QCD Amplitude Calculations and LHC Physics

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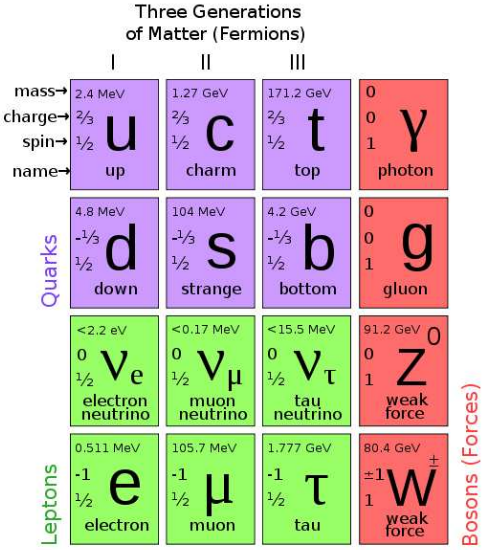
Introdution: Standard Model



Introduction

Standard Model:

1. lepton: e, μ, τ, involved in electroweak interaction
   neutrino ν_e, ν_μ, ν_τ, involved in weak interaction
2. Quark: u, d, c, s, t, b, involved in both electroweak and strong interaction.
3. Gauge boson: photon γ mediate electromagnetic interaction, U(1) gauge interaction
   Z^0 and $W^\pm$ mediate weak interaction, $SU(2)$ gauge symmetry, Spontaneously broken.
   gluon g mediate color interaction (strong interaction), $SU(3)$ gauge interaction
4. Higgs H boson: scalar, no electric charge, weak interaction. Important in spontaneous sysmmetry breaking of $SU(2)$ and mass generation.

Introduction

QCD: Color $SU(3)$ gauge field theory. Gluon mediates the color interaction.

- Describe the strong interaction between *quarks* and *gluon*. Quark $\sim$ 3-dimensional fundamental representation of $SU(3)$. Gluon $\sim$ 8-dimensional adjoint representation.
- Color confinement and asymptotic freedom: Quarks and gluons can not propagate freely, they are always combined into color singlet, Hadron. The interaction is stronger when quarks are pulled away from each other.
- Spontaneous and explicit breaking of chiral symmetry: peudo-goldstone boson (π and K)
- Hadron: A meson is composed of a quark and an anti-quark (valence quarks), with zero baryon number, such as $\pi^{\pm}, \pi^0 \sim u\bar{d}, u\bar{u}, d\bar{d}, d\bar{u},\ K^{\pm} \sim u\bar{s} \ldots$.
  A baryon is composed of 3 quarks and has baryon number 1, proton $p \sim uud$.

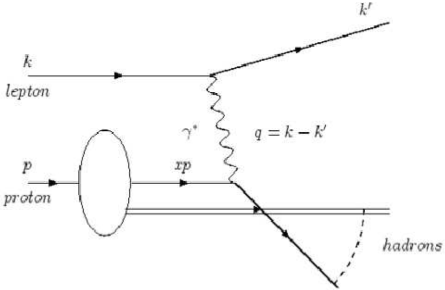
Introdution

- Our understanding of the micro world are approached by the collision of particles:
  - Deep inelastic scattering confirms the quark model: high energy lepton probing inner structure of the proton
  - W and Z boson were discovered by SPS (Super Proton Synchrotron) at CERN: proton anti-proton collision
  - LEP: electron and positron collision, precise test of standard model
  - Top quark was discovered at Tevatron: Proton antiproton collision.
  - ...

Introduction

- LHC: Proton-Proton and heavy-ion collision, the energy of each proton in the two beams has already reached $\sim 3.5Tev$, the final purpose is $\sqrt{s} \sim 14Tev$, $v \sim 0.99999999c$, $9.7km/hour$ slower than c
  - ALICE: collide lead ion, Study quark-gluon plasma.
  - ATLAS, CMS: general-purpose detectors, search for Higgs, physics beyond standard model: SUSY, extra-dimension, dark matter, origin of mass ...
  - LHCb: B-physics, CP violation. Why matter not anti-matter.
  - LHCf: simulate Cosmic rays.
  - TOTEM: precise measurement of the proton-proton interaction cross section, in-depth study of the proton structure

Introduction



Proton:

- Probe $Q^2 > (2Gev)^2$: 3 valence quarks uud, gluons, sea quarks (all kinds of quarks). These are called partons.
- Each parton carries a proportion of the longitudinal momentum of Proton $p_i = x_i P$, a bit transverse momentum, some angular momentum.
- Parton distribution function (pdf): $f_i(x, Q^2) \to f_i(x)$ for $Q^2 \to \infty$

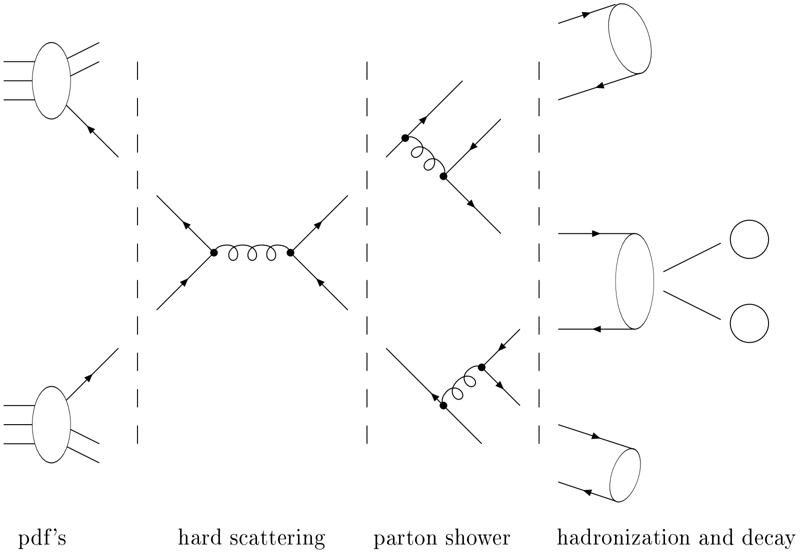
Introduction

A typical QCD process in proton-proton scattering

- Choose two partons from the two incoming protons: pdf
- High energy hard scattering
- Parton shower: radiation of some soft partons from the hard scattering final states.
- Hardronization and decay: the partons from parton shower recombine to form hadrons—jets: a cone of hadrons, coming from a parton after the hard process in leading order. Some unstable hadrons may decay.

Introduction

Diagram from (Stefan Weinzierl, PoS ACAT:005,2007, arxiv:0707.3342)



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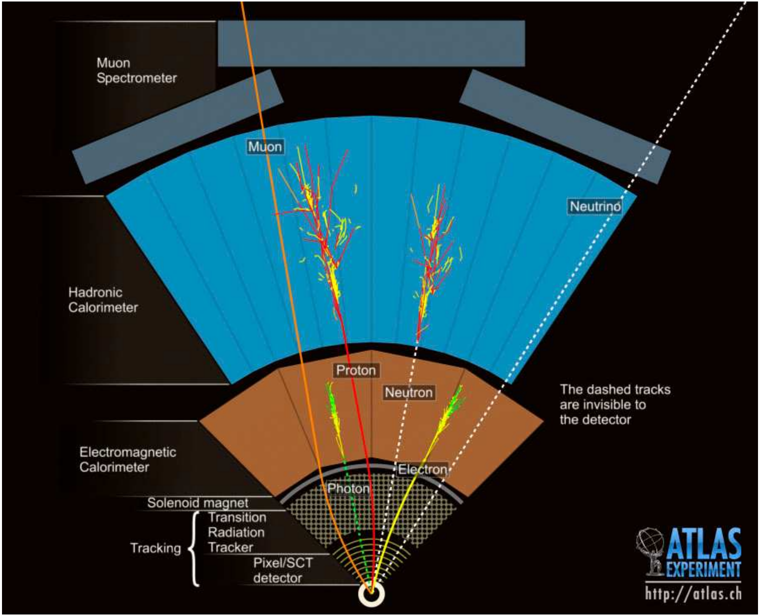

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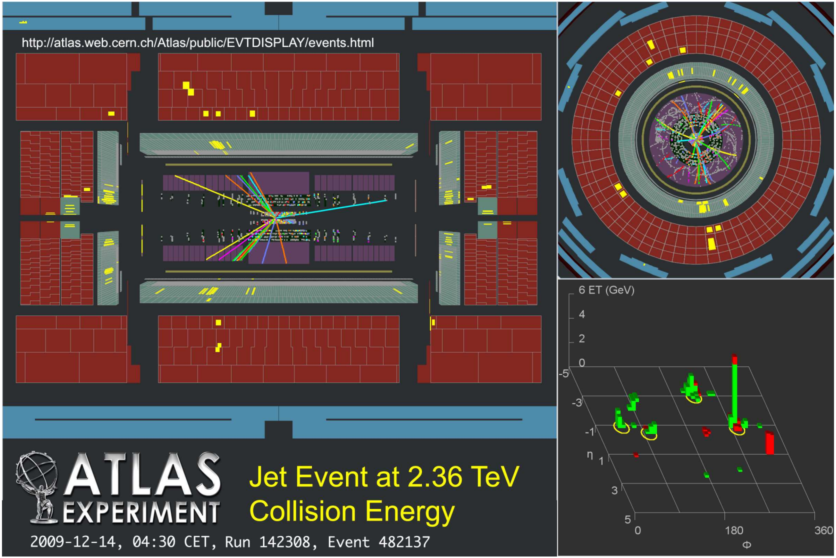

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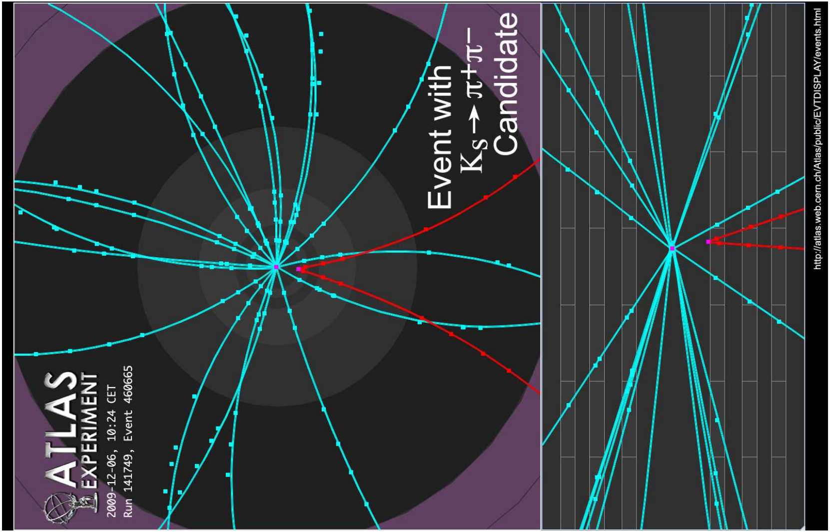


Introduction

- Observable relating the experiments and the theory in high energy physics: cross section

$$\text{NO. of events} = \frac{\sigma N_A N_B}{A}$$

N_A, N_B number of particles in the incoming two beams. A the area of the beam. σ is cross section.

Introduction

- To calculate the cross section, we need first calculate the amplitude.
  Differential cross section (Hard part):

$$d\sigma = \frac{1}{2s}\left(\prod_{i=1}^{N}\frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}\right) \times (2\pi)^4\delta^4(p_A + p_B - \sum p_i) \\ \times |\mathcal{M}(p_A, p_B \to \{p_i\}|^2 ,$$

$\mathcal{M}$: scattering amplitude.

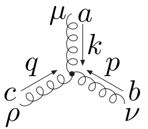
Introduction

Multi-leg amplitudes are needed because:

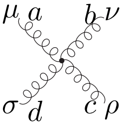
- LHC processes always have multi-jets final states—multi-parton hard process.
- In order to test the theory with the experiments precisely, we need multi-leg amplitudes.
- Some new physics particles also produce multi-jet final states. To confirm it is a new physics phenomenon we needs to first understand the SM background of this process, which always involves QCD multi-leg amplitudes.

Feynman rules

- Calculating amplitudes in Quantum Field Theory: Feynman diagrams and Feynman rules
  QCD: Nonabelian gauge field theory.
  For pure Yang-Mills, Feynman Rules for self-interaction:



$$= g f^{abc}\Big(\eta^{\mu\nu}(k-p)^{\rho} + \eta^{\nu\rho}(p-q)^{\mu} + \eta^{\rho\mu}(q-k)^{\nu}\Big)$$



$$= \left\{ \begin{array}{rl} -ig^2[& f^{abe}f^{ecd}(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}) \\ + & f^{ade}f^{ebc}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\rho}\eta_{\nu\lambda}) \\ + & f^{ace}f^{ebd}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho})] \end{array} \right.$$

Difficulties

Difficulties in calculating the amplitudes in Yang-Mills (Y-M) theory and QCD. For multi-gluon amplitudes

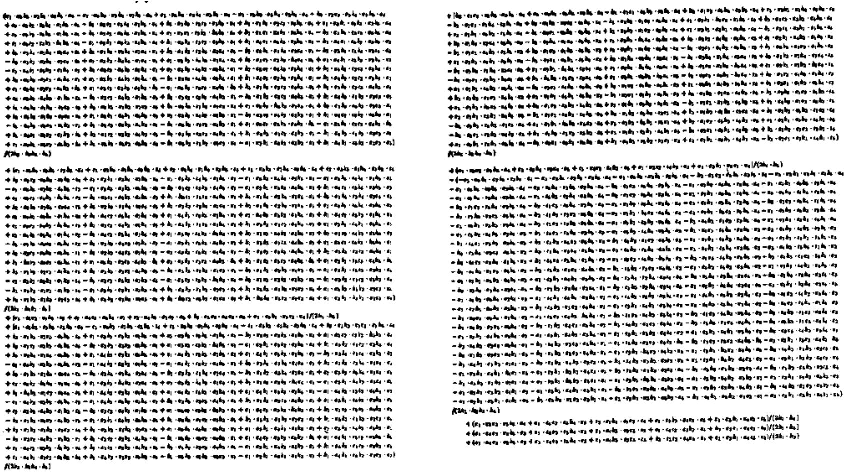
- Too many diagrams

external gluons	4	5	6	7	8	9	10
diagrams	4	25	220	2485	34300	559405	10525900

- Too many terms in each diagram: comes from the complexity of vertices
- Too many kinetic variables: $k_i \cdot k_j$

A 5-gluon tree-level amplitude

A typical brute force calculation of five-gluon tree level amplitude:
(from Z. Bern hep-ph/9304249)



Color ordering

- The structure constants in the vertices of pure Yang-Mills theory:

$$f^{abc} = 2\mathrm{Tr}(T^a T^b T^b - T^c T^b T^a)$$

- The full amplitude can be decomposed:

$$\mathcal{A}[(p^h, a)_1, \dots, (p^h, a)_n] =$$
$$g^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}[T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}]\, A[(p^h)_{\sigma(1)}, \dots, (p^h)_{\sigma(n)}]$$

  A is called partial amplitude. Feynman rules can be formulated directly in color ordered version.

- **Reversal.**

$$A(1, \dots, n) = (-1)^n A(n, \dots, 1)$$

- **Dual Ward identity.**

$$A(1, 2, 3, \dots, n) + A(2, 1, 3, \dots, n) + A(2, 3, 1, \dots, n) + \cdots + A(2, 3, \dots, 1, n) = 0$$

Spinor momentum

- It is convenient to express a Lorentz 4-vector as a bispinor

$$p_{a\dot{a}} = p_\mu \, \sigma^\mu_{a\dot{a}}$$

where σ^μ are the components of the Pauli 4-vector

$$\sigma^\mu \to \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

- $2p \cdot q = p_{a\dot{a}} q^{a\dot{a}}$ and $p^2 = \det p_{a\dot{a}}$. If $p^2 = 0$, we can **factorise** the bispinor thus:

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}},$$

often abbreviated $p = \lambda\tilde{\lambda}$. λ and $\tilde{\lambda}$ transform in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations, respectively.

- For **real** momenta, $\lambda_a = (\lambda_{\dot{a}})^*$; for **complex** momenta or under the $(-,-,+,+)$ metric, the two are **independent**.

Spinor helicity

- We can also express the polarisation vectors in this formalism. For a gluon with momentum $p = \lambda\tilde{\lambda}$, we have positive and negative helicity polarisations given by

$$\epsilon^{+}_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu\ \lambda \rangle} \quad \text{and} \quad \epsilon^{-}_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\lambda\ \mu]}.$$

- $\mu\tilde{\mu}$ is a **reference momentum** not parallel to p. This freedom amounts to a **gauge choice**.
- Clearly, $p \cdot \epsilon^{\pm} = 0$.

Five point amplitude

Example: Five point amplitude: We need only $A(--+++)$ amplitudes. The others can be related to these two amplitudes.

$$A(--+++) = i\frac{\langle 1\ 2\rangle^4}{\langle 1\ 2\rangle\langle 2\ 3\rangle\ldots\langle 5\ 1\rangle} \sim A(++---)^*$$

$A(-+-++)$ can be obtained from $A(--+++)$ by dual ward identity.

$$A(-+-++) = i\frac{\langle 1\ 3\rangle^4}{\langle 1\ 2\rangle\langle 2\ 3\rangle\ldots\langle 5\ 1\rangle}$$

$A(+++++) = A(-++++) = 0$

Some simple n-point partial amplitudes

For general n point partial amplitudes:

- $$A(1^+, \ldots, n^+) = A(1^-, 2^+, \ldots, n^+) = 0$$
- Pure Y-M MHV (Maximal Helicity Violating) amplitude, proposed by Parke and Taylor [Phys. Lett. B **157** 81 (1985)], proved by Berends and Giele [Nucl. Phys. B **306** 759 (1988)]. : Simple & Beautiful

$$A(1^+ \cdots i^- \cdots j^- \cdots n^+) = i \frac{\langle i\, j \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n-1\, n \rangle}$$

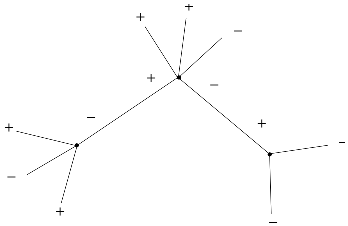
CSW RULES

CSW rules (F. Cachazo, P. Svrcek, E. Witten, JHEP 0409:006,2004):

- Analytically continue the MHV amplitude to off-shell. For off-shell momenta, we define

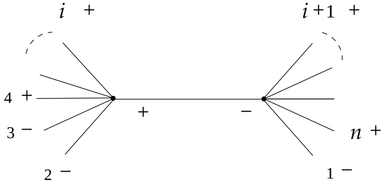
$$\lambda_a = P_{a\dot{a}} \eta^{\dot{a}}.$$

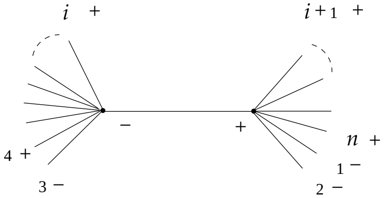
  η is an arbitrary spinor. Use this λ to construct off-shell MHV vertices.
- Construct tree level non-MHV amplitudes using MHV vertices connected with scalar propagator, $\frac{1}{P^2}$.
- Adding one MHV vertex increases one "$-$" helicity external leg. $n_- - 1$ vertices for diagrams with n_- "$-$" helicity legs.



CSW rules

- Advantage: Reduce the number of diagrams and the number of terms for each diagram.
  The number of diagrams grows at most as n^2 for large n.
  A typical example: $A(---+\cdots+)$, n external legs:
  $2(n-3)$ diagrams





CSW rules

- Only useful in tree level amplitude and can also be used in one loop supersymmetric amplitude.
- For one loop non-supersymmetric amplitude, there are some missing pieces.

BCFW recursion relation

$$A_n(1,2,\ldots,(n-1)^-,n^+) = \sum_{i=1}^{n-3}\sum_{h=+,-}\Big(A_{i+2}(\hat{n},1,2,\ldots i,-\hat{P}^h_{n,i}) \times \frac{1}{P^2_{n,i}} A_{n-i}(+\hat{P}^{-h}_{n,i}, i+1,\ldots,n-2, n\hat{-}1)\Big).$$

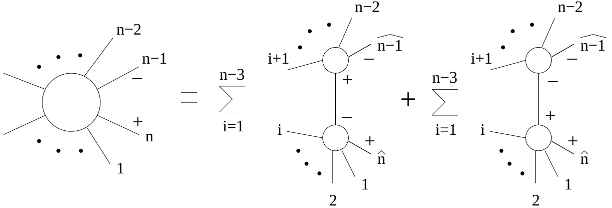
where

$$\begin{aligned}
P_{n,i} &= p_n + p_1 + \ldots + p_i, \\
\hat{P}_{n,i} &= P_{n,i} + \frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n\rangle}\lambda_{n-1}\tilde{\lambda}_n, \\
\hat{p}_{n-1} &= p_{n-1} - \frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n\rangle}\lambda_{n-1}\tilde{\lambda}_n, \\
\hat{p}_n &= p_n + \frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n\rangle}\lambda_{n-1}\tilde{\lambda}_n.
\end{aligned}$$

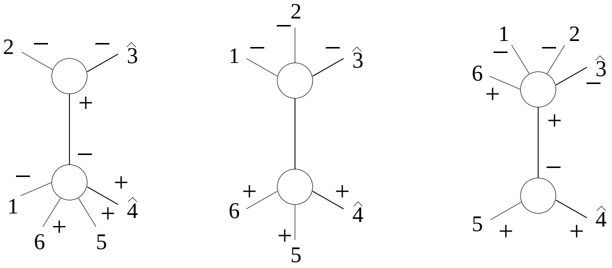
All the amplitudes involved in the recursion relation are onshell.

BCFW recursion relation

In diagram, the recursion relation can be represented as:



An example: $A(- - - + + +)$, only two diagrams are non zero



(a) (b) (c)

MHV Lagrangian: Introduction

CSW rules was first proposed inspired by the considerations in twistor string theory. To derive them from field theory side

- There were some indirect proof of CSW rules. Using the BCFW recursion relation.
- MHV-lagrangian approach to derive CSW rules:
  - Paul Mansfield: Propose a framework to deduce MHV lagrangian from Pure Yang-Mills under canonical transformation of the fields (JHEP0603:037,2006). Each MHV vertex comes from one term in the lagrangian

    $$L^{-+}[B,\bar{B}]+L^{--+}[B,\bar{B}]+L^{--++}[B,\bar{B}]+L^{--+++}[B,\bar{B}]\cdots$$

    The vertices from the lagrangian are MHV amplitudes continued to off-shell.
  - The canonical transformation can be explicitly solved. The missing pieces of the CSW rules is from the Completion vertices. (Tim and James,JHEP0608:003,2006, JHEP 0705:011,2007.)
  - The MHV lagrangian can be extended to include fermions and to SQCD (JHEP 0808:103,2008, JHEP 0812:028,2008.) .

Massless Light-cone Y-M Lagrangian

LCYM: Starting from massless Y-M Lagrangian in terms of $\check{A}$, $\hat{A}$, $\tilde{A}$, $\bar{A}$.

- Choose a gauge: $\mu \cdot A = \hat{A} = 0$, reference momentum $\mu^{\mu} = (1, 0, 0, 1)$ The lagrangian is quadratic in $\check{A}$.
- LCYM Lagrangian: Integrate out nondynamical fields: $\check{A}$. The fields left are A $\bar{A}$.

$$L_{LCYM} \quad = \quad L_{YM}^{-+} + L_{YM}^{++-} + L_{YM}^{--+} + L_{YM}^{--++}$$

Canonical Transformation for light-cone Y-M

- From

$$L_{YM}^{-+} = \mathrm{tr}(\check{\partial} A \hat{\partial} \bar{A} - \partial A \bar{\partial} \bar{A})$$

  Canonical Fields (up to constant coeff.):

$$(A, \hat{\partial} \bar{A})$$

- Canonical Transformation: $(A, \hat{\partial}\bar{A}) \to (B, \hat{\partial}\bar{B})$, *s.t.*

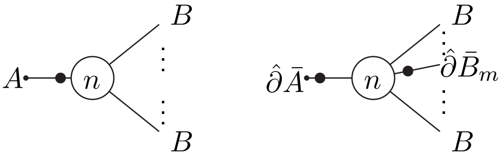
$$L_{YM}^{-+}[A, \bar{A}] + L_{YM}^{++-}[A, \bar{A}] = L_{YM}^{-+}[B, \bar{B}]$$

Canonical Transformation for light-cone Y-M

- MHV and charge conservation (preserve − helicity fields) requirements:

$$A_0 \sim \sum_n \int \Upsilon_{01\cdots n} B_1 \cdots B_n,$$

$$\hat{\partial}\bar{A}_0 \sim \sum_{m,n} \int \Xi^m_{01\cdots n} B_1 \cdots \hat{\partial}\bar{B}_m \cdots B_n$$



- If we can solve these coefficients and then we can substitute these into the LCYM and we obtain MHV Lagrangian

$$L^{-+}[B,\bar{B}] + L^{--+}[B,\bar{B}] + L^{--++}[B,\bar{B}] + L^{--+++}[B,\bar{B}]\cdots$$

Solving the equation for A

From equation:

$$L_{YM}^{-+}[A,\bar{A}] + L_{YM}^{++-}[A,\bar{A}] = L_{YM}^{-+}[B,\bar{B}]$$

Change to momentum space, we obtain an integral equation:

$$\omega_1 A_1 - i\int_{23}\zeta_3[A_2,A_3]\,(2\pi)^3\delta^3(\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3) \quad = \quad \int_{\mathbf{p}}\omega_{\mathbf{p}}B^b_{\mathbf{p}}\frac{\delta A_1}{\delta B^b_{\mathbf{p}}},$$

where $\omega_i = p_i\bar{p}_i/\hat{p}_i$ and $\zeta_{i,j} = (\bar{p}_i+\cdots+\bar{p}_j)/(\hat{p}_i+\cdots+\hat{p}_j)$.

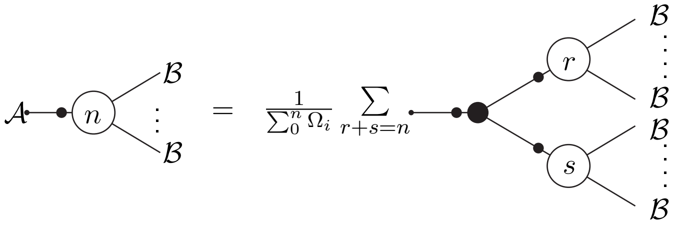
- Expand A in B

$$A_1 = B_1 + \sum_{n=3}^{\infty}\int_{2\cdots n}\Upsilon_{12\cdots n}\,B_{\bar{2}}\cdots B_{\bar{n}},$$

- Substitute this into the integral equation, we have a recursion relation for Υ:

$$\Upsilon(1\cdots n) \quad = \quad \frac{i}{\omega_1+\cdots+\omega_n}\sum_{j=2}^{n-1}(\zeta_{j+1,n}-\zeta_{2,j})$$
$$\times\Upsilon(-,2,\cdots,j)\Upsilon(-,j+1,\cdots,n),$$

Solving the equation for A



This can be solved:

$$\Upsilon(1\cdots n) \quad = \quad -g^{n-2}\frac{\hat{1}}{\sqrt{\hat{2}\,\hat{n}}\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle}, \quad n\geq 3,$$

Solving the equation for $\bar{A}$

We expand $\hat{\partial}\bar{A}$

$$\hat{1}\bar{A}_{\bar{1}} = \sum_{m=2}^{\infty} \sum_{s=2}^{m} \int_{2\cdots m} \hat{s}\; \Xi^{s-1}_{\bar{1}2\cdots m}\, B_{\bar{2}} \cdots \bar{B}_{\bar{s}} \cdots B_{\bar{m}},$$

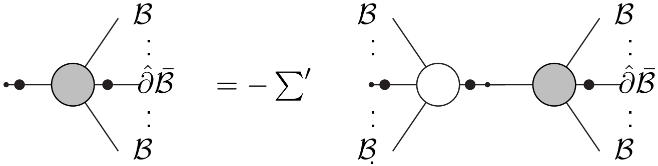
From

$$\int d^3\vec{x}\,\check{\partial}B^a(x^0,\vec{x})\hat{\partial}\bar{B}^a(x^0,\vec{x}) = \int d^3\vec{x}\,\check{\partial}A^a(x^0,\vec{x})\hat{\partial}\bar{A}^a(x^0,\vec{x})$$

We obtain a recursion relation for Ξ

$$\Xi^l(1\cdots n) = -\sum_{r=2}^{n+1-l} \sum_{m=\max(r,3)}^{r+l-1} \Upsilon(-, n-r+3, \cdots, m-r+1) \times \Xi^{l+r-m}(-, m-r+2, \cdots, n-r+2),$$

Solving the equation for $\bar{A}$



At least two left legs on the white blob. Solution:

$$\Xi^{s-1}(1 \cdots n) \quad = \quad -\frac{\hat{s}}{\hat{1}} \Upsilon(1 \cdots n), \qquad (s = 2, \cdots, n \text{ and } n \geq 2).$$

CSW rules from MHV Lagrangian

- We have obtained the canonical transformation for A and $\hat{\partial}\bar{A}$
- Substituting these into the LCYM and collecting the similar helicity terms, we would obtain MHV lagrangian:

  $$L^{-+}[B,\bar{B}]+L^{--+}[B,\bar{B}]+L^{--++}[B,\bar{B}]+L^{--+++}[B,\bar{B}]\cdots$$

  the vertices for B fields are the off-shell continuation of the MHV amplitudes. This is proved in paper arXiv:0908.0020, Fu.
- The propagators of B are only the scalar propagators.

  $$L_{YM}^{-+}=\mathrm{tr}(\check{\partial}B\hat{\partial}\bar{B}-\partial B\bar{\partial}\bar{B})$$

  $$\langle B(p)\bar{B}(-p)\rangle=\frac{i}{p^2}$$

  This is CSW rule.

Summary and Discussion

- We gave some introduction to the Standard Model and physics on LHC. We need to calculate multi-leg amplitude from the theory side to match with the experiment.
- We review some techniques used in tree-level calculation of QCD amplitudes: Color ordering, Spinor method, CSW rules, BCFW recursion relation.
- We briefly sketched the proof of the CSW rule from MHV Lagrangian in Yang-Mills theory. This can also be extended to full QCD.
- Tree-level amplitude is the first step. In fact, now the bottleneck is in the one-loop calculation.
- How to extend the CSW rules to highly simplify the one-loop nonsupersymmetric amplitude calculation is still not available.

THANK YOU!