1°: $\mathcal{N}=1^*$ 超对称规范理论

$$\mathcal{N}=4 \Longrightarrow \mathcal{N}=1^*$$

$\mathcal{N}=4$ 矢量多重态 含有一个 $\mathcal{N}=1$ 的矢量多重态、3个 $\mathcal{N}=1$ 的手征多重态.

超对称破缺:

增加质量、使得超对称部分破缺

手征多重态 Φ_i $i=1,2,3$.

质量 m_i $i=1,2,3$.

相互作用超势:

$$W=\frac{2\sqrt{2}}{g_{YM}^2}\,\mathrm{tr}\left([\Phi_1,\Phi_2]\Phi_3\right)$$

$$\Delta W=\frac{1}{g_{YM}^2}\left(m_1\,\mathrm{tr}\,\Phi_1^2+m_2\,\mathrm{tr}\,\Phi_2^2+m_3\,\mathrm{tr}\,\Phi_3^2\right)$$

a°. $m_1=m_2$, $m_3=0$ $\mathcal{N}=2$

b° $m_1=m_2$ $m_3\neq 0$ $\mathcal{N}=1$

为了区分于通常的 $\mathcal{N}=1$ 的超对称规范理论、这种超对称规范理论称为 $\mathcal{N}=1^*$. or $\mathcal{N}=2^*$

2°: $AdS_5 \times S^5$ 上的微扰.

IIB 理论 (Einstein 标架)

$$S = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}\,R - \frac{1}{4\kappa^2}\int\Big(d\phi\wedge * d\phi + e^{2\phi}\,dC\wedge * dC$$

$$+\, g e^{-\phi} H_3\wedge * H_3 + g e^{\phi}\tilde{F}_3\wedge *\tilde{F}_3 + \frac{g^2}{2}\tilde{F}_5\wedge *\tilde{F}_5$$

$$+\, g^2 C_4\wedge H_3\wedge F_3\Big)$$

解的构象中: $*\tilde{F}_5 = \tilde{F}_5$

$$\tilde{F}_3 = F_3 - C H_3, \quad F_3 = dC_2$$

$$\tilde{F}_5 = F_5 - C_2\wedge H_3 \qquad F_5 = dC_4$$

$$(G_{\mu\nu})_{\text{Einstein}} = g^{1/2} e^{-\phi/2}\,(G_{\mu\nu})_{\text{string}}$$

$$2\kappa^2 = (2\pi)^7\alpha'^4 g^2$$

场方程: (EOM)

$$\nabla^2\phi = e^{2\phi}\partial_M C\,\partial^M C - \frac{g e^{-\phi}}{12} H_{MNP}H^{MNP} + \frac{g e^{\phi}}{12}\tilde{F}_{MNP}\tilde{F}^{MNP}$$

$$\nabla^M\left(e^{2\phi}\partial_M C\right) = -\frac{g e^{\phi}}{6} H_{MNP}H^{MNP}$$

$$d*\left(e^{\phi}\tilde{F}_3\right) = g F_5\wedge H_3$$

$$d*\left(e^{-\phi}H_3 - C e^{\phi}\tilde{F}_3\right) = -g F_5\wedge F_3$$

$$d*\tilde{F}_5 = -F_3\wedge H_3$$

$$\Big(\cancel{R_{MN} = \frac{1}{2}\partial_M\phi\,\partial_N\phi + \frac{e^{2\phi}}{2}\partial_M C\,\partial_N C}$$

$$\cancel{+\frac{g^2}{96}\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQRS}}$$

$$R_{MN} = \frac{1}{2}\partial_M \phi \partial_N \phi + \frac{e^{2\phi}}{2}\partial_M C \partial_N C + \frac{g^2}{96}\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQRS}$$

$$+ \frac{g}{4}\left(e^{-\phi} H_{MPQ} H_N{}^{PQ} + e^{\phi}\tilde{F}_{MPQ}\tilde{F}_N{}^{PQ} \right.$$

$$\left. - \frac{g}{48}\left(e^{-\phi} H_{PQR} H^{PQR} + e^{\phi}\tilde{F}_{PQR}\tilde{F}_B{}^{PQR} \right) G_{MN}$$

Bianchi 等式：

$$d\tilde{F}_3 = - dC \wedge H_3$$

$$d\tilde{F}_5 = - dC_2 \wedge H_3 = - F_3 \wedge H_3$$

4° 经典解：

$$ds^2 = ds^2_{string} = Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}dx^m dx^m$$

$$\tilde{F}_5 = d\chi_4 + * d\chi_4 \qquad \chi_4 = \frac{1}{gZ}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$e^{\phi} = g \qquad C = \theta/2\pi.$$

$\mu, \nu = 0, 1, 2, 3,$

$m, n = 4, 5, 6 \ldots 9.$

Z 是调和函数 (x^m) $\qquad \partial^m \partial^m Z = 0.$

对于 $AdS_5 \times S^5$

$$Z = \frac{R^4}{r^4} \qquad R^4 = 4\pi g N \alpha'^2$$

其余场为零.

B° 线性化的微扰解：(B

引入： $G_3 = F_3 - \hat{\tau} H_3$

$\tau = C + i e^{-\phi}$.

代入经典解就给出 $\hat{\tau} = \theta/2\pi + i/g$

约定：用"^"代表经典背景的各种量的定义 则 G_3 满足.

$$\begin{cases} d\hat{*} G_3 + i g\, G_3 \wedge \hat{\tilde{F}}_5 = 0 \\ d G_3 = 0 \end{cases}$$

对横向 有，$(m.n.p)$ $(4..9)$

$$\hat{*} G_3 = Z^{-1} *_6 G_3 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

平坦度规 δ_{mn} 形成的"*"变换.

可以给出：

$$d\left[Z^{-1}\left(*_6 G_3 - i G_3\right)\right] = 0 \qquad (*)$$

对偶的场： $-*\tilde{F}_3 = dC_6 - H_3 \wedge C_4$

C° 附录：张量球谐函数与费米质量

$\mathcal{N}=4$ 理论含有 Weyl 费米子 λ_α，它们形成 SO(6) R-symmetry 的 4 维表示

质量项 $\quad m^{\alpha\beta}\lambda_\alpha\lambda_\beta + h.c.$

对角情形：$\quad m^{\alpha\beta} = m_\alpha \delta^{\alpha\beta}$

当一个质量为零时，哈密顿量具有 $\mathcal{N}=1$ 的超对称，我们可令 $m_4=0$. $\quad \lambda_4 \longrightarrow$ 胶微子 gluino

费米子的双线性变换是按 SO(6) 的对称表示变换.

$$(4\otimes 4)_{sym} = 10.$$

因此质量矩阵应该是 $\overline{10}$，它们是虚的自对偶反对称张量

$$*_6 T_{mnp} \equiv \frac{1}{3!}\epsilon_{mnp}{}^{qrs}T_{qrs} = \pm i T_{mnp}$$

$$+ \longrightarrow 10$$

$$- \longrightarrow \overline{10}$$

引入复坐标

$$z^1 = x^4 + ix^7/\sqrt{2}, \qquad z^2 = x^5 + ix^8/\sqrt{2} \qquad z^3 = x^6 + ix^9/\sqrt{2}$$

在坐标转动：$z^i \Rightarrow e^{i\phi_i} z^i$，旋量：

$$\lambda_1 \longrightarrow \exp\{i(\phi_1-\phi_2-\phi_3)/2\}\lambda_1$$

$$\lambda_2 \longrightarrow \exp\{i(-\phi_1+\phi_2-\phi_3)/2\}\lambda_2$$

$$\lambda_3 \longrightarrow \exp\{i(-\phi_1-\phi_2+\phi_3)/2\}\lambda_3$$

$$\lambda_4 \longrightarrow \exp\{i(\phi_1+\phi_2+\phi_3)/2\}\lambda_4$$

要求: T_3 的变化规律与之推论: 讨论出 T_3

$$T_3 = m_1\, dz^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + m_2\, d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}^3 + m_3\, d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^3$$
$$+ m_4\, dz^1 \wedge dz^2 \wedge dz^3$$

对 $N=1$: $m_4 = 0$ 故

$$T_{1\bar{2}\bar{3}} = m_1, \quad T_{\bar{1}2\bar{3}} = m_2 \quad T_{\bar{1}\bar{2}3} = m_3.$$

Let $m_1 = m_2 = m_3 = m$

$$\begin{cases} T_{i\bar{j}\bar{k}} = T_{\bar{i}j\bar{k}} = T_{\bar{i}\bar{j}k} = m\,\epsilon_{ijk} \\ *_6 T = -iT \end{cases}$$

利用这些 T 形的构造出

$$\begin{cases} T_3 = \frac{1}{3!} T_{mnp}\, dx^m \wedge dx^n \wedge dx^p \\ V_3 = \frac{1}{3!} V_{mnp}\, dx^m \wedge dx^n \wedge dx^p, \qquad V_{mnp} = \frac{x^q}{r^2}\left(x_m T_{qnp} + x_n T_{mqp} + x_p T_{mnq}\right) \\ S_2 = \frac{1}{2} T_{nmp} X^n dx^m \wedge dx^p \end{cases}$$

~~$S_2 = \frac{1}{2}$~~

确定:

$$dS_2 = 3T_3, \qquad d(\ln r) \wedge S_2 = V_3 \qquad d(r^p S_2) = r^p(3T_3 + pV_3)$$

$$dT_3 = 0 \qquad dV_3 = -3\, d(\ln r) \wedge T_3$$

$$*_6 T_3 = \pm i T_3, \qquad *_6 V_3 = \pm i (T_3 - V_3)$$

D: 线性解:

Ansatz: $G_3 = r^p(\alpha T_3 + \beta V_3)$

由 Bianchi 等式

$$dG_3 = 0 \quad \Longrightarrow \quad \beta = p\alpha/3$$

$$*_6 G_3 - iG_3 = -i r^p \alpha/3 \left[(3 \mp p \mp 3) T_3 + (p \pm p) V_3 \right]$$

$$d\left[(*_6 G_3 - i G_3) Z^{-1} \right] = 0$$

$$\Longrightarrow$$

$$\begin{cases} p^2 + 10p = 0 & 10 \\ p^2 + 10p + 24 = 0 & \overline{10} \end{cases}$$

$(\zeta = -3\sqrt{2})$

$\overline{10}$: $p = -4$, $G_3 = \alpha r^{-4}(T_3 - 4V_3/3) \rightarrow d\left(\frac{\zeta}{3g}\left(\frac{R}{r}\right)^4 S_2\right)$

$p = -6$ $\qquad G_3 = \alpha r^{-6}(T_3 - 2V_3)$

10: $p = 0$ $\qquad G_3 = \alpha T_3$

$p = -10$ $\qquad G_3 = \alpha r^{-10}(T_3 - 10V_3/3)$

$3°$: D_5 Probe: $\quad (AdS_5\times S^5$. $\quad$ 于: $R^4\times S^2$. $\quad$ D_3-膜荷 $n \ll N$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \gg \sqrt{gN}$

$$S = -\frac{\mu_5}{g}\int d^6\xi\left[-\det(G_{\parallel})\det\left(g^{-1/2}e^{\Phi/2}G_\perp + 2\pi\alpha'\mathcal{F}\right)\right]^{1/2}$$

$$+\mu_5\int\left(C_6 + 2\pi\alpha'\mathcal{F}_2\wedge C_4\right)$$

$$2\pi\alpha'\mathcal{F}_2 = 2\pi\alpha' F_2 - B_2$$

$G_{\parallel}$: R^4. $\qquad\qquad \det G_{\parallel} = Z^{-2}$

$G_\perp$: S^2 $\qquad\qquad \det\mathcal{F} = \frac{1}{2}\mathcal{F}_{ab}\mathcal{F}^{ab}\det G_\perp$

D_3-膜荷 $\qquad \int_{S^2} F_2 = 2\pi n$

若 $G_\perp$ 是球对称: $\qquad F_{\theta\varphi} = \frac{1}{2}n\sin\theta$, $F_{ab}F^{ab} = n^2/2Zr^4$.

$$4\pi^2\alpha'^2 F_{ab}F^{ab} = 2\pi^2\alpha'^2 n^2/2R^4 \sim n^2/gN \gg 1$$

大场强时 $g^{-1/2}e^{\Phi/2} = 1$

$$\left[\det(G_\perp + 2\pi\alpha' F)\right]^{1/2} = 2\pi\alpha'\sqrt{\det F} + \det G_\perp\Big/4\pi\alpha'\sqrt{\det F}$$

若 D_5 膜是球对称的:则非叠加的领头项为 $(对\perp)$

$$\frac{\mu_5}{g}\int_{S^2} d^2\xi\,\frac{\sqrt{\det G_{\parallel}}\,\det G_\perp}{4\pi\alpha'\sqrt{\det F}} = \frac{\mu_5}{g}\int_{S^2} d\cos\theta\, d\varphi\,\frac{r^4}{2\pi\alpha' n} = \frac{\mu_5 r^4}{g n\alpha'}$$

由 $G_3 = d\left[\frac{5}{3g}\left(\frac{R}{r}\right)^4 S_2\right]$

$$\Longrightarrow\quad C_2 - \hat{\tau} B_2 = \frac{5}{3g}\left(\frac{R}{r}\right)^4 S_2$$

D5 膜中的 B_2 效应是次级项可略去.

$$B_{ab}/2\pi\alpha' F_{ab} \sim \frac{mR^4}{r^3} \Big/ \frac{\alpha' n}{r^2} \sim \frac{m g N \alpha'}{n r} \ll 1$$

$$r \sim m n \alpha'$$

因此在 Ads 粘合 $r = m n\alpha'$ 或更大的处,场强 F_{ab} 在 $\mathcal{F}_{ab}$ 中起支配作用的地位。

C_6 的贡献:

$$-\frac{\Delta S}{V} = -\frac{\mu_5}{g}\int_{S^2} \frac{2\zeta}{g} \mathrm{I_m}(S_2) \quad \propto \quad r^3, \quad m.$$

对标量规范场:

$$\frac{\mu_5 r^4}{g n \alpha'} \longrightarrow \phi^4$$

$$\frac{\Delta S}{V} \longrightarrow m\phi^3$$

$$? \longrightarrow m^2\phi^2$$

↓

二阶微扰 (dilaton, metric, C_4)

但超对称可以进一步计算,最后:

$$-\frac{S}{V} = \frac{\mu_5}{g}\left\{ \int_{S^2} d^2\xi \frac{\sqrt{\det G_{\parallel}}\ \det G_{\perp}}{4\pi\alpha'\sqrt{\det F}} - \frac{\zeta}{g}\int_{S^2} \mathrm{I_m}\left(T_{mnp}\, x^m dx^n \wedge dx^p\right) + \frac{\pi\alpha'\zeta^2}{18} T_{ij\bar{k}}\, \bar{T}_{\bar{i}jk} \int_{S^2} F_2\, z^i \bar{z}^{\bar{i}} \right\}$$

(c, d) probe.

$$\tau' = \frac{a\tau+b}{c\tau+d} \qquad \text{S-dual}$$

$$g' = g|M|^2, \quad G'_{MN} = G_{MN}|M| \quad C_4' = C_4$$

$$G_3' = G_3 M^{-1} \quad B_6' - C_6'\hat{\tau}' = (B_6 - \hat{\tau} C_6) M^{-1}$$

$(M = c\tau + d)$

在新的耦合中:

$$-S = \mu_5 \int d^4x \left\{ \int_{S^2} d^2\xi \frac{\sqrt{\det G'_{\parallel}} \det G'_{\perp}}{g' 2\pi\alpha' \sqrt{\det F}} - \int_{S^2} C_6' + O(T^2) \right\}$$

$$\longrightarrow -\frac{S}{V} = \frac{\mu_5}{g} \left\{ |M|^2 \int_{S^2} d^2\xi \frac{\sqrt{\det G_{\parallel}} \det G_{\perp}}{4\pi\alpha' \sqrt{\det F}} - \xi/g \int_{S^2} \mathrm{Im}(\bar{M} T_{mnp} x^m dx^n \wedge dx^p) + \frac{\pi\alpha'\xi^2}{18} T_{i\bar{j}\bar{k}} \bar{T}_{\bar{i}jk} \int_{S^2} F_2 z^i \bar{z}^{\bar{i}} \right\}$$

耦合到 $C_6' = -g' \mathrm{Im}(B_6' - \hat{\tau} C_6') = -g\, \mathrm{Im}(\bar{M}[B_6 - \hat{\tau} C_6])$

$$= B_6 c + C_6 d$$

(c, d) 5膜耦合: c NS5膜, d与D_5膜

$$\mathcal{L} = R * 1 - \frac{1}{2} F_4 \wedge * F_4 + \frac{1}{6} F_4 \wedge F_4 \wedge C_3$$

$$F_4 = dC_3 \quad , \quad F_7 \equiv dC_6 = * F_4 + \frac{1}{2} C_3 \wedge F_4$$

经典解: $ds^2 = Z^{-1/3} \eta_{\mu\nu} dx^\mu dx^\nu + Z^{2/3} dx^i dx^i$

$$F_7 = \partial_i Z^{-1} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^4 \wedge dx^{11} \cdot \wedge dx^i$$

$$Z = R^3/r^3 \qquad r^2 = x^i x^i \qquad R^3 = \pi N l_p^3$$

$$F_4 = \frac{1}{4!} \epsilon_{ijklm} \partial_m Z \, dx^i \wedge dx^j \wedge dx^k \wedge dx^l$$

线性扰动运动方程:

$$h \equiv h_{11,i} = \delta g_{i11}$$

$$\begin{cases} d\left[Z^{-1} \left(*_5 d(h Z^{1/3}) + G_3 \right) \right] = 0 \\ d\left[Z^{-1} \left(d(h Z^{1/3}) + *_5 G_3 \right) \right] = 0 \end{cases}$$

SO(5):

$$T_2 = - \mathrm{Re} \left[m_1 dz_1 \wedge d\bar{z}_2 + m_2 dz_1 \wedge dz_2 \right]$$

$$T_3 = *_5 T_2 = \mathrm{Re} \left[m_1 dz_1 \wedge d\bar{z}_2 + m_2 dz_1 \wedge dz_2 \right] \wedge dx^7$$

$$V_2 = \frac{1}{2} \left[x^8 x^i / r^2 \, T_{8j} + 1 \text{ more} \right] dx^i \wedge dx^j$$

$$V_3 = \frac{1}{3!} \left[\frac{x^8 x^m}{r^2} T_{8np} + 2 \text{ more} \right] dx^m \wedge dx^n \wedge dx^p$$

$$F_2 = \left(\frac{r}{R}\right)^p (a V_2 + b T_2) \qquad pb = 2a$$

$$G_3 = \left(\frac{r}{R}\right)^p (A V_3 + B T_3) \qquad pB = 3A$$

$(p = 0, \ b/B = -1)$, $(p = -8, \ b/B = -1)$

$(p = -5, \ b/B = 2/3)$, $(p = -3, \ b/B = 2/3)$

5: $(4+1)$-维 $N=1^*$

背景:

$$g_s^{-4/3} e^{4\phi/3} = Z^{-1/3} + B^2 Z^{2/3}(\rho_1^2+\rho_2^2) \equiv \Lambda$$

$$ds_{10}^2 = \Lambda^{1/2}\left(Z^{-1/3} dx_{\parallel}^2 + Z^{2/3} dx_{\perp}^2\right) - B^2\Lambda^{-1/2} Z^{4/3}\left(\rho_1^2 d\varphi_1 + \rho_2^2 d\varphi_2\right)^2$$

$$g_s C_{\varphi_1} = \Lambda^{-1}\rho_1^2 B Z^{2/3}, \quad g_s C_{\varphi_2} = \Lambda^{-1}\rho_2^2 B Z^{2/3}$$

$$g_s F_4 = *_5 dZ^{-1} \qquad *_5 H_3 = B(\rho_1^2 d\varphi_1 + \rho_2^2 d\varphi_2)\wedge dZ$$

$\cancel{Z = R_{M5}^3/r^3}$ $\quad r^2 = x^i x^i$. $\quad i = 5.6..9,$

0, 1, 2, 3, 4, 10

两种方法获得: 1°: 直接计算

2°: 11维扭动. M_5 brane. 约化: (紧致化)

$$\begin{cases} x^{11} \sim x^{11} + 2\pi R^{11} n_1 \\ \varphi_1 \sim \varphi_1 + 2\pi n_2 + 2\pi n_1 R^{11}\beta_1 \\ \varphi_2 \sim \varphi_2 + 2\pi n_3 + 2\pi n_1 R^{11}\beta_2 \end{cases} \quad \beta_1 = \beta_2 = B$$

$$Z = \int \frac{\rho(y)\,dy}{\left[\rho_1^2+\rho_2^2+(x_7-y)^2\right]^{3/2}}$$

6. 矩阵模型和D膜，A型拓扑弦

$$W[\Phi_i] = \mathrm{Tr}\left[i\,\Phi_1 D\Phi_2 + m\,\Phi_1\Phi_2 + \mu \cosh(\beta\Phi_3)\right]$$

$$Z = \int \prod_{i=1}^{3} d\Phi_i \; \exp\left\{ -(\beta g_s)^{-1} \int dx\, W(\Phi_i) \right\}$$

$$D\Phi_2 = \partial_x \Phi_2 + [\Phi_3, \Phi_2]$$

积分 Φ_1 和 Φ_2

$$Z = \int dU \; \frac{\exp\{ -g_s^{-1}\mu\, \mathrm{Tr}\cosh(\beta\Phi_3)\}}{\det \sinh\frac{1}{2}(im + \Phi_3 \otimes 1 - 1\otimes \Phi_3)}$$

$$U = \exp\{\beta\Phi_3\}$$

$$\det(\partial_x + \varphi) = \det \sinh(\tfrac{1}{2}\beta\varphi)/(\tfrac{1}{2}\beta)$$

SU(N) 对角规范：$\quad U \simeq (e^{\varphi_1} \dots e^{\varphi_N})$

$$Z = \int \prod_i d\varphi_i \prod_{i\neq j} \frac{\sinh[(\varphi_i - \varphi_j)/2]}{\mathrm{sh}\left[\frac{\varphi_i - \varphi_j + i\beta m}{2}\right]} \exp\left\{ -g_s^{-1}\mu \sum_i \cosh\varphi_i \right\}$$

定义 $S = g_s N$，$\quad \omega(z) = \frac{1}{2}\int_{-a}^{a} d\varphi \, \frac{\rho(z)}{\tanh\frac{z-\varphi}{2}} \qquad \int_{-a}^{a} \rho(z)\, dz = 1$

$$\omega(\phi + i\varepsilon) - \omega(\varphi - i\varepsilon) = -2\pi i\, \rho(\varphi), \quad \varphi \in [-a, a]$$

Z 可以写成：$\quad Z = \int \prod_i d\varphi_i \; \exp\{ W(\varphi_i)\}$

鞍点 $\quad \dfrac{\partial W(\varphi_i)}{\partial \varphi_j} = 0 \quad \Longrightarrow$

$$\frac{\mu \sinh\phi}{S'} = \omega(\phi+i\varepsilon) + \omega(\varphi - i\varepsilon) - \omega(\varphi + i\beta m) - \omega(\varphi - i\beta m)$$

def: $$G(\zeta) = \frac{\mu}{2\,\mathrm{sh}\,\frac{\beta m}{2}} \cosh\zeta + iS\left[\omega(\zeta + i\beta m/2) - \omega(\zeta - i\beta m/2)\right]$$

$G(\zeta)$: 解析 : $-\pi \le \zeta \le \pi$, ，两条割线: $\begin{cases} -a + i\beta m/2, & a + i\beta m/2 \\ -a - i\beta m/2 & a - i\beta m/2 \end{cases}$

粘连条件可以写成:

$$G(\phi + i\beta m/2 \pm i\varepsilon) = G(\phi - i\beta m/2 \mp i\varepsilon)$$

渐近行为: $$\lim_{\mathrm{Re}\,\zeta \to \pm\infty} G(\zeta) \longrightarrow \frac{\mu}{4\sin\frac{\beta m}{2}} e^{\pm\zeta} + O(e^{\mp\zeta})$$

坐标变换: $\zeta(u)$: $\zeta \longrightarrow u$

A: $\zeta(u + 2\pi i) = \zeta(u)$ $\quad G(\zeta(u+2\pi i)) = G(\zeta(u))$

B: $\zeta(u + 2\pi i\tilde{\tau}) = \zeta(u) + i\beta m$ $\quad G(\zeta(u + 2\pi i\tilde{\tau})) = G(\zeta(u))$

$$\exp\{\zeta(u)\} = \theta_1\left(\tfrac{u}{2i} - \beta m/4 \,\middle|\, \tilde{\tau}\right) \Big/ \theta_1\left(\tfrac{u}{2i} + \beta m/4 \,\middle|\, \tilde{\tau}\right)$$

$$G(\zeta(u)) = \frac{i}{2\sin\beta m/2} \frac{\theta_1(\frac{\beta m}{2}|\tilde{\tau})}{\theta_1'(0|\tilde{\tau})} \Big[\zeta(u - i\beta/2) - \zeta(u + i\beta/2) + 2\zeta(i\beta) - \frac{\beta}{\pi}\zeta(\pi i) \Big]$$

根据 Dijkgraaf – Vafa:

$$2\pi i S = -i \int_A G(\zeta(u)) \frac{d\zeta(u)}{du} du = \frac{dh(\tilde{\tau})}{d\tilde{\tau}}$$

$$\frac{\partial F_0}{\partial S} = -i \int_B G(\zeta(u)) \frac{d\zeta(u)}{du} du = \tilde{\tau}\frac{dh(\tilde{\tau})}{d\tilde{\tau}} - h(\tilde{\tau})$$

with: $h(\tilde{\tau}) = \dfrac{\mu}{\sin \frac{\beta m}{2}} \dfrac{\theta_1(\frac{\beta m}{2}|\tilde{\tau})}{\theta_1'(0|\tilde{\tau})}$

有效超势: $W_{eff} = N \dfrac{\partial \mathcal{F}_0}{\partial S} - 2\pi i \tau S$

$$\frac{\partial W_{eff}}{\partial S} = 0 \implies \tilde{\tau} = \frac{\tau}{N}$$

$$W_{eff} = -N h(\tau/N) = -\frac{N\mu}{\sin\frac{\beta m}{2}} \frac{\theta_1(\frac{\beta m}{2}|\tau/N)}{\theta_1'(0|\tau/N)}$$

$$\xrightarrow[\beta \to 0]{} -N\mu - \frac{N\mu m^2}{24} E_2\left(\frac{\tau}{N}\right)\beta^2 + \cdots$$

$$\tau = \theta/2\pi + i/g = \frac{\theta}{2\pi} + \frac{i\,2\pi}{g_4^2}$$

因为θ破坏5-dim的Lorentz不变性，可令 $\theta = 0$

$$\tau = \frac{i2\pi}{g_4^2} = \frac{i2\beta\pi}{g_5^2}$$

质量m的虚部对应于5维理论参考系的质量，故可令 $m = im_0$

令: $z = iz_2 = im_0 \beta/2$, $\tilde{\tau} = i\tilde{\tau}_2 = i\, 2\pi\beta / g_5^2 N$

$$W_{eff} = -\frac{N\mu}{\sin z} \frac{\theta_1(z|\tilde{\tau})}{\theta_1'(0|\tilde{\tau})}$$

$z_2 \longrightarrow \infty$

$\tilde{\tau}_2 \longrightarrow \infty$

$$A = \frac{z_2}{\tilde{\tau}_2 \pi} = \text{fixed}$$

$$W_{eff} = -\frac{N\mu}{\tilde{\tau}_2 \sinh \zeta_2} e^{\frac{1}{2}\beta m_0 A} \frac{\theta_1(-A \mid -1/\tilde{\tau})}{\theta_1'(0 \mid \tilde{\tau})}$$

$$= -\frac{N\mu}{g_s^2 N \,\mathrm{sh}\, \frac{\beta m_0}{2}} e^{\beta m_0 A/2} \sin(\pi A) \prod_{n=1}^{\infty} \left(1 + \frac{\sin^2(\pi A)}{\mathrm{sh}^2\left(\frac{n g_s^2 N}{2\beta}\right)}\right)$$

若 $A \in \mathbb{Z}$, $W_{eff} = 0$

7: 可积系统:

4-dim:

$W_{eff} \Longleftrightarrow H_{elliptic}$.

$$H_{elliptic} = \mu\left[\sum_a \left(\frac{1}{2} p_a^2\right) - m^2 \sum_{a \neq b} \wp(x_a - x_b)\right]$$

Elliptic Calogero-Moser model. 可积

5-dim:

Elliptic-Ruijsenaars-Schneider 模型: 也是可积.

$$H_{RS} = C\mu \sum_a \cosh(\beta p_a) \prod_{b \neq a} \sqrt{\wp(i\beta m) - \wp(x_a - x_b)}$$

能量极小: $p_a = 0$, $x_a = 2\pi i a \tau / N$ $a = 1 \ldots N$

$$H_{RS} = NC\mu \prod_{a=1}^{N-1} \sqrt[N]{\wp(i\beta m/2) - \wp(i2\pi a\tau/N)}$$

利用等式：

$$\frac{\theta_1(\beta m/2 \mid \tau/N)}{\theta_1'(0 \mid \tau/N)} = (2i)^{N-1} \left(\frac{\theta_1(\beta m/2 \mid \tau)}{\theta_1'(0 \mid \tau)} \right)^N \prod_{a=1}^{N-1} \sqrt{\wp(i\beta m \mid \tau) - \wp\left(\frac{i2\pi a\tau}{N} \Big| \tau\right)}$$

$$= NC\mu \cdot \frac{1}{(2i)^{N-1}} \left(\frac{\theta_1'(0 \mid \tau)}{\theta_1(\beta m/2 \mid \tau)} \right)^N \cdot \left(\frac{\theta_1(\beta m/2 \mid \tau/N)}{\theta_1'(0 \mid \tau/N)} \right)$$

? $N=4$ 可能. 如何证明 .. pp wave