, and Vennin, JCAP06 (2013) 021; Lorentz, Martin, and Ringeval, PRD78,083513(2008);TZ, AW, Cleaver, Kirsten, Sheng, QW, BFL, PRD90 (2014)063503; PRD90(2014)103517; APJL807(2015)L17; JCAP10(2015)052; JCAP03(2016)046; PRD97 (2018)103502; PRD99 (2019) 103536; arXiv:1911.01580]
- Treatments for two turning points problem [Olver, Phil. Trans. R. Soc. A 278, 137 (1975).]
  - Applications to inflationary cosmology [TZ, AW, Cleaver, Kirsten, Sheng, and QW, PRD89(2014)043507; IJMPA29(2014)1450142; PRD93(2016)123525, JCAP02 (2018)018; JCAP09 (2019) 064; Phys.Dark Univ. 26 (2019) 100373 ; arXiv:1907.13108]
- Treatments with extreme points [TZ and AW, arXiv:1902.09675 [quant-ph]] .

Standard Form of the Equation

$$\frac{d^2\mu_k(y)}{dy^2} = [g(y) + q(y)]\mu_k(y),$$

with

$$g(y) + q(y) \equiv -\Omega^2(y).$$

Note: functions $g(y)$ and $q(y)$ should be determined by the analysis of the error bounds of the approximate solutions of the above equation.

Error Control functions

- For poles

$$\mathscr{F}(\mathsf{y}) = \int \left[\frac{1}{|\mathsf{g}|^{1/4}} \frac{\mathsf{d}^2}{\mathsf{dy}^2} \left(\frac{1}{|\mathsf{g}|^{1/4}} \right) - \frac{\mathsf{q}}{|\mathsf{g}|^{1/2}} \right] \mathsf{dy}, \quad (24)$$

- For single turning points

$$\mathscr{H}(\xi) = \int^{\xi} \frac{\psi(\mathsf{v})}{|\mathsf{v}|^{1/2}} \mathsf{dv} = \mathscr{F}(\mathsf{y}) \pm \frac{5}{24(\pm\xi)^{3/2}} \quad (25)$$

- For two turning points

$$\begin{aligned} \mathscr{I}(\zeta) &= \int^{\xi} \frac{\psi(\mathsf{v})}{|\mathsf{v}^2 - \zeta_0^2|^{1/2}} \mathsf{dv} \\ &= \mathscr{F}(\mathsf{y}) \pm \int_{\zeta_0}^{\xi} \left[\frac{5\zeta_0^2}{4(|\mathsf{v}^2 - \zeta_0^2|)^{5/2}} + \frac{3}{4(|\mathsf{v}^2 - \zeta_0^2|)^{3/2}} \right] \mathsf{dv}. \end{aligned} \quad (26)$$

Convergence of $\mathscr{F}(\mathrm{y})$ for second-order pole

The convergence of the error control function $\mathscr{F}(\mathrm{y})$ at $\mathrm{y} \to 0^+$,

$$\mathscr{F}(\mathrm{y}) \quad = \quad \int \left[\frac{1}{|\mathrm{g}|^{1/4}} \frac{\mathrm{d}^2}{\mathrm{dy}^2} \left(\frac{1}{|\mathrm{g}|^{1/4}} \right) - \frac{\mathrm{q}}{|\mathrm{g}|^{1/2}} \right] \mathrm{dy}, \qquad (27)$$

requires that one must choose

$$\mathbf{q}(\mathbf{y}) = -\frac{\mathbf{1}}{\mathbf{4y^2}}. \qquad (28)$$

With this choice, it is easy to see that $\mathscr{H}(\xi)$ and $\mathscr{I}(\zeta)$ are both convergence near the second-order pole.

Convergence of $\mathscr{I}(y)$ for extreme point

We observe that $\mathscr{I}(y)$ associated with two turning point,

$$\mathscr{I}(y) \sim \left.\frac{7p'^2 - 6pp''}{32|p|^{5/2}}\right|_{y_{1,2}} \ln|y_2 - y_1| - \int \frac{q(y)}{\sqrt{|g(y)|}} dy.$$

Here $g(y) = p(y)(y - y_1)(y - y_2)$ with $p(y)$ being regular. When $y_1 = y_2$(one **double turning point**), $\mathscr{I}(y) \to \pm\infty$.

This divergence could be cured if one takes $q(y)$ that satisfies

$$\int \frac{q(y)}{\sqrt{|g(y)|}} dy \sim \left.\frac{7p'^2 - 6pp''}{32|p|^{5/2}}\right|_{y_{1,2}} \ln|y_2 - y_1|, \tag{29}$$

which depends on the properties of function $-\Omega^2(y)$.

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Exact energy levels for Hydrogen atom

Radial Schrödinger equation for hydrogen atom

$$\frac{d^2\psi(r)}{dr^2} + \left[\frac{2m}{\hbar^2}\left(E + \frac{Ze^2}{r}\right) - \frac{l(l+1)}{r^2}\right]\psi(r) = 0. \tag{30}$$

The exact energy levels are given by

$$E_n = -\frac{mZ^2e^4}{2\hbar^2 n^2} = -\frac{mZ^2e^4}{2\hbar^2(n_r + 1/2 + l + 1/2)^2} \tag{31}$$

1930, Young and Uhlenbeck found the JWKB method gives
(Young and Uhlenbeck, Phys. Rev. 36, 1154 (1930).)

$$E_n^{JWKB} = -\frac{mZ^2e^4}{2\hbar^2(n_r + 1/2 + \sqrt{l(l+1)})^2}. \tag{32}$$

Exact result can be obtained by replacing $l(l+1) \to (l + \frac{1}{2})^2$.

Langer modification for radial problem

In 1937, Langer suggested a replacement to the radial Schrödinger equation (R. E. Langer, Phys. Rev. 51, 669 (1937)).

$$\frac{d^2\psi(r)}{dr^2} + \left[\frac{2m}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2}\right]\psi(r) = 0. \qquad (33)$$

The **Langer modification** is defined as

$$l(l+1) \longrightarrow (l+1/2)^2 \qquad (34)$$

Then the JWKB method gives correct results for the radial problem. This is a standard procedure in quantum mechanics.

Interpretation of Langer modification

The centrifugal potential term in radial equation leads to a second-order pole at the origin

$$\Omega^2(r) = \frac{2m}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2} \tag{35}$$

And the Langer modification can be expressed as

$$\frac{l(l+1)}{r^2} = \frac{(l+1/2)^2}{r^2} - \frac{1}{4r^2} \tag{36}$$

Here the last term is exactly the choice of $q(r)$ in the uniform asymptotic approximation for eliminating the divergence in the error control functions.

Improved Quantization Condition

For bound states problem with two turning point, the JWKB approximation gives the JWKB quantization rules

$$\int_{r_1}^{r_2} \sqrt{\frac{2m}{\hbar^2}(E - V)} = (n_r + 1/2)\pi \tag{37}$$

The uniform asymptotic approximation gives

$$\int_{\tilde{r}_1}^{\tilde{r}_2} \sqrt{\frac{2m}{\hbar^2}(E - V) + q(r)} = (n_r + 1/2)\pi. \tag{38}$$

Here $q(y)$ is chosen for reducing the errors of the approximation near extreme points, which depends on the specific form of the potential $V(r)$.

- Our improved quantization condition gives the exact energy spectra for all these potentials.

TABLE I. Some exactly solvable potentials and the choices of $q(x)$.

Potentials	$V(x)$	$q(x)$
Hydrogen	$-\frac{e^2}{x}+\frac{\hbar^2 l(l+1)}{2mx^2}$	$-\frac{1}{4x^2}$
Harmonic oscillator	$\frac{1}{2}m\omega^2x^2+\frac{\hbar^2 l(l+1)}{2mx^2}$	$-\frac{1}{4x^2}$
Morse potential	$v_0e^{-2\alpha x}+v_1e^{-\alpha x}$	0
Pöschl-Teller potential	$\frac{v_0}{\cosh^2(\alpha x)}$	$\frac{\alpha^2}{4\cosh^2(\alpha x)}$
Eckart potential	$\frac{v_0}{\sinh^2(\alpha x)}+\frac{v_1}{\tanh(\alpha x)}$	$-\frac{\alpha^2}{4\sinh^2(\alpha x)}$

(Tao Zhu and Anzhong Wang, arXiv:1902.09675 [quant-ph])

Quasi–normal modes

For Schwarzchild black hole,

$$\frac{d^2R(r)}{dr^2} + \Big(\omega^2 - V(r,\omega)\Big)R(r) = 0. \tag{39}$$

The improved quantization condition read

$$\int_{r_1}^{r_2} \frac{\sqrt{g(r)}}{1-2M/r} dr = i(n+1/2)\pi \tag{40}$$

This new quantization condition leads to

$$\begin{aligned} \omega^2 \quad &\simeq \quad V_m - q_m - i(n+1/2)\sqrt{-2g''_m} \\ &\times \left[1 + \frac{i(n+1/2)}{8\sqrt{-2g''_m}} \left(-\frac{5}{9}\frac{g'''^2_m}{g''^2_m} + \frac{1}{3}\frac{g''''_m}{g''_m}\right)\right] + \cdots\cdots \end{aligned} \tag{41}$$

(TZ, AW, WZ, KL, et. al, in preparation.)

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It is in general impossible to get the exact inflationary observables analytically in inflation models.

$$\mu_k''(\eta) + \left(\omega_k^2(\eta) - \frac{z''}{z}\right)\mu_k(\eta) = 0$$

Some alternative approximations:

- Bessel function approximation [Schwarz et. al, PLB517(2001)243];
- Green's function method [Stewart and Gong, PLB510(2001)1;Wei, Cai, and Wang, PLB603(2004)95];
- WKB approximation [Martin and Schwarz, PRD67,083512(2005); Casadio et.al., PLB625(2005)1];
- Improved WKB approximation [Casadio et. al. PRD72,103516(2005)];
- Phase integral method [Rojas and Villalba, PRD 75, 063518 (2007)];
- Uniform asymptotic approximation [Hibab et.al. PRL89, 281301(2002); PRD70,083507(2004); PRD89(2014)043507; IJMPA29(2014) 1450142; PRD90(2014)063503;PRD90(2014)103517; APJL807(2015)L17; JCAP10(2015)052; JCAP03(2016)046; PRD93(2016)123525; PRD97(2018) 103502; JCAP02(2018)018; PRD99 (2019)103536; JCAP09 (2019) 064; Phys.Dark Univ. 26 (2019)100373 ; 1902.09675; 1907.13108; 1911.01580.]

General formulas of power spectra for single turning point

$$\Delta^2(k) = \frac{k^2}{4\pi^2}\frac{-k\eta}{z^2(\eta)\nu(\eta)c_s(\eta)}\exp\left(2\int_{y}^{\bar{y}_0}\sqrt{g(\hat{y})}d\hat{y}\right) \times \left[1+\frac{\mathscr{H}(+\infty)}{\lambda}+\frac{\mathscr{H}^2(+\infty)}{2\lambda^2}+\dots\right]$$

where the error control function $\mathscr{H}(\xi)$ reads as

$$\frac{\mathscr{H}(\xi)}{\lambda} = \int_{y}^{y_0}\left(\frac{5}{16\hat{\xi}^3(y')}+\frac{q(y')}{\hat{g}(y')}-\frac{5\hat{g}'^2(y')}{16\hat{g}^3(y')}+\frac{\hat{g}''(y')}{4\hat{g}^2(y')}\right)\sqrt{\hat{g}(y')}dy'$$

Table: Errors to be expected in the uniform approximation

Quantity	1st-order	2nd-order	3th-order
Power spectrum: $\Delta^2(k)$	$\lesssim 15\%$	$\lesssim 1.5\%$	$\lesssim 0.15\%$

Applications to slow-roll inflation

- k-inflation [PRD90(2014)103517]
- Inflation with Nonlinear dispersion relation[PRD90(2014)063503.]
- Inverse-volume and holonomy corrections in LQC [JCAP10(2015)052; JCAP03(2016)046; APJL807(2015)L17]
- Inflation with Gauss-Bonent corrections [PRD97 (2018)103502]
- Power spectra in closed algebra approach in LQC [PRD99 (2019)103536]
- Effective field theory of inflation [arXiv:1907.13108]
- Polarized PGWs in ghost-free parity violating gravities [J. Qiao, TZ, W. Zhao, AW, arXiv:1911.01580]
- ……

General formulas of power spectra with extra two turning points

$$\Delta^2(k) \simeq \mathscr{A}(k)\frac{k^2}{4\pi^2}\frac{-k\eta}{z^2(\eta)\nu(\eta)}\exp\left(2\lambda\int_y^{y_0}\sqrt{\hat{g}(y')}dy'\right) \times\left[1+\frac{\mathscr{H}(+\infty)}{2\lambda}+\frac{\mathscr{H}^2(+\infty)}{8\lambda^2}+\mathscr{O}\left(\frac{1}{\lambda^3}\right)\right], \quad (42)$$

where $\mathscr{A}(k)$ denotes the modified factor due to the presence of the two extra turning points y_1 and y_2, which reads

$$\mathscr{A}(k) = 1+2e^{\pi\lambda\zeta_0^2}+2e^{\pi\lambda\zeta_0^2/2}\sqrt{1+e^{\pi\lambda\zeta_0^2}} \times\left\{\cos 2\mathfrak{B}-\frac{\mathscr{I}(\zeta)+\mathscr{H}(\xi)}{\lambda}\sin 2\mathfrak{B} -\frac{\left[\mathscr{I}(\zeta)+\mathscr{H}(\xi)\right]^2}{2\lambda^2}\cos 2\mathfrak{B}\right\}. \quad (43)$$

Applications to inflation

- Inflation with Nonlinear dispersion relation[IJMPA29 (2014)1450142; PRD89 (2014) 043507; PRD90 (2014)027304]
- Extended effective field theory of inflation [JCAP 09 (2019)064]
- Schwinger effects during inflation [JCAP1802 (2018) 018]
- Inflation with sound speed changes [in preparation (2019)]
- Ultra-Slow-Roll inflation [in preparation (2019)]
- . . . . . .

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Three types of resonance

Consider a simple equation

$$\frac{d^2\mu_k(y)}{dy^2} + (A_k - 2p\cos 2y)\,\mu(y) = 0, \tag{44}$$

the mode $\mu_k(y)$ can be amplified by the parametric resonance due to non-adiabatic excitations driven by $2p\cos 2y$.

The strength of the resonance depends on A_k and p.

- Tachyonic resonance, which corresponds to $A_k < 2p$. It can occur for most of the parametric space and momentum modes;
- Broad resonance, which corresponds to $p \gg 1$. It can occur for a wide range of the parametric space and momentum modes;
- Narrow resonance, which corresponds to $p \ll 1$. It can only occur in some narrow bands near $A_k \simeq l^2$, $\quad l = 1, 2, 2$ and each band has width about p^l. Thus the most important band is $l = 1$.

Mathieu Equation and turning points

The parametric resonance is described by the Mathieu equation

$$\frac{du_k^2}{dx^2} + (A_k - 2p\cos 2x)u_k = 0. \tag{45}$$

In the Uniform asymptotic approximation, it is written into

$$\frac{du_k^2}{dx^2} = (g + q)u_k, \tag{46}$$

$$g(x) = 2p\cos 2x - A_k, \quad q(x) = 0. \tag{47}$$

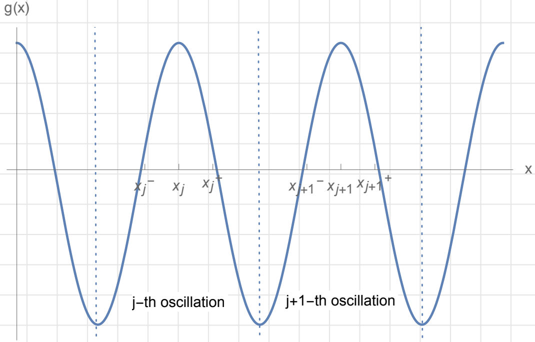
Obviously $g(x)$ has turning points,

$$x_j^{\pm} = x_j \pm \frac{1}{2}\arccos\frac{A_k}{2p}. \tag{48}$$

[TZ, QW, AW, Phys.Dark Univ. 26 (2019)100373]

Nature of turning points (TPs)

$$x_j^{\pm} = x_j \pm \frac{1}{2} \arccos \frac{A_k}{2p} = \begin{cases} \text{two real TPs}, & A_k < 2p, \\ \text{one double TP}, & A_k = 2p, \\ \text{two complex TPs}, & A_k > 2p \end{cases}$$



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Solutions around pairs of turning points

In each oscillation, the approximate solution is given by

$$u_k = \left(\frac{\zeta_j^2 - \zeta_0^2}{-g(x)}\right)^{1/4} \left[a_j W(\zeta_0^2/2, \sqrt{2}\zeta_j) + b_j W(\zeta_0^2/2, \sqrt{2}\zeta_j)\right] \quad (49)$$

The relation between two oscillations can be connected via

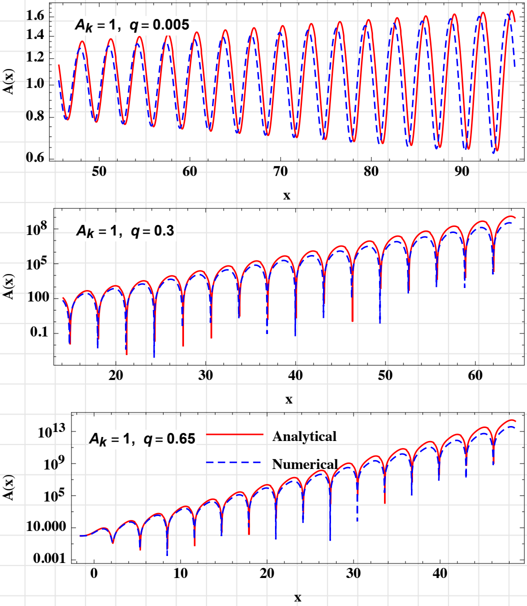
$$\begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix} = \begin{pmatrix} \kappa \sin \mathfrak{B}, & -\cos \mathfrak{B} \\ \cos \mathfrak{B}, & \kappa^{-1} \sin \mathfrak{B} \end{pmatrix} \begin{pmatrix} a_j \\ b_j \end{pmatrix}. \quad (50)$$

where $\kappa = \sqrt{1 + e^{\pi\zeta_0^2}} - e^{\pi\zeta_0^2/2}$ and

$$\mathfrak{B} \equiv \int_{\mathrm{Re}(x_j^+)}^{\mathrm{Re}(x_{j+1}^-)} \sqrt{-g(x)}dx + \frac{\pi}{2} + 2\phi\left(\frac{\zeta_0^2}{2}\right). \quad (51)$$

$$\zeta_0^2 \equiv \pm\frac{2}{\pi}\left|\int_{x_j^-}^{x_j^+} \sqrt{g(x)}dx\right| \quad (52)$$

Analytical and Numerical Solutions

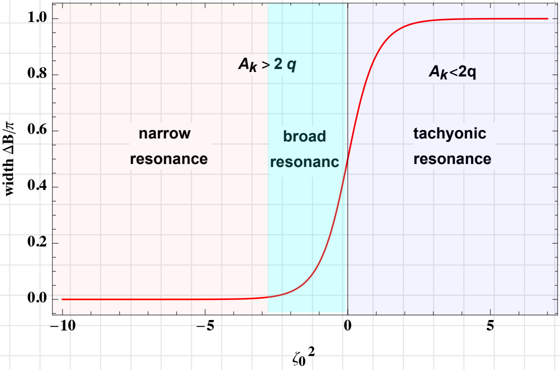


Condition for parametric resonance

With the above solution, we obtain the condition for parametric resonance

$$n\pi + \arctan\left(e^{-\pi\zeta_0^2/2}\right) \quad < \quad \mathfrak{B} < n\pi + \pi - \arctan\left(e^{-\pi\zeta_0^2/2}\right),$$

$$\Delta\mathfrak{B} \quad = \quad \pi - 2\arctan\left(e^{-\pi\zeta_0^2/2}\right).$$



Amplification factor and particle productions

The amplification of field after N-oscillations

$$\left|\frac{u_k(x_N)}{u_k(x_0)}\right| \sim \frac{2^{3/4}}{\sqrt{\kappa}}\left|\frac{(a_1+b_1 Y_-)(\sin\Delta_N+\kappa Y_+\cos\Delta_N)}{Y_+-Y_-}Z_-^{N-1}\right.$$
$$\left.+\frac{(a_1+b_1 Y_+)(\sin\Delta_N+\kappa Y_-\cos\Delta_N)}{Y_--Y_+}Z_+^{N-1}\right|,$$

and the particle production rate is

$$|\beta_k^{(N)}|^2 = \frac{1}{2\kappa}\left|\frac{(\kappa Y_+ + i)(a_1+b_1 Y_-)}{Y_+-Y_-}Z_-^{N-1}\right.$$
$$\left.+\frac{(\kappa Y_- + i)(a_1+b_1 Y_+)}{Y_--Y_+}Z_+^{N-1}\right|^2.$$

[TZ, QW, AW, Phys.Dark Univ. 26 (2019)100373]

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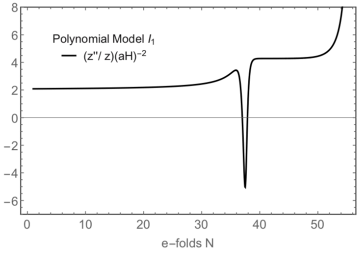
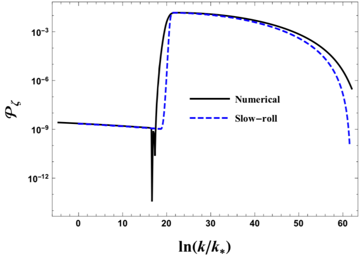
Primordial modes in ultra slow-roll inflation

[H. Di and Y. Gong, arXiv: 1707.09578; I. Dalianis, A. Kehagias, and G. Tringas, arXiv: 1805.09483; C.Fu, P.Wu, H. Yu, arXiv: 1907.05042 ......]

$$\frac{d^2 u_k(\eta)}{d\eta^2} + \left(k^2 - \frac{z''}{z}\right) u_k(\eta) = 0 \tag{53}$$

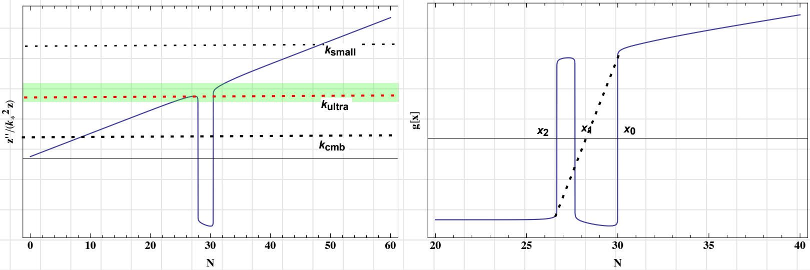




Turning points in ultra slow–roll inflation

$$\frac{d^2\mu(x)}{dx^2} + \left(\frac{k^2}{k_*^2} - \frac{z''}{k_*^2 z}\right)\mu_k = 0 \quad (54)$$

$$g(x) = -\left(\frac{k}{k_*} - \frac{z''}{k_*^2 z}\right) + \frac{1}{4x^2}, \quad x = -k_*\tau. \quad (55)$$



Primordial spectra for ultra–slow–roll inflation

$$\mathcal{P}_{\mathcal{R}}^{2}(k) \simeq \mathscr{A}(k)\frac{H^2}{8\pi^2 M_{Pl}^2\epsilon_1}, \quad (56)$$

$$\mathscr{A}(k) = 1 + 2e^{\pi\lambda\zeta_0^2} + 2e^{\pi\lambda\zeta_0^2/2}\sqrt{1+e^{\pi\lambda\zeta_0^2}}\cos 2\mathfrak{B} \quad (57)$$

- for $k_{small} \gg k_{ultra}$, $g(x)$ only has one single TP, $\mathscr{A}(k) = 1$;
- for $k \sim k_{ultra}$, $g(x)$ only has one single TP and two extra coalescing TPs, $\mathscr{A}(k) \simeq 1 + 2e^{\pi\zeta_0^2}\cos 2\mathfrak{B}$;
- for $k_{cmb} \ll k_{ultra}$, $g(x)$ has three real TPs. If we evaluate H and ϵ_1 at ultra-slow-roll phase (at x_0), we have $\mathscr{A}(k) \gg 1$ and $\epsilon_1 \ll \epsilon_1^{cmb}$. If we evaluate at the slow-roll phase (at x_2), we will have $\mathscr{A}(k) = 1$ and $\epsilon_1 \simeq \epsilon_1^{cmb}$.

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Outlooks

- **How to describe accurately the non-adiabatic evolutions of quantum fields in the curved spacetime?**

  - Perturbations modes during inflation with abrupt changes of some parameters [fast roll/ultra slow-roll; sound speed changes; abrupt inflation potentials];
  - Gravitational waves in extreme mass ratio system;
  - Quantum fields in various black hole spacetimes [Stability/ instability; Hawking radiation; greyfactor; quasi-normal modes; ringdown gravitational waves];
  - Schwinger effects by periodic laser pulses
  - . . . .

Thanks!