边界场论的Hamiltonian表述及其在开弦理论中的应用

西北大学　赵柳

1. 边界场论的一些特点

a. 存在非平凡的关空边界，边界条件独立于运动方程，但一般可以从作用量变分得出；

——“作用量就是物理”

b. 边界条件一般会破坏naive的Poisson括号

⟹ 正则Hamilton表述不自洽，导致存在约束；

c. 边界约束是“不完整”约束。

2. Dirac方法在边界场论中应用出现的问题

由于边界约束的不完整性，直接将Dirac处理约束系统的方法应用于边界场论会出现一些微妙的问题。主要的问题在于Dirac矩阵（约束$\{G^a\}$构成的矩阵$\Delta^{ab}\equiv\{G^a, G^b\}$）的逆不好定义。

例：(1+1)-维 Klein-Gordon 场 + 边界

$$S=\frac{1}{2}\int_{-\infty}^{\infty}dt\int_0^{\infty}dx\,[(\partial_t\varphi)^2-(\partial_x\varphi)^2-m^2\varphi^2]-\boxed{\frac{1}{2}\gamma\int_{-\infty}^{\infty}dt\,\varphi^2\Big|_{x=0}}$$

边界项

$$\delta S=-\int_{-\infty}^{\infty}dt\int_0^{\infty}dx\,(\partial_t^2\varphi-\partial_x^2\varphi+m^2\varphi)\delta\varphi-\int_{-\infty}^{\infty}dt\,(\partial_x-\gamma)\varphi\delta\varphi\Big|_{x=0}^{\infty}$$

（∞ — 扔掉）

$$+\int_0^{\infty}dx\,\partial_t\varphi\,\delta\varphi\Big|_{t=-\infty}^{\infty}$$ 扔掉

$$\delta S=0\Rightarrow\quad(\partial_t^2-\partial_x^2+m^2)\varphi=0\qquad\text{运动方程}$$

$$(\partial_x-\gamma)\varphi\Big|_{x=0}=0\qquad\text{边界条件（约束）}$$

处理上述约束体系可以采纳两种不同的~~思路~~观点：Ⓐ连续场论的观点；Ⓑ离散化再处理。

Ⓐ 连续场论的观点：

正则变量：$\{\varphi(x,t),\ \pi(x,t)\}$，其中

$$\pi \equiv \frac{\delta \mathcal{L}}{\delta \partial_t \varphi} = \partial_t \varphi .$$

$$H = \int_0^\infty dx\ \pi\dot{\varphi} - \mathcal{L} = \frac{1}{2}\int_0^\infty dx \left[\pi^2 + (\partial_x\varphi)^2 + m^2\varphi^2\right] + \frac{1}{2}\eta\varphi^2\Big|_{x=0} .$$

标准的等时 Poisson 括号：

$$\{\varphi(x),\ \varphi(y)\} = \{\pi(x),\ \pi(y)\} = 0 ,\quad \{\varphi(x),\ \pi(y)\} = \delta(x-y) .$$

初级约束（primary constraint）

$$G^{(0)} = (\partial_x - \eta)\varphi\Big|_{x=0} = \lim_{\epsilon\to 0^+}\int_0^\infty dy\ \delta(y-\epsilon)(\partial_y - \eta)\varphi(y) \simeq 0 .$$

第一类 Hamiltonian：

$$H_I = H + \lambda G^{(0)} , \qquad \lambda:\ \text{Lagrangian multiplier}$$

$$\Rightarrow \quad \partial_t G^{(0)} \simeq \{G^{(0)},\ H_I\} = (\partial_x - \eta)\pi(x)\Big|_{x=0} \simeq 0$$

得到 Secondary constraint

$$G^{(1)} = (\partial_x - \eta)\pi(x)\Big|_{x=0} = \lim_{\epsilon\to 0^+}\int_0^\infty dy\ \delta(y-\epsilon)(\partial_y - \eta)\pi \simeq 0 .$$

$\partial_t G^{(1)}$ fixes λ, no more secondary constraints.

Dirac 矩阵 $\Delta^{ij} = \{G^i,\ G^j\}$,

$$\Delta^{01} = -\Delta^{10} = (\partial_x^2 - \eta^2)\delta(x)\Big|_{x=0} .$$

△中含有 $\delta(0)$，无法求逆，方案④失败。

③ 离散化方案：

取格点间距 h，将 $x\in[0,\infty)$ 作离散化：$x_n \equiv nh$，$n\in\mathbb{N}$.

空间微商变为差分：

$$\partial_x \varphi(x)\Big|_{x=x_n} \longrightarrow \frac{\varphi_{n+1}-\varphi_n}{h}.$$

$$S=\int_{-\infty}^{\infty} dt\,\mathcal{L} = \frac{1}{2}\int_{-\infty}^{\infty} dt \sum_{n=0}^{\infty}\left[h\dot{\varphi}_n^2-\frac{1}{h}(\varphi_{n+1}-\varphi_n)^2-m^2h\varphi_n^2\right]$$

$$-\frac{1}{2}\int_{-\infty}^{\infty} dt\,\varphi_0^2.$$

$$\pi_j \equiv \delta\mathcal{L}/\delta\dot{\varphi}_j = h\dot{\varphi}_j.$$

$$\delta_j S = \int dt\left\{\sum_{n=0}^{\infty}\left[-h\ddot{\varphi}_n\delta_{jn}-\frac{1}{h}(\varphi_{n+1}-\varphi_n)(\delta_{j,n+1}-\delta_{jn})-m^2h\varphi_n\delta_{jn}\right]\right.$$

$$\left.-\eta\varphi_0\delta_{j0}\right\}\delta\varphi_j$$

$j=0$：$h\ddot{\varphi}_0+m^2h\varphi_0-\frac{1}{h}(\varphi_1-\varphi_0)+\eta\varphi_0=0$.

前两项与其余项相比属高阶小量，可舍去.

$$\Rightarrow \frac{1}{h}(\varphi_1-\varphi_0)+\eta\varphi_0=0 \qquad (\text{边界条件的离散形式})$$

$j>0$：$\ddot{\varphi}_j-\frac{1}{h^2}(\varphi_{j+1}-2\varphi_j+\varphi_{j-1})+m^2\varphi_j=0$ （运动方程）

初级约束： $G_0 \equiv \frac{1}{h}(\varphi_1 - \varphi_0) + \eta\varphi_0 \simeq 0$ （完整力学约束）

$$H_I = H + \lambda G_0, \quad H = \frac{1}{2h}\sum_{j=0}^{\infty}\left[\pi_j^2 + (\varphi_{j+1} - \varphi_j)^2 + m^2 h^2 \varphi_j^2\right] + \frac{1}{2}\eta\varphi_0^2$$

~~正则~~ 正则 Poisson 括号：

$$\{\varphi_i, \varphi_j\} = \{\pi_i, \pi_j\} = 0, \quad \{\varphi_i, \pi_j\} = \delta_{ij}.$$

$\Rightarrow$

$$\dot{G}_0 = \frac{1}{h^2}\left[\pi_1 - (1 + \eta h)\pi_0\right] \simeq 0,$$

i.e. $G_1 \equiv \frac{1}{h^2}\left[\pi_1 - (1 + \eta h)\pi_0\right] \simeq 0.$

$\dot{G}_1$ fixes λ, no more constraints.

$$(\Delta_{ij}) = (\{G_i, G_j\}), \quad \Delta_{01} = -\Delta_{10} = \frac{1}{h^3}\left[1 + (1 + h\eta)^2\right],$$

Δ 可逆：

$$\Delta^{-1} = \frac{h^3}{1 + (1 + h\eta)^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$\Rightarrow$ Dirac Poisson 括号 （ $\{A, B\}_D \equiv \{A, B\} - \{A, G_i\}(\Delta^{-1})_{ij}\{G_j, B\}$

$$\{\varphi_i, \varphi_j\}_D = \{\pi_i, \pi_j\}_D = 0,$$

$$\{\varphi_i, \pi_j\}_D = \delta_{ij} - \frac{h^2}{1 + (1 + h\eta)^2}\left[\frac{\delta_{i1} - \delta_{i0}}{h} - \eta\delta_{i0}\right]\left[\frac{\delta_{1j} - \delta_{0j}}{h} - \eta\delta_{0j}\right].$$

在 $\{\,,\,\}_D$ 下，G_0，G_1 均强等于零。

问题：连续极限不自洽，

$$\{\varphi(x), \pi(y)\}_D = \delta(x-y) - \lim_{h\to 0}\frac{h^3}{2}[\delta'(x)-\gamma\delta(x)][\delta'(y)-\gamma\delta(y)],$$

$$\{G^{(0)}, \pi(y)\}_D \neq 0.$$

问题的核心：简单的差分不保持Poisson结构！

解决问题的办法：ⓐ寻找保Poisson结构的离散化方法

ⓑ放弃Dirac方法另寻途径

3. 处理边界条件的新方法：

Dirac括号的核心特点：$\{G^a, \text{everything}\}_D = 0$.

新方法：不必遵循标准的Dirac过程，直接寻找新的Poisson括号，使之满足 $\{G^a, \text{everything}\} = 0$.

基本原则之一：locality

若靶空间的整体对称性不被边界条件破坏，则只需对正则Poisson括号引入边界修正。

假定某一场论系统包含基本场 $\varphi_i(x)$，系统仅在某一类空方向 x^A 存在边界：$x^A \in [0, \infty)$. 在不考虑边界条件时，系统的自洽等时Poisson括号为

$$\{\varphi_i(x), \varphi_j(y)\} = \alpha_{ij}(\varphi, \pi)\,\underline{\delta(x-y)},$$

$$\{\varphi_i(x), \pi_j(y)\} = \beta_{ij}(\varphi, \pi)\,\underline{\delta(x-y)},$$

$$\{\pi_i(x), \pi_j(y)\} = \gamma_{ij}(\varphi, \pi)\,\underline{\delta(x-y)}.$$

体现 locality

引入边界条件 $G_i \simeq 0$ 后，可以将 Poisson 括号修改为

$$\{\varphi_i(x), \varphi_j(y)\}_N = \alpha_{ij}(\varphi,\pi)\,\delta(x-y) + \mathcal{A}_{ij}\,\bar{\delta}(x-y)\,\delta(x^A+y^A), \quad ①$$

$$\{\varphi_i(x), \pi_j(y)\}_N = \beta_{ij}(\varphi,\pi)\,\delta(x-y) + \mathcal{B}_{ij}\,\bar{\delta}(x-y)\,\delta(x^A+y^A), \quad ②$$

$$\{\pi_i(x), \pi_j(y)\}_N = \gamma_{ij}(\varphi,\pi)\,\delta(x-y) + \mathcal{C}_{ij}\,\bar{\delta}(x-y)\,\delta(x^A+y^A), \quad ③$$

其中，$\delta(x-y) \equiv \prod_{i=1}^{D-1} \delta(x^i-y^i)$，$\bar{\delta}(x-y) = \prod_{\substack{i=1\\ i\neq A}}^{D-1} \delta(x^i-y^i)$.

$$\mathcal{A}_{ij} = -\mathcal{A}_{ji}, \quad \mathcal{C}_{ij} = -\mathcal{C}_{ji}.$$

注意 $x^A, y^A \geq 0$，故含 $\mathcal{A}, \mathcal{B}, \mathcal{C}$ 的项仅在边界 $(x^A=0)$ 处是非平凡的。这一要求体现了 locality 原则。

为使边界条件在新 Poisson 括号 $\{,\}_N$ 下自洽，需要求

$$\{G_i(\varphi,\pi), \varphi_j\}_N = 0$$

$$\{G_i(\varphi,\pi), \pi_j\}_N = 0$$

若上述两组 Poisson 括号均为零，则可得到关于 $\mathcal{A}, \mathcal{B}, \mathcal{C}$ 的两组方程。

另外，新括号 $\{\ ,\ \}_N$ 必须满足 Jacobi 等式。由

$$\{\varphi_i, \varphi_j, \varphi_k\}, \quad \{\varphi_i, \varphi_j, \pi_k\},$$

$$\{\varphi_i, \pi_j, \pi_k\}, \quad \{\pi_i, \pi_j, \pi_k\}$$

满足的 Jacobi 等式又可得到关于 A, B, C 的四组方程。

结论：关于算符 A, B, C 的方程组一般是超定的，解的存在性需针对具体问题来分析。如果证明解不存在，说明边界条件与系统的动力学不相容。

下面分析几个例子。

例 1. 标量场 ~~Klein-Gordon 系统~~ （D+1 维时空，边界在 x^D 方向）

$$S[\varphi] = \frac{1}{2}\int d^D x \int_0^\infty dx^D \left[\partial_M \varphi \partial^M \varphi - m^2 \varphi^2 + 2g V(\varphi)\right]$$

$$+ \lambda_B \int d^D x\, V_B(\varphi) \Big|_{x^D=0}.$$

$$\pi(x) \equiv \delta\mathcal{L}/\delta\partial_0\varphi = \partial_0\varphi(x).$$

$$\delta S = 0 \quad \Rightarrow \quad \partial_M \partial^M \varphi + m^2 \varphi - g\frac{\delta V}{\delta g} = 0,$$

$$\partial_D \varphi + \lambda_B \frac{\delta V_B(\varphi)}{\delta \varphi} = 0 \Big|_{x^D=0}.$$

正则等时括号

$$\{\varphi(x), \varphi(y)\} = 0, \qquad \{\pi(x), \pi(y)\} = 0$$

$$\{\varphi(x), \pi(y)\} = \delta(x-y), \qquad \delta(x-y) = \prod_{i=1}^{D} \delta(x^i - y^i) = \delta^{(D)}(x-y)$$

边界约束

$$G = \lim_{\epsilon \to 0^+} \int_0^\infty dx^D \, \delta(x^D - \epsilon) \left[\partial_D \varphi + \lambda_B \frac{\delta V_B}{\delta \varphi} \right] \simeq 0$$

不依赖于 $\pi(x)$，方程 $\{G, \varphi(x)\} = 0$ 自动成立。因此，只需修改 $\{\varphi, \pi\}$ 这一个括号：

$$\{\varphi(x), \pi(y)\}_N = \delta(x-y) + \mathcal{B}(y)\, \delta(x - \sigma(y)),$$

$$y = (y^1, \cdots, y^{D-1}, y^D), \quad \sigma(y) = (y^1, \cdots, y^{D-1}, \underline{-y^D}).$$

$\{G, \pi(y)\}_N = 0 \Rightarrow$

$$\left[\mathcal{B}(y) \left(\partial_{y^D} + \lambda_B \frac{\delta^2 V_B}{\delta \varphi^2} \right) - \left(\partial_{y^D} - \lambda_B \frac{\delta^2 V_B}{\delta \varphi^2} \right) \right] \delta^{(D-1)}(x-y)\, \delta(y^D) = 0.$$

$$\Rightarrow \quad \mathcal{B}(y) = \frac{\partial_{y^D} - \lambda_B \frac{\delta^2 V_B}{\delta \varphi^2}}{\partial_{y^D} + \lambda_B \frac{\delta^2 V_B}{\delta \varphi^2}}.$$

Jacobi identities hold automatically.

特别地，当 $V_B = \frac{1}{2}\varphi^2$ 时，$\mathcal{B}(y) = \dfrac{\partial_{y^D} - \lambda_B}{\partial_{y^D} + \lambda_B}$.

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$\lambda_B = 0 \Rightarrow$ 边界条件 $\partial_{x^0}\varphi \big|_{x^0=0} = 0$. Neumann 条件

$$\{\varphi(x), \pi(y)\}_N = \delta(x-y) + \delta(x-\sigma(y)) \qquad (\mathcal{B}=1)$$

$\lambda_B = \infty \Rightarrow$ 边界条件 $\varphi = \text{const}\big|_{x^0=0}$ Dirichlet 条件

$$\{\varphi(x), \pi(y)\}_N = \delta(x-y) - \delta(x-\sigma(y)), \qquad (\mathcal{B}=-1).$$

例 2. $O(N)$ 非线性 σ 模型 (1+1 时空, $x \geq 0$)

$$S = \frac{1}{2}\int d^2x \left[\partial_\mu n^T \partial^\mu n + \omega(n^T \cdot n - 1)\right]$$

$n = (n_1, n_2, \cdots, n_N)^T$, ω: Lagrangian multiplier

运动方程: $\partial_\mu \partial^\mu n^T - \omega n^T = 0$, $n^T \cdot n = 1$

$$\Rightarrow \partial_\mu \partial^\mu n + (\partial_\mu n^T \cdot \partial^\mu n) n = 0.$$

共轭动量

$$\pi_i \equiv \frac{\delta \mathcal{L}_B}{\delta \partial_t n^i} = \partial_t n_i \qquad (\mathcal{L}_B: \text{bulk Lagrangian})$$

Bulk Poisson 括号 (Dirac 括号):

$$\{n_i(x), n_j(y)\} = \alpha_{ij}\,\delta(x-y), \qquad \alpha_{ij} = 0$$

$$\{n_i(x), \pi_j(y)\} = \beta_{ij}\,\delta(x-y), \qquad \beta_{ij} = \delta_{ij} - n_i n_j$$

$$\{\pi_i(x), \pi_j(y)\} = \gamma_{ij}\,\delta(x-y), \qquad \gamma_{ij} = \pi_i n_j - n_i \pi_j$$

五种边界条件:

i) Neumann boundaries along all target space directions

$$\partial_x n_i \big|_{x=0} = 0 , \quad i = 1, \cdots, N ;$$

ii) Dirichlet boundaries along all target space directions

$$\partial_t n_i \big|_{x=0} = 0 , \quad i = 1, \cdots, N ;$$

iii) mixture of Neumann and Dirichlet boundaries, e.g.

$$\partial_x n_i = 0 , \quad i = 1, \cdots, p ,$$

$$\partial_t n_i = 0 , \quad i = p+1, \cdots, D ;$$

iv) mixed boundary conditions along all target space directions,

$$(\partial_x n_i + M_{ij} \partial_t n_j)\big|_{x=0} = 0 , \quad i = 1, \cdots, N ,$$

M: antisymmetric, invertible $N \times N$ matrix

$$M = g_1 (i\sigma^2) \oplus g_2 (i\sigma^2) \oplus \cdots \oplus g_K (i\sigma^2) \qquad (N = 2K)$$

σ^2: Pauli matrix

v) mixture of mixed and Dirichlet boundaries

$$\partial_x n_i + M_{ij} \partial_t n_j \big|_{x=0} = 0 , \quad i, j = 1, \cdots, p = 2K ,$$

$$\partial_t n_i = 0 , \quad i = p+1, \cdots, N .$$

M. Moriconi, hep-th/9809178 (PLB 447 (1999) 292-297)

hep-th/0111195

边界条件的统一形式：

$$\left(\partial_t n_i + W_{ij}\,\partial_x n_j\right)\Big|_{x=0} = 0, \qquad (*)$$

$$W = \begin{pmatrix} \mathcal{W} & \\ & 0_{N-p} \end{pmatrix}.$$

i) $\longleftrightarrow$ $p = N$, $\mathcal{W}$ diagonal & $\mathcal{W}_{ii} \to \infty$; } $O(N)$ invariant

ii) $\longleftrightarrow$ $p = 0$; } $O(N)$ invariant

iii) $\longleftrightarrow$ generic p, $\mathcal{W}$ diagonal & $\mathcal{W}_{ii} \to \infty$; } $O(p) \times O(N-p)$ invariant

iv) $\longleftrightarrow$ ~~gene~~ $p = 2K = N$, $\mathcal{W} = M^{-1}$; } ?

v) $\longleftrightarrow$ $p = 2K < N$, $\mathcal{W} = \mathcal{M}^{-1}$. } ?

边界条件依赖于正则坐标和正则动量：

$$G_i \equiv \pi_i + W_{ij}\,\partial_x n_j \Big|_{x=0} = 0 .$$

假定新 Poisson 括号的形式为

$$\{n_i(x), n_j(y)\}_N = \alpha_{ij}\,\delta(x-y) + \mathcal{A}_{ij}\,\delta(x+y)$$

$$\{n_i(x), \pi_j(y)\}_N = \beta_{ij}\,\delta(x-y) + \mathcal{B}_{ij}\,\delta(x+y)$$

$$\{\pi_i(x), \pi_j(y)\}_N = \gamma_{ij}\,\delta(x-y) + \mathcal{C}_{ij}\,\delta(x+y)$$

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$\{G_i, n_j(y)\}_N = 0 \Rightarrow$

① $$(\alpha - \mathcal{A}) W \partial_y + (\beta + \mathcal{B}) = 0,$$

$\{G_i, \pi_j(y)\}_N = 0 \Rightarrow$

② $$\gamma + \mathcal{C} - W(\beta - \mathcal{B})\partial_y = 0.$$

Jacobi identities $\Rightarrow$

③ $$\frac{\delta(\alpha+\mathcal{A})_{ij}}{\delta n_m}(\alpha+\mathcal{A})_{mk} - \frac{\delta(\alpha+\mathcal{A})}{\delta \pi_m}(\beta+\mathcal{B})_{km} + \frac{\delta(\alpha+\mathcal{A})_{jk}}{\delta n_m}(\alpha+\mathcal{A})_{mi}$$
$$- \frac{\delta(\alpha+\mathcal{A})_{jk}}{\delta \pi_m}(\beta+\mathcal{B})_{im} + \frac{\delta(\alpha+\mathcal{A})_{ki}}{\delta n_m}(\alpha+\mathcal{A})_{mj} - \frac{\delta(\alpha+\mathcal{A})_{ki}}{\delta \pi_m}(\beta+\mathcal{B})_{jm} = 0,$$

④ $$\frac{\delta(\alpha+\mathcal{A})_{ij}}{\delta n_m}(\beta+\mathcal{B})_{mk} + \frac{\delta(\alpha+\mathcal{A})_{ij}}{\delta \pi_m}(\gamma+\mathcal{C})_{mk} + \frac{\delta(\beta+\mathcal{B})_{jk}}{\delta \pi_m}(\alpha+\mathcal{A})_{mi}$$
$$- \frac{\delta(\beta+\mathcal{B})_{jk}}{\delta \pi_m}(\beta+\mathcal{B})_{im} - \frac{\delta(\beta+\mathcal{B})_{ik}}{\delta n_m}(\alpha+\mathcal{A})_{mj} + \frac{\delta(\beta+\mathcal{B})_{ik}}{\delta \pi_m}(\beta+\mathcal{B})_{jm} = 0,$$

⑤ $$\frac{\delta(\beta+\mathcal{B})_{ij}}{\delta n_m}(\beta+\mathcal{B})_{mk} + \frac{\delta(\beta+\mathcal{B})_{ij}}{\delta \pi_m}(\gamma+\mathcal{C})_{mk} + \frac{\delta(\gamma+\mathcal{C})_{jk}}{\delta n_m}(\alpha+\mathcal{A})_{mi}$$
$$- \frac{\delta(\gamma+\mathcal{C})_{jk}}{\delta \pi_m}(\beta+\mathcal{B})_{im} - \frac{\delta(\beta+\mathcal{B})_{ik}}{\delta n_m}(\beta+\mathcal{B})_{mj} - \frac{\delta(\beta+\mathcal{B})_{ik}}{\delta \pi_m}(\gamma+\mathcal{C})_{mj} = 0,$$

⑥ $$\frac{\delta(\gamma+\mathcal{C})_{ij}}{\delta n_m}(\beta+\mathcal{B})_{mk} + \frac{\delta(\gamma+\mathcal{C})_{ij}}{\delta \pi_m}(\gamma+\mathcal{C})_{mk} + \frac{\delta(\gamma+\mathcal{C})_{jk}}{\delta n_m}(\beta+\mathcal{B})_{mi}$$
$$+ \frac{\delta(\gamma+\mathcal{C})_{jk}}{\delta \pi_m}(\gamma+\mathcal{C})_{mi} + \frac{\delta(\gamma+\mathcal{C})_{ki}}{\delta n_m}(\beta+\mathcal{B})_{mj} + \frac{\delta(\gamma+\mathcal{C})_{ki}}{\delta \pi_m}(\gamma+\mathcal{C})_{mj} = 0.$$

求解①—⑥需考虑几种不同情况：

A：O(N)不变的边界条件（i) 和 ii) 类边界条件）

取 $\alpha_{ij}=0$，$\beta_{ij}=\delta_{ij}-n_in_j$，$\gamma_{ij}=\pi_in_j-\pi_jn_i$，①和②变为

①′ $\quad \mathcal{A}_{im}W_{mj}\partial_y-(I-n\cdot n^T+\mathcal{B})_{ij}=0,$

②′ $\quad (\pi\cdot n^T-n\cdot\pi^T+\mathcal{C})_{ij}-W_{im}(I-n\cdot n^T-\mathcal{B})_{mj}\partial_y=0.$

ii) 对应 $p=0$ 或 $W_{ij}=0$，$\Rightarrow$ $\mathcal{B}=-(I-n\cdot n^T)$，$\mathcal{C}=-(\pi\cdot n^T-n\cdot\pi^T)$.

i) 对应 $p=N$，W 对角且 $W_{ii}\to\infty$. $\Rightarrow$ 仅有含 W 的项起作用

$\Rightarrow$ $\mathcal{A}=0$，$\mathcal{B}=I-n\cdot n^T$.

以上两种情形的解分别代入③—⑥，可得以下 Poisson 括号：

$$\{n_i(x),n_j(y)\}_N=0,$$
$$\{n_i(x),\pi_j(y)\}_N=(\delta_{ij}-n_in_j)[\delta(x-y)\pm\delta(x+y)],$$
$$\{\pi_i(x),\pi_j(y)\}_N=(\pi_in_j-\pi_jn_i)[\delta(x-y)\pm\delta(x+y)],$$

式中取 + 对应边界条件 i)，即 Neumann 边界条件，取 − 对应边界条件 ii)，即 Dirichlet 边界条件。

B 破坏 O(N) 对称的边界条件 iii)

O(N) 变换不能将 Dirichlet 条件变为 Neumann 条件，因此边界条件 iii) 明显破坏靶空间的整体 O(N) 对称性。这一事实也可以由①，②两个方程中体现出来。取 ~~$\mathcal{A}_{ij}=0$, $\mathcal{B}_{ij}=\delta_{ij}-n_in_j$, $\mathcal{C}$~~

$$\alpha_{ij}=0,\quad \beta_{ij}=\delta_{ij}-n_in_j,\quad \gamma_{ij}=\pi_in_j-\pi_jn_i,$$

同时令 $p=2K<N$，$W_{ii}\to\infty$，$W_{ij}=0$ for $i\neq j$，则方程②′的第一项反称，第二项不反称，须各自取值为零，即

$$\mathcal{B}=I-n\cdot n^T,\qquad \mathcal{C}=-(\pi\cdot n^T-n\cdot\pi^T).$$

这个结果代入①′，将发现 $\mathcal{A}$ 不能反对称，故①′和②′联立无解。

基本原则之二：

> 当边界条件破坏靶空间的整体对称性时，需先对 Bulk 部分作对称约化，使边界与 Bulk 的整体对称性保持一致。

对于边界条件 iii)，靶空间整体对称群 O(N) 中仅有子群 $O(p)\times O(N-p)$ 被保持。因此需先约化 O(N) 非线性 σ 模型为 $O(p)\times O(N-p)$ 模型，再按上述方法处理。

定义指标集 $\{a, b \cdots = 1, \cdots, p\}$，$\{\mu, \nu, \cdots = p+1, \cdots N\}$.

约化的结果：

$$\alpha_{ab} = \alpha_{a\mu} = \alpha_{\mu b} = \alpha_{\mu\nu} = 0,$$

$$\beta_{ab} = \delta_{ab} - n_a n_b, \quad \beta_{a\mu} = 0, \quad \beta_{\mu b} = 0, \quad \beta_{\mu\nu} = \delta_{\mu\nu} - n_\mu n_\nu$$

$$\gamma_{ab} = \pi_a n_b - \pi_b n_a, \quad \gamma_{a\mu} = 0, \quad \gamma_{\mu b} = 0, \quad \gamma_{\mu\nu} = \pi_\mu n_\nu - \pi_\nu n_\mu.$$

代入 ①－⑥ $\Rightarrow$

$$\{n_a(x), n_b(y)\}_N = 0$$

$$\{n_a(x), \pi_b(y)\}_N = (\delta_{ab} - n_a n_b)(\delta(x-y) + \delta(x+y)),$$

$$\{\pi_a(x), \pi_b(y)\}_N = (\pi_a n_b - \pi_b n_a)(\delta(x-y) + \delta(x+y));$$

$$\{n_\mu(x), n_\nu(y)\}_N = 0$$

$$\{n_\mu(x), \pi_\nu(y)\}_N = (\delta_{\mu\nu} - n_\mu n_\nu)(\delta(x-y) - \delta(x+y)),$$

$$\{\pi_\mu(x), \pi_\nu(y)\}_N = (\pi_\mu n_\nu - \pi_\nu n_\mu)(\delta(x-y) - \delta(x+y)).$$

<u>C</u>：动力学不自洽的边界条件 iv) 和 v)

混合边界条件 $(\partial_x n_i + M_{ij} \partial_t n_j)\big|_{x=0} = 0$ 可以通过在作用量中

加边界项

$$S_b = \int dt \, M_{ij} n_i \partial_t n_j \Big|_{x=0}$$

得出。

在 $O(N)$ 转动

$$n_i \to O_{ij} n_j$$

下，M_{ij} 的变换规则为

$$M_{ij} \to O_{ik} M_{kl} O^T_{lj}.$$

在 $O(N)$ 群中使 S_b 不变的最大子群为 $O(2)^{\otimes K}$，而且 S_b 的不变性要求 M 只能取 $g_1(i\sigma^y) \oplus \cdots \oplus g_K(i\sigma^y)$ 这样的形式。

据基本原则二，首先约化 $O(N)$ 对称性到 $O(2)^{\otimes K}$ 对称性。

令 $n^{(l)} = (n_{2l-1}, n_{2l})$，$n^{(l)T} \cdot n^{(l)} = u_l$，$\sum_l u_l = 1$.

取 $\alpha^{(l)} = 0$，$\beta^{(l)} = I_{2\times2} - n^{(l)} \cdot n^{(l)T}$，$\gamma^{(l)} = \pi^{(l)} \cdot n^{(l)T} - n^{(l)} \cdot \pi^{(l)T}$，

$$\alpha = \bigoplus_l \alpha^{(l)}, \quad \beta = \bigoplus_l \beta^{(l)}, \quad \gamma = \bigoplus_l \gamma^{(l)},$$

① $\Rightarrow$ $\mathcal{A}_{im} W^{(l)}_{mj} \partial_y - (I - n^{(l)} \cdot n^{(l)T} + \mathcal{B})_{ij} = 0$,

② $\Rightarrow$ $(\pi^{(l)} \cdot n^{(l)T} - n^{(l)} \cdot \pi^{(l)T} + \mathcal{C})_{ij} - W^{(l)}_{im} (I - n^{(l)} \cdot n^{(l)T} - \mathcal{B})_{mj} \partial_y = 0$.

$$W = M^{-1} = \bigoplus_l W^{(l)}$$

$\Longrightarrow$ No solution !

事实上，当 $N=2$，$M = g\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$，$\partial_x n_i + M_{ij} \partial_t n_j \big|_{x=0} = 0 \Longrightarrow$

$\partial_x n_i = \partial_t n_i = 0$，边界即自由又固定，矛盾！

4. NS 背景中的开弦

作用量：(world sheet)

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-g}\,\Big[g^{ab} G_{ij}(X)\partial_a X^i \partial_b X^j + 2\pi\alpha' B_{ij}(X)\varepsilon^{ab}\partial_a X^i \partial_b X^j + \alpha' \Phi(X) R^{(2)} \Big]$$

$G_{ij}(X)$, $B_{ij}(X)$, $\Phi(X)$: NS fields

假定 G_{ij}, B_{ij}, Φ 保证量子 Weyl 不变性 (i.e. β-functionals vanish)

选 flat gauge (利用 Worldsheet conformal trasformation)

$\Rightarrow$

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \Big[g^{ab} G_{ij}(X)\partial_a X^i \partial_b X^j + 2\pi\alpha' B_{ij}(X)\varepsilon^{ab}\partial_a X^i \partial_b X^j \Big],$$

$$g^{ab} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

$\delta S = 0 \Rightarrow$

$$\partial^a\partial_a X^k + \Gamma^k_{ij}(X) g^{ab}\partial_a X^i \partial_b X^j - \pi\alpha' G^{kl}(X) B_{(ij;l)}(X)\varepsilon^{ab}\partial_a X^i \partial_b X^j = 0,$$

$$\Big[G_{ik}(X)\partial_\sigma X^i + 2\pi\alpha' B_{kj}(X)\partial_\tau X^j \Big]\Big|_{\sigma=0,\pi} = 0.$$

$$B_{(ij,l)} \equiv \frac{\partial B_{ij}}{\partial X^l} + \frac{\partial B_{jl}}{\partial X^i} + \frac{\partial B_{li}}{\partial X^j}.$$

共轭动量

$$P_i \equiv \frac{\delta \mathcal{L}}{\delta \partial_\tau X^i} = \frac{1}{2\pi\alpha'}\left[G_{ij}(X)\partial_\tau X^j + 2\pi\alpha' B_{ij}(X)\partial_\sigma X^j \right]$$

$$\Rightarrow \partial_\tau X^k = 2\pi\alpha' G^{ki}(X)\left[P_i - B_{ij}(X)\partial_\sigma X^j \right].$$

正则 Poisson 括号：

$$\{X^i(\sigma), X^j(\sigma')\} = \{P_i(\sigma), P_j(\sigma')\} = 0,$$

$$\{X^i(\sigma), P_j(\sigma')\} = \delta^i{}_j\, \delta(\sigma-\sigma').$$

边界修正：

$$\{X^i(\sigma), X^j(\sigma')\}_N = (\mathcal{A}_L)^{ij}\delta(\sigma+\sigma') + (\mathcal{A}_R)^{ij}\delta(2\pi-\sigma-\sigma'),$$

$$\{X^i(\sigma), P_j(\sigma')\}_N = \delta^i{}_j\delta(\sigma-\sigma') + (\mathcal{B}_L)^i{}_j\,\delta(\sigma+\sigma') + (\mathcal{B}_R)^i{}_j\,\delta(2\pi-\sigma-\sigma')$$

$$\{P_i(\sigma), P_j(\sigma')\}_N = (\mathcal{C}_L)_{ij}\delta(\sigma+\sigma') + (\mathcal{C}_R)_{ij}\delta(2\pi-\sigma-\sigma').$$

边界约束：

$$(G_L)^i = \lim_{\epsilon\to 0^+}\int_0^\pi d\sigma\, \delta(\sigma-\epsilon)\left[\partial_\sigma X^i + (2\pi\alpha')^2 B^{im}(X)\left(P_m - B_{mk}(X)\partial_\sigma X^k\right)\right] \simeq 0,$$

$$(G_R)^i = \lim_{\epsilon\to 0^+}\int_0^\pi d\sigma\, \delta(\pi-\epsilon-\sigma)\left[\partial_\sigma X^i + (2\pi\alpha')^2 B^{im}(X)\left(P_m - B_{mk}(X)\partial_\sigma X^k\right)\right] \simeq 0.$$

$$\mathcal{D}^i{}_n = \frac{\delta B^{im}}{\delta X^n} P_m$$

$$\mathcal{X}^i{}_n = \frac{\delta (B^2)^i{}_k}{\delta X^n} \partial_\sigma X^k$$

$$\{(G_{L,R})^i, X^j(\sigma')\}_N = 0$$

$$\Rightarrow (2\pi\alpha')^2 (\mathcal{D} - \mathcal{X}) \mathcal{A}_{L,R} + (I - (2\pi\alpha')^2 B^2) \mathcal{A}_{L,R} \partial_{\sigma'}$$
$$- (2\pi\alpha')^2 B (1 + \mathcal{B}_{L,R}) = 0,$$

$$\{(G_{L,R})^i, P_j(\sigma')\}_N = 0$$

$$\Rightarrow [(2\pi\alpha')^2 B^2 - I](I - \mathcal{B}_{L,R}) \partial_{\sigma'} + (2\pi\alpha')^2 [(\mathcal{D} - \mathcal{X})(I + \mathcal{B}_{L,R}) + B\,\mathcal{C}_{L,R}] = 0.$$

Jacobi identities $\Rightarrow$ 4 more sets of equations

这6组算符矩阵方程很难求解. 但是, 存在特解

$$(\mathcal{A}_{L,R})^{ij} = 0,$$

$$(\mathcal{B}_{L,R})^i{}_j = -\delta^i{}_j.$$

$$(\mathcal{C}_{L,R})_{ij} = \frac{2}{(2\pi\alpha')^2} (G + 2\pi\alpha' B)_{ik} (B^{-1})^{kl} (G - 2\pi\alpha' B)_{lj} \partial_{\sigma'}$$

得出 Poisson 括号

$$(*) \begin{cases} \{X^i(\sigma), X^j(\sigma')\}_N = 0, \\ \{X^i(\sigma), P_j(\sigma')\}_N = \delta^i{}_j [\delta(\sigma - \sigma') - \delta(\sigma + \sigma') - \delta(\sigma + \sigma' - 2\pi)], \\ \{P_i(\sigma), P_j(\sigma')\}_N = \frac{2}{(2\pi\alpha')^2} (G + 2\pi\alpha' B)_{ik} (B^{-1})^{kl} (G - 2\pi\alpha' B)_{lj} \\ \qquad \times \partial_{\sigma'} [\delta(\sigma + \sigma') + \delta(2\pi - \sigma - \sigma')]. \end{cases}$$

回忆：当 $G_{ij}=\eta_{ij}$，$B_{ij}=0$ 时，边界条件

$$\partial_\sigma X^i\Big|_{\sigma=0,\pi}=0 , \quad i=1,\cdots,p$$

$$\partial_\tau X^i\Big|_{\sigma=0,\pi}=0 , \quad i=p+1,\cdots,D-1$$

对应 Dp-brane。加入背景场 $G_{ij}(X)$，$B_{ij}(X)$ 后，D-brane 是怎样描述的？

在前面的特解 ⊛ 下，计算 $\partial_\tau X^i\Big|_{\sigma=0,\pi}$。由于 $X^i(0,\pi)$ 与任何正则变量 Poisson 交换，故有

$$\partial_\tau X^i\Big|_{\sigma=0,\pi}=0 , \quad i=0,1,\cdots,D-1.$$

这是 D0-brane！

其他 D-brane 对应的 Poisson 括号是什么样的？ Open question.

问题的简化：G_{ij} flat，B_{ij} = constant $\Rightarrow \mathcal{D}^i{}_n=\mathcal{X}^i{}_n=0.$

$$\Rightarrow (2\pi\alpha')^2 B(I+\mathcal{B}_{L,R}+B\mathcal{A}_{L,R}\partial_{\sigma'})-\mathcal{A}_{LR}\partial_{\sigma'}=0,$$

$$(2\pi\alpha')^2 B(\mathcal{C}_{L,R}-B(-I+\mathcal{B}_{L,R})\partial_{\sigma'}]+(-I+\mathcal{B}_{L,R})\partial_{\sigma'}=0.$$

Jacobi identities become trivial $\Rightarrow$ infinite many solutions