Blackhole. Entropy, Shock waves.

& Stability in Nonlinear Gravity.

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" Colliding Plane wave ~~wave~~ solution in.

11D SG "

NLG: universal feature of BH physics in NLG.

- BH entropy
- shock wave solution
- positive energy

- $\mathcal{L}_{NLG} = \frac{1}{16\pi G_N}\sqrt{-g}\,F(R,\ R_{\mu\nu}R^{\mu\nu},\ R_{\kappa\lambda\mu\nu}R^{\kappa\lambda\mu\nu}, \ldots)$

$I = \int \mathcal{L}$

QG for String theory.

Einstein-Hilbert Action: $F = R$

If QG effect. $\mathcal{L} \to \mathcal{L}_{NLG}$

No complete knowledge on the form of $\mathcal{L}_{NLG}$.

On the other hand, phenomenologically.

- $F(R) = R + \lambda R^2$ Starobinsky inflation.
  - R large. early universe. λR^2 important.
- $F(R) = R - \frac{\mu^4}{R} + \ldots$ Late-time inflation. (S. Carroll
  - R small. $1/R$ important.

$\Rightarrow$ study $\mathcal{L}_{NLG}$.

Mainly BH. physics in $\mathcal{L}_{NLG}$. $F = F(R)$

- Area law? $S = \frac{A}{4G}$ modified?
  - in other words $\rightarrow$ Holographic property.
- S-Matrix Ansatz:
  - $\Rightarrow$ renormalization of Newton constant

- positive energy ~~condition~~ theorem??

$$\mathcal{L} = \frac{1}{16\pi G_N} \sqrt{-g}\, F(R)$$

E.O.M:

$$F'(R) R_{\mu\nu} - \frac{1}{2} F(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F'(R) + g_{\mu\nu} \nabla^\alpha \nabla_\alpha F'(R) = -8\pi G_N T^{(m)}_{\mu\nu}$$

$$F'(R) = \frac{\partial F(R)}{\partial R}.$$

· vacuum solutions of constant curvature. $\nabla F'(R) = 0$ $T^{(m)}_{\mu\nu} = 0$

$\therefore$

$$F'(R) R - \frac{D}{2} F(R) = 0$$

D: space-time dimen..

$\Rightarrow R_0$ $\left\{ \begin{array}{ll} \text{Schwarzschild. BH.} & \checkmark \\ \text{Sch -- AdS. BH.} & \checkmark \\ \text{charged BH.} & \checkmark \ (g_{\mu\nu}.\ A_\mu). \end{array} \right.$

Sch... dS. $\times$

Entropy ? Euclidean. QG. time. $\longrightarrow$

$$I = \frac{1}{16\pi G} \int_M \sqrt{-g}\, R + \frac{1}{8\pi G} \oint_{\partial M} K$$

$\delta g_{\mu\nu} = 0.$ but $\partial_n (\delta g_{\mu\nu}) \neq 0$

Gibbons-Hawking term.

trace of 2nd fundamental form

$k = \nabla_\mu n^\mu$

∂M.

· $I_p = I(BH) - I(\text{flat}) = F$

· Schw.. $R = 0.$ $I_{bulk} = 0$

$$I_b = \frac{1}{8\pi G} \oint_{\partial M} (K - K_0)$$

· AdS - Schwarz.: $I_b = 0$

I_{bulk}: dominate.

$$S = \frac{A}{4G_N}$$

4-dim

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$r = 2M$. singular. but it's coordinate singularity

$r = 0$ curvature singularity. unavoidable.

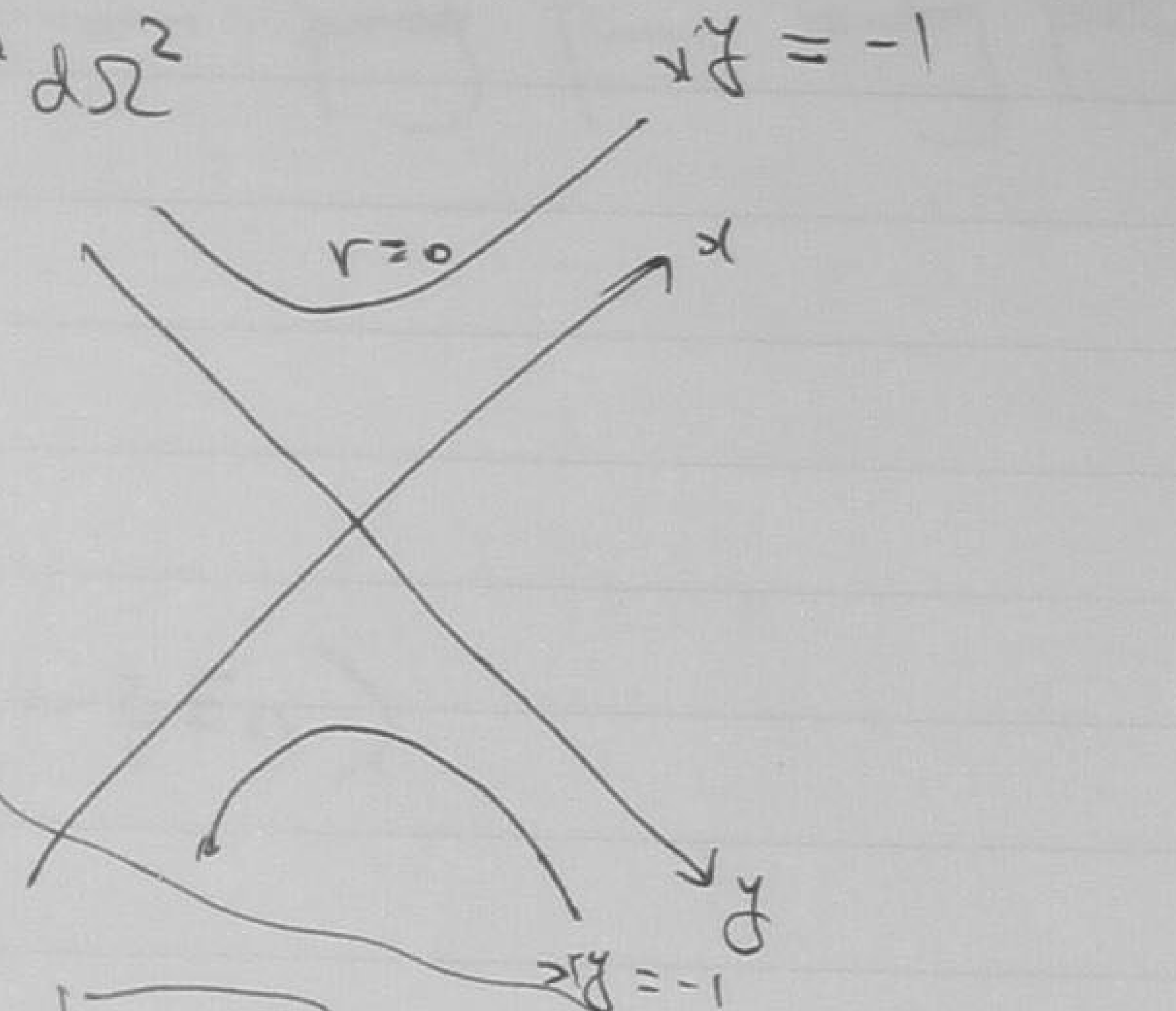
"censorship"

→ coordinate transformation.

· Kruskal coordinate.

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}\, dx\, dy + r^2 d\Omega^2$$

$$\begin{cases} xy = \left(\frac{r}{2M} - 1\right) e^{r/2M} \\ x/y = e^{t/2M} \end{cases}$$



$$ds^2 = \left(\frac{x}{4M}\right)^2 d\tau^2 + \left(\frac{r^2}{4M^2}\right)^2 dx^2 + r^2 d\Omega^2$$

$$x = 4M\,(1 - 4Mr^{-1})^{1/2}.$$

$r = 2M$. $x = 0$

$x \geq 0$ $r \geq 2M$

Euclideanize: $\tau = it$

$\Rightarrow$ τ periodic with period $\beta = \boxed{8\pi M}$

($r \sim 2M$. metric $\Rightarrow$ Rindler spacetime.

τ. looks like. angular. coordinate about. its axis $r = 2M$.

$\Rightarrow$ $8\pi M$ $(2\pi K^{-1})$)

In Euclideanized space, nonsingular.

$I|_Y$. ∂Y : $S^1 \times S^2$ compact.

Y: bounded by a surface. $r = r_0$ $(r_0 \to \infty)$

$$\begin{cases} \langle E \rangle = \frac{\partial I}{\partial \beta_0} \\ S = \beta_0 E - I. \end{cases}$$

β_0: T^{-1} $T = \beta^{-1}$

Canonical ensemble.

$Z = e^{-I}$ the dominant contribution come from metric near g_0 (stationary phase)

$$Z = \Sigma e^{-\beta E_n}.$$

$$\langle E \rangle = \frac{\Sigma E_n e^{-\beta E_n}}{\Sigma e^{-\beta E_n}}$$

$$= -\frac{\partial}{\partial\beta}\log Z. \quad = M$$

$$T = 1/\beta$$

$$\log Z = -\beta^2/16\pi. = -I_0$$ ← come purely from boundary part.

$$S = -\Sigma p_n \log p_n.$$

$$p_n = Z^{-1} e^{-\beta E_n}$$

$$\Rightarrow S = -\Sigma Z^{-1} e^{-\beta E_n}. (-\log Z - \beta E_n)$$

$$= \beta \langle E \rangle + \log Z$$

$$\langle \phi_2 . t_2 | \phi_1 . t_1 \rangle = \int D(\phi) \exp(i I(\phi))$$

$$= \langle \phi_2 | e^{-iH(t_2 - t_1)} | \phi_1 \rangle.$$

$$t_2 - t_1 = -i\beta$$

QFT at finite temperature.

$$= \Sigma e^{-\beta E_n}.$$

$$= \langle e^{-\beta H} \rangle.$$

$$\therefore Z = \langle e^{-\beta H} \rangle = \int D(\phi) e^{i I(\phi)}.$$

AdS-Sch. case: same strategy. $\beta = T^{-1}$

$$\log Z = -I.$$

$$\Rightarrow S = \frac{A.}{4G}$$

· I_{NLG}

$$\delta I_g|_{\partial V} = \frac{1}{8\pi G_N}\int_{\partial V} \underbrace{F'(R)\,\delta K}_{\delta(\ ?\ )}$$

In general very difficult to get I_b s.t δI_b

Fortunately: Maximal symmetric solution.

$$I_g = \frac{1}{8\pi G_N}\int_{\partial V}\sum_{n=1}^{\infty}(-)^n\left(\frac{2+2D}{D}\right)^{n-1}\prod_{k=0}^{n-1}\left(\frac{1}{2k+1}\right)K^{2n-1}F^{(n)}(R)$$

Ref: M.S. Madsen & J. D. Barrow "De Sitter, Ground states and Boundary terms in Generalized Gravity" NPB323 (1989) 242.

· Schwarzschild. $R_0 = 0$ $\quad F(R_0) = 0$

$$\int_{\partial V}\left(K^n - K_0^n\right) \sim \frac{1}{r_0^{n-1}} + \ldots$$

$\therefore$ $n=1$ dominate.

$$F = \frac{1}{8\pi G_N}\int_{\partial V} F'(0)\,(K-K_0) = \frac{F'(0)}{8\pi G_N}(K-K_0)$$

$$\Rightarrow \quad S_{NLG} = F'(0)\,\frac{A}{4G_N}$$

· AdS-Schwarz.

$$\int_{\partial V}\left(K^{2n-1} - K_0^{2n-1}\right) = \int_{\partial V}(K-K_0)(K_0^{2n-2} + \cdots\cdots) = 0$$

K. $K_0 \sim O(1)$. $\qquad \sim r_0^{-2}$

$$\int_{\partial V}(K-K_0) = 0$$

$$I_{NLG} = \frac{F(R_0)}{16\pi G_N}\int d^Dx\sqrt{g}.$$

$$S_{NLG} = F'(R_0)\,\frac{A}{4G_N}$$

$$\boxed{G_N^{eff} = \frac{G_N}{F'(R_0)}} \quad \text{then} \quad S_{NLG} = \frac{A}{4G_N^{eff}}.$$

$$ds^2 = \left(\frac{r^2}{b^2}+1-\frac{\omega_n M}{r^{n-2}}\right)dt^2 + \left(\quad\right)^{-1}dr^2 + r^2 d\Omega^2$$

AdS-Sch.

$(n+1)$-dim.

$$\omega_n = \frac{16\pi G_N}{(n-1)\,\mathrm{Vol}(S^{n-1})}.$$

$$\begin{cases} \Lambda = n b^{-2}(n-1) \\ R = -n b^{-2}(n+1) \end{cases}$$

$$R_0 = -\frac{n+1}{n-1}\Lambda$$

$$= -\cancel{\ }\,2\Lambda$$

$$I = -\frac{1}{16\pi G_N}\int d^{n+1}x\sqrt{g}\,(R+\Lambda) = \frac{n}{8\pi G_N b^2}\int d^{n+1}x\sqrt{g}$$

MG: $I = -\dfrac{f(R)}{16\pi G}\displaystyle\int d^{n+1}(x)\sqrt{g}$

$$f'(R)R - \frac{n+1}{2}f(R) = 0.$$

$$\Rightarrow f(R) = \frac{2}{n+1}f'(R)R = \frac{2f'(R)}{n+1}\cdot(-n)b^{-2}(n+1)$$

$$= \frac{-2n f'(R)}{b^{+2}}$$

$$\Rightarrow I = \frac{n f'(R_0)}{8\pi G_N b^2}\int d^{n+1}x\sqrt{g}$$

$$= f'(R_0)\, I_{AdS}$$

$$\frac{f(R)}{f'(R)} = \frac{2R}{n+1} = \left(f\, 2n^{\cancel{\ }} b^{-2}\right) = -\frac{2\Lambda}{n-1}$$

. R. Wald. "Noether charge method"

Remarks:

- $F(R)$. quantum effective action.

  G_N^{eff} : renormalized Newton constant.

- Linear Newton Limit: (Ref: R. Dick. gr-qc/0307052)

  $|F(R_0) F''(R_0)| \ll 1$

  then

  $$F'(R_0)\left(\delta R_{\mu\nu} - \frac{1}{2} g_{0\mu\nu}\, \delta R\right) = -8\pi G_N T^{m}_{\mu\nu}$$

  ∴ $G_N{}^{eff}$ as physical quantity to recover Newton limit.

- Area $\sim (G_N M)^{D-2}$

  $r_0 \sim G_N M$

  G_N or $G_N{}^{eff}$ to use?

Physically, one can ~~decide~~ make the choice from ADM mass.

In EH: ADM mass at ∞

Formally Hamiltonian to decide!

In NLG. usually hard to construct a Hamiltonian formulat

1) $\Box(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h) = -16\pi G_N^{eff} T_{\mu\nu}$

$h_{00} \simeq \frac{M}{r^{D-3}} G_N^{eff}$.

2) Wald's Noether charge method.

Canonical ~~density~~ energy: $\mathcal{E} \propto \frac{f'(R_0)}{G_N}(\quad) = \frac{(\quad)M}{G_N{}^{eff}} = M$

$\Rightarrow$ $G_N{}^{eff}$!

~~3) $\langle E \rangle = -\frac{\partial}{\partial\beta}\log Z = \ldots M$~~

Wald's Noether charge method:

Stationary BH. event horizon generated by a killing vector field.

The diffeomorphism of the killing vector field ⇒ Noether charge.

S: local geometric density integrated over a space-like section of the horizon.

$$\chi^a \partial_a = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$$

hypersurface Σ: from ∞ to horizon. (bifurcation surface)

on the horizon $\chi^a = 0$.

$$\delta H = \underbrace{\delta \int_\Sigma dV_a J^a}_{\delta M} - \underbrace{\int_\Sigma dV_a \nabla_b (\xi^a \theta^b - \xi^b \theta^a)}_{\Omega \delta J}$$

$$L = L(\psi, \nabla_a \psi, g_{ab}, R_{abcd}, \ldots)$$

$$S = -2\pi \oint d^2x \sqrt{h} \sum_{m=0}^{n} (-)^m \nabla_{(e_1} \ldots \nabla_{e_m)} \pi^{e_1 \ldots e_m; abcd} \epsilon_{ab} \epsilon_{cd}.$$

$$\pi^{e_1 \ldots abcd} \equiv \frac{\partial L}{\partial \nabla_{(e_1} \ldots \nabla_{e_m)} R_{abcd}}.$$

for $L = F(R)$.

$$S = \frac{1}{4G} \oint d^2x \, (F'(R))$$

- $R = 0$ (Sch case).
- G_{eff}. universal property.

· $R_{\mu\nu} - \frac{1}{D-2} \Lambda_{eff}\, g_{\mu\nu} = -8\pi G_{Neff}\, T_{\mu\nu}^{max.}$ (?)

$R_0 = -\frac{D}{2} \frac{F(R_0)}{F'(R_0)} = \frac{-2D}{D-2} \Lambda_{eff}$

{ AdS - Sch. BH !
{ charged BH. !

· { EH ⟶ NCG.
{ $G_N \longrightarrow G_{Neff}$.

physics around a constant curvature solution !

As a nontrivial check. let's go to another topic.

Shock wave in nonlinear gravity !

· t' Hooft. S-matrix Ansatz:

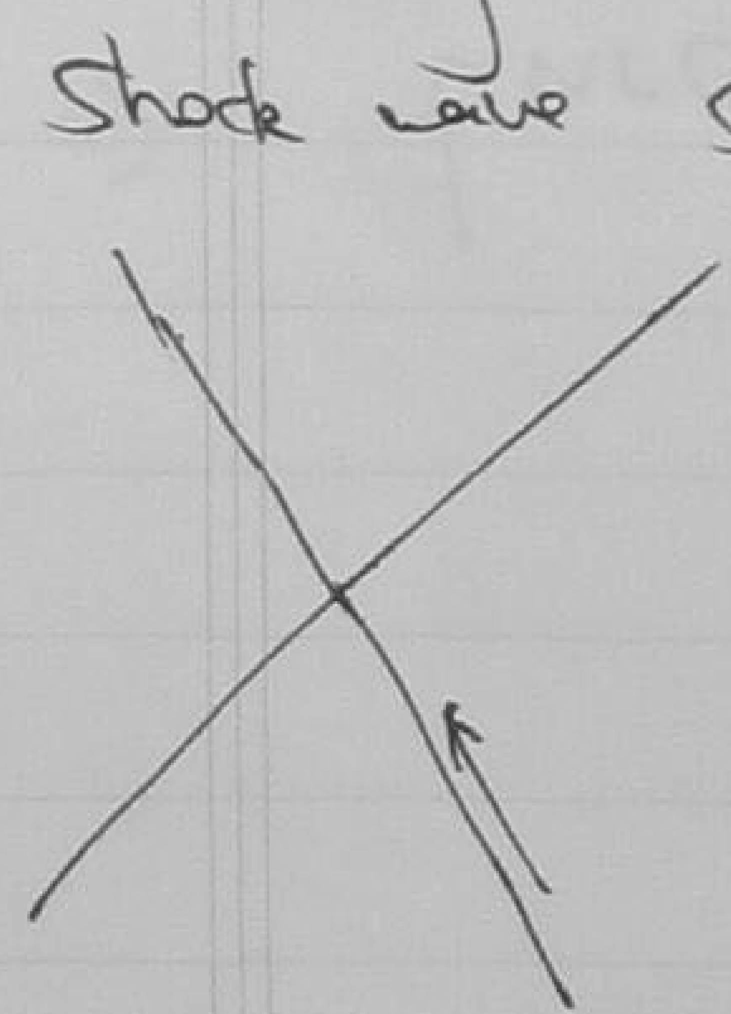
QM should be true.

Near BH horizon: infalling particle got boosted.

⟶ massless particle with speed of light

Interaction between BH & infalling particle become very strong. ⇒ backreaction

Shock wave solution exist.!



⇒ S-matrix of infalling particle with BH.

· S-matrix

· quantization. ⇒ NC. light cone coordinate.

$$ds^2 = 2A(u,v)\,du\,dv + g(u,v)\,h_{ij}(x)\,dx^i dx^j$$ Kruskal coordi

shock wave. $T_{uu} = -4p\,\delta^{(d-2)}(x)\,\delta(u)$

spacetime metric modified !

$\Rightarrow$ V- coordinate. shift.

Ansatz :

$$ds^2 = 2A(u,v)\,du\,dv - 2A(u,v)\,f(x)\,\delta(u)\,du^2$$
$$+ g(u,v)\,h_{ij}(x)\,dx^i dx^j.$$

$$v \to v + f(x)\,\Theta(u).$$

$f(x)$: shift. function.

· EH : $T_{vv} = 0$ $\quad g_{,v} = 0 = A_{,v}$

· NLG : still. true.

$R_{\mu\nu}$ components. δf- terms. should be vanish.

· R_0. inv.

· Solve. ~~E~~ E.O.M.

$$\Box_h f - R^{(D-2)} f - 2g\Lambda_{eff}\, f = 32\pi p.\ \underbrace{\frac{G_N}{F'(R_0)}}_{G_{N\,eff}} A g\,\delta^{(D-2)}(x).$$

$$\Rightarrow \quad f^{NLG} = F'(R_0)\, f^{Einstein}.$$

Conclusion:

"Stationary axial-symmetric vacuum solution of E.O.M."

one needs "renormalize" Newton constant.

$$G_N \longrightarrow G_{N\,eff} = \frac{G_N}{F'(R_0)}$$

- BH entropy $S = \frac{A}{4G_{N\,eff}}$, $G_N M \longrightarrow G_{N\,eff} M$.

  Schwarzschild AdS-Sch. charged.

- Shock wave solution

- Newton-limit: around vacuum solution. (AdS. Minkowski).

  $G_N \longrightarrow G_{N\,eff}$.

- Area law always true for $F(R)$

  Holography holds!

- $F(R)$ + Lovelock gravity (Myers & Simon).

$$\frac{\lambda}{2}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right).$$

Gauss-Bonnet density.

- in 4-dim. topological term, trivial.
- $D \geq 5$. modify gravity

boundary term. can be worked out

$$S = \frac{A_{D-2}}{4G} r_h^{D-2}\left[1 + \underbrace{\frac{D-2}{D-4}\frac{\hat{\lambda}}{r_h^2}}_{\text{correction}}\right]$$

BH solution. not the form of Sch. or AdS-Sch.

$$\hat{\lambda} = \lambda(D-3)(D-4)$$

Stability of constant vacuum solutn in MG.

1. $R_0 = 0$. Minkowski spacetm as vacuum.

(Magnano & Sokolowski: A. Strominger $R + \alpha R^2$)

- Jordan frame: $G_{\mu\nu} \equiv \Theta_{\mu\nu}(g, p)$

  $\Theta_{\mu\nu}$ does not satisfy the dominant energy conditn.

  $\exists$ negative-energy solutn.

- Einstein frame:

  $f'(R) > 0$ & $f''(R) \neq 0$

  $$\begin{cases} g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1/2-\epsilon}) \\ g_{\mu\nu,\alpha} = O(r^{-3/2-\epsilon}) \end{cases} \qquad \epsilon > 0$$

  $\Rightarrow E_{ADM}[g] \geq 0$.

  $\tilde{g}_{\mu\nu} = p\, g_{\mu\nu}$. $\qquad p \sim 1 + O(R)$

  $$L = [\tilde{R} - \tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2V(\phi)]\sqrt{-\tilde{g}}$$

  If $V \geq 0$ the dominant energy holds.

  ϕ : $\Rightarrow$ stress tensor.

  $\Rightarrow$ energy is nonnegative.

  $\Leftrightarrow$ R^2 term. coefficient ~~...~~ should be positive (after taylor expansn).

2. AdS. dS

   Abbott & Deser. (1982).

   $\Lambda > 0$: small fluctutn. inside the horizon. (subtle).

   $\Lambda < 0$. AdS: for all asymptotically AdS metrics.

"killing energy"

$$\mathcal{L} = \left[\tilde{R} - \tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2V(\phi)\right]\sqrt{-\tilde{g}}$$

P. Breitenlohner & D. Z. Freedman.

$$\Rightarrow \quad \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = T_{\mu\nu}$$

$$= \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\tilde{g}_{\mu\nu}$$

$$= \frac{1}{2}\left[2\partial_\mu\phi\partial_\nu\phi - \tilde{g}_{\mu\nu}(\partial^\sigma\phi\partial_\sigma\phi + 2\tilde{V}(\phi)\right]$$

$$- \tilde{g}_{\mu\nu}\tilde{V}(\phi_0).$$

$$\Rightarrow \quad \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} + \underline{\tilde{g}_{\mu\nu}\tilde{V}(\phi)} = 0$$

ϕ_0, critical point

$$\begin{cases} \tilde{V}(\phi_0) > 0 & \text{AdS} \\ \tilde{V}(\phi_0) = 0 & \text{Mink} \\ \tilde{V}(\phi_0) < 0 & \text{dS} \end{cases}$$

$\left.\frac{\partial^2 V}{\partial\phi^i\partial\phi^j}\right|_{\phi_0}$ maximum.

new result: minimum . . .

if $\frac{\partial^2 V}{\partial\phi^i\partial\phi^j}$ not too large.

CAdS

$$ds^2 = (a^2\cos^2\rho)^{-1}\left[dt^2 - d\rho^2 - \sin^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$-\infty < t < \infty \qquad 0 \le \theta \le \pi \qquad 0 \le \rho < \pi/2.$

$R = 12a^2$

$$\tilde{R} - 2\tilde{R} + 4V(\phi_0) = 0$$

$$\Rightarrow \quad \tilde{R} = 4V(\phi_0)$$

$$\Lambda = -2V(\phi_0)$$

$$\Rightarrow \quad a^2 = \frac{1}{3}V(\phi_0)$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) \qquad \phi(x) = \phi_0(x) + \lambda(x)$$

$$E = \int d^3x\sqrt{-\bar{g}}\left[t^{0\nu} + T^{0\nu} + \bar{g}^{0\nu}V(\phi_0)\right]\xi_\nu$$

$$\xi^\nu = (1\ 0\ 0\ 0)$$

$E[t_{\mu\nu}] = \int d^3x\sqrt{-\bar{g}}\,t^{0\nu}\xi_\nu$ R Jaeen. metric fluctuation.

$E[T_{\mu\nu}] = \int d^3x\sqrt{-\bar{g}}\left[T^{0\nu} + \bar{g}^{0\nu}V(\phi_0)\right]\xi_\nu$ scalar fluctuation.

$$\begin{cases} \sqrt{-\det\bar{g}} = \frac{1}{a^4}\tan^2\rho\,\frac{\sin\theta}{\cos^2\rho} \\ \xi_\nu = \frac{1}{a^2\cos^2\rho}(1\ 0\ 0\ 0) \end{cases}$$

$$T^{00} = \left\{(\partial^t\lambda)^2 - \frac{1}{2}\bar{g}^{00}\left(\partial^t\lambda\partial_t\lambda + \partial^\rho\lambda\partial_\rho\lambda + \partial^\theta\lambda\partial_\theta\lambda + \partial^\phi\lambda\partial_\phi\lambda\right)\right.$$

$$\left. - \bar{g}^{00}\left(V(\phi_0+\lambda) - V(\phi_0)\right)\right]$$

$$= a^4 \cos^4\rho \, (\partial_t \lambda)^2 - \frac{1}{2} a^2 \cos^2\rho \left[a^2\cos^2\rho\,(\partial_t\lambda)^2 - a^2\cos^2\rho\,(\partial_\rho\lambda)^2 - \frac{a^2\cos^2\rho}{\sin^2\rho}(\partial_\theta\lambda)^2 - \frac{a^2\cos^2\rho}{\sin^2\rho\,\sin^2\theta}(\partial_\phi\lambda)^2 \right]$$

$$- a^2\cos^2\rho \left[\frac{1}{2} \frac{\partial^2 V}{\partial\phi\,\partial\phi} \lambda^2 \right]$$

$$= \frac{1}{2} a^4 \cos^4\rho \left[(\partial_t\lambda)^2 + (\partial_\rho\lambda)^2 + \frac{1}{\sin^2\rho}\left[(\partial_\theta\lambda)^2 + \frac{1}{\sin^2\theta}(\partial_\phi\lambda)^2\right] - (a^2\cos^2\rho)^{-1} \frac{\partial^2 V}{\partial\phi^2}\lambda^2 \right]$$

$$\Rightarrow E = \frac{1}{2a^2}\int \sin\theta\, d\theta\, d\phi\, d\rho \, \tan^2\rho \left\{ (\partial_t\lambda)^2 + (\partial_\rho\lambda)^2 + \frac{1}{\sin^2\rho}\left((\partial_\theta\lambda)^2 + \frac{1}{\sin^2\theta}(\partial_\phi\lambda)^2\right) - \frac{1}{a^2\cos^2\rho} V'' \lambda^2 \right\}$$

$$\begin{cases} V'' = a^2 \alpha & \text{eigenvalues} \\ \vec{e} \quad \text{eigenvector} & \lambda \end{cases}$$

1) $\alpha < 0$ $\quad E > 0$ $\quad$ Sufficient condition

2) $\alpha > 0$.

$\lambda = (\cos\rho)^\mu \, h(\cdot)$

Focus on: $(\partial_\rho\lambda)^2 - \frac{\alpha}{\cos^2\rho}\lambda^2$

$$= -\frac{1}{\cos^2\rho}\left[\alpha - 3\mu + \mu^2\sin^2\rho\right]$$

$$\Rightarrow E = \frac{1}{2a^2}\int d\Omega\, d\rho\, \tan^2\rho\,(\cos\rho)^{2\mu} \left\{ (\partial_\rho h)^2 - \underline{\left[(\alpha - 3\mu + \mu^2\sin^2\rho)/\cos^2\rho\right]} h^2 \right\}$$

+ boundary term

Require $\boxed{\alpha \leq 9/4}$ by reality condition.

$\mu_\pm = \frac{3}{2} \pm \frac{1}{2}(9 - 4\alpha)^{1/2}$ $\quad$ from $\underbrace{\mu^2 - 3\mu + \alpha = 0}$

$\mu > \frac{3}{2}$ $\quad\quad$ $(\mu - \frac{3}{2})^2 + \alpha - \frac{9}{4} = 0$