U S T C

General Relativity

x^μ, $\mu = 0, 1, 2, 3$. $\eta^{\mu\nu} = \eta_{\mu\nu} = (1, -1, -1, -1)$

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

$x^\mu \to x'^\mu = x'^\mu(x)$: GL(4) group of general coord. transf.

Covariant derivatives :

$$V^\lambda{}_{;\mu} = \partial_\mu V^\lambda + \Gamma^\lambda{}_{\mu\nu} V^\nu$$

$$V_{\nu;\mu} = \partial_\mu V_\nu - \Gamma^\lambda{}_{\nu\mu} V_\lambda$$

$$\boxed{g_{\mu\nu}(x)\ ,\quad \Gamma^\lambda{}_{\mu\nu}}$$

Riemannian : $\begin{cases} \Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} \\ g^{\mu\nu}{}_{;\lambda} = g_{\mu\nu;\lambda} = 0 \end{cases}$

$$\Downarrow$$

$\left\{ {\lambda \atop \mu\nu} \right\}$ $\qquad \Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})$

Christoffel connection

General Relativity as a gauge theory:

$$\begin{cases} \text{Yang-Mills} \quad \vec{A}_\mu(x) \;:\; SU(2) \text{ gauge field} \\ \qquad\qquad\qquad\qquad \text{" connection} \\ \text{GR} \quad \Gamma^\lambda{}_{\mu\nu} \;:\; GL(4) \text{ connection} \\ \qquad\qquad\qquad\qquad \text{" gauge field} \end{cases}$$

$$\begin{cases} \vec{F}_{\mu\nu} = \vec{A}_{\mu,\nu} - \vec{A}_{\nu,\mu} + \vec{A}_\mu \times \vec{A}_\nu \\ R^{\lambda\rho}{}_{\mu\nu} = g^{\rho\sigma}(\Gamma^\lambda{}_{\sigma\mu,\nu} - \Gamma^\lambda{}_{\sigma\nu,\mu} - \Gamma^\lambda{}_{\alpha\mu}\Gamma^\alpha{}_{\sigma\nu} + \Gamma^\lambda{}_{\alpha\nu}\Gamma^\alpha{}_{\sigma\mu}) \end{cases}$$

Hilbert-Einstein:

$$W = -\frac{c^3}{16\pi G}\int \sqrt{-g}\; R \, d^4x + W_{matter}$$

$$R = R^{\mu\nu}{}_{\mu\nu}, \quad g = \det(g_{\mu\nu})$$

$$\delta W = 0 \;:\quad R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^3}$$

General Relativity: GL(4) gauge theory

Dirac Equation (1928)

$$(i\hbar \gamma^\mu \partial_\mu - m)\psi = 0. \qquad \hbar = 1.$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Under Lorentz transf. $x^\mu \to x^\mu + \varepsilon^\mu{}_\nu x^\nu$, $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$.

$$\psi(x) \to \psi'(x) = e^{-i\varepsilon^{\mu\nu}\frac{\sigma_{\mu\nu}}{4}}\psi(x), \quad \left(\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]\right)$$

How to formulate Dirac equation in curved space?

H. Weyl, Z. Phys. 56, 330 (1929)
Proc. Nat. Acad. Sci. 15, 323 (1959)
V. Fock, Z. Phys. 57, 261 (1959).

Dirac spinor $\psi(x)$: 2-valued representation of Lorentz group $SO(1,3)$

$\left\{\begin{array}{l}\text{In flat space, spinor can be uniformly defined.}\\ \text{In curved space, spinor can only be defined with respect to } \underline{\text{local Cartesian frames}}\end{array}\right.$

local Cartesian frame at x:

$e^a{}_\mu(x)$, $a = 0, 1, 2, 3$ Cartesian index

vierbein field

$$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^a{}_\nu$$

$$\boxed{\begin{array}{l} e_a{}^\mu : \text{inverse of } e^a{}_\mu \\ e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu \\ e_a{}^\mu e^b{}_\mu = \delta_a{}^b \end{array}}$$

$$\{ \gamma_a , \gamma_b \} = 2 \eta_{ab} \quad , \quad \eta_{ab} = (1, -1, -1, -1)$$

$$\sigma_{ab} = \frac{i}{2} [\gamma_a , \gamma_b] .$$

The set of $\frac{1}{2} \sigma_{ab}$ satisfy <u>Lorentz algebra</u>:

$$\left[\frac{\sigma_{ab}}{2} , \frac{\sigma_{cd}}{2} \right] = i \left(\eta_{bc} \frac{\sigma_{ad}}{2} - \eta_{ac} \frac{\sigma_{bd}}{2} + \eta_{ad} \frac{\sigma_{bc}}{2} - \eta_{bd} \frac{\sigma_{ac}}{2} \right)$$

At each point x^μ, and with respect to local Cartesian frame $e^a{}_\mu(x)$, the is the local freedom of defining spinor:

$$\psi(x) \longrightarrow \psi'(x) = e^{-\frac{i}{4}\varepsilon^{ab}(x)\sigma_{ab}}\psi(x)$$

local Lorentz gauge freedom

To define a Lorentz covariant derivative, a connection field $\omega^{ab}{}_\mu(x)$ is introduced:

$$\boxed{D_\mu\psi(x) = \left(\partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu(x)\sigma_{ab}\right)\psi(x)}$$

Under a local Lorentz transformation

$$\psi(x) \longrightarrow e^{-\frac{i}{4}\varepsilon^{ab}(x)\sigma_{ab}}\psi(x),$$

the connection field $\omega^{ab}{}_\mu(x)$ is required to transform in such a way that

$$D_\mu\psi(x) \longrightarrow e^{-\frac{i}{4}\varepsilon^{ab}(x)\sigma_{ab}}D_\mu\psi(x),$$

leading to

$$\omega_\mu(x) \longrightarrow \omega'_\mu(x) = e^{-i\varepsilon(x)}\omega_\mu(x)e^{i\varepsilon(x)} - \left[i\partial_\mu e^{-i\varepsilon(x)}\right]e^{i\varepsilon(x)}.$$

where

$$\omega_\mu(x) \equiv \frac{1}{4}\sigma_{ab}\omega^{ab}{}_\mu(x),$$

$$\varepsilon_\mu(x) \equiv \frac{1}{4}\sigma_{ab}\varepsilon^{ab}(x).$$

Dirac Equation in curved space.

$$D_\mu \psi = (\partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab})\psi \quad \text{covariant}$$

γ^a : transf. according to "a".

$\gamma^a e_a{}^\mu$: " " to "μ".

$\bar{\psi} i \gamma^a e_a{}^\mu (\partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab})\psi$: invariant under $SO(1,3)$ and $GL(4)$. but <u>not</u> hermitian

It's hermitian adjoint is

$$-\bar{\psi}(\overleftarrow{\partial}_\mu + \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab}) i\gamma^a e_a{}^\mu \psi .$$

$$\therefore \quad \boxed{\begin{aligned} W &= \int d^4x \, h\Big[\tfrac{1}{2}\big(\bar{\psi} i\gamma^a e_a{}^\mu D_\mu \psi - \bar{\psi}\bar{D}_\mu i\gamma^a e_a{}^\mu \psi\big) \\ &\qquad\qquad - m\bar{\psi}\psi \Big] \\ D_\mu &= \partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab} \\ \bar{D}_\mu &= \overleftarrow{\partial}_\mu + \frac{i}{4}\omega^{ab}{}_\mu \sigma_{ab} \end{aligned}}$$

$$h = \det(e^a{}_\mu) = \sqrt{-g}$$

We have in the theory, in addition to $\psi(x)$,

$$\boxed{e^a{}_\mu(x), \quad \omega^{ab}{}_\mu(x)} \quad \text{independent field variables}$$

In terms of these field variables, what are the geometric entities

$$\boxed{g_{\mu\nu}(x), \ \Gamma^\lambda{}_{\mu\nu}(x)} \qquad ?$$

Already, we have the metric tensor

$$\boxed{g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu(x)\, e^b{}_\nu(x).}$$

What is the affine connection

$$\Gamma^\lambda{}_{\mu\nu}(x) \qquad ?$$

Recall :

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})$$

in General Relativity.

Geometry is Riemannian.

To proceed, we need to define covariant derivatives with respect to local Lorentz transf. and general coordinate transf. for a generic entity $\chi^a{}_\mu$:

$$\boxed{D_\nu \chi^a{}_\mu = \chi^a{}_{\mu;\nu} = \chi^a{}_{\mu,\nu} + \omega^a{}_{b\nu}\chi^b{}_\mu - \Gamma^\lambda{}_{\mu\nu}\chi^a{}_\lambda .}$$

Similarly,

$$\boxed{D_\nu \chi_a{}^\mu = \chi_a{}^\mu{}_{;\nu} = \chi_a{}^\mu{}_{,\nu} - \omega^b{}_{a\nu}\chi_b{}^\mu + \Gamma^\mu{}_{\lambda\nu}\chi_a{}^\lambda .}$$

To construct $\Gamma^\lambda{}_{\mu\nu}$ in terms of $e^a{}_\mu(x)$ and $\omega^{ab}{}_\mu(x)$, let's first impose the condition of "metricity" :

$$D_\lambda g^{\mu\nu} = D_\lambda g_{\mu\nu} = 0 ,$$

or, equivalently,

$$\boxed{D_\lambda e^a{}_\mu = D_\lambda e_a{}^\mu = 0 .}$$

$$D_\nu e^a{}_\mu = e^a{}_{\mu,\nu} + \omega^a{}_{b\nu} e^b{}_\mu - \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda$$

$$\overset{\|}{0} \Longrightarrow \boxed{\Gamma^\lambda{}_{\mu\nu} = e_a{}^\lambda \left(e^a{}_{\mu,\nu} + \omega^a{}_{b\nu} e^b{}_\mu\right)}$$

$$\boxed{\omega^{ab}{}_\mu = e^a{}_\lambda e^{b\nu} \Gamma^\lambda{}_{\nu\mu} - e^{b\nu} e^a{}_{\nu,\mu}}$$

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^{\rho}{}_{\mu\lambda} g_{\rho\nu} - \Gamma^{\rho}{}_{\nu\lambda} g_{\mu\rho} = 0 \qquad \times(-1/2)$$

$$g_{\nu\lambda;\mu} = g_{\nu\lambda,\mu} - \Gamma^{\rho}{}_{\nu\mu} g_{\rho\lambda} - \Gamma^{\rho}{}_{\lambda\mu} g_{\nu\rho} = 0 \qquad \times(+1/2)$$

$$g_{\lambda\mu;\nu} = g_{\lambda\mu,\nu} - \Gamma^{\rho}{}_{\lambda\nu} g_{\rho\mu} - \Gamma^{\rho}{}_{\mu\nu} g_{\lambda\rho} = 0 \qquad \times(+1/2)$$

Sum :

$$\frac{1}{2}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$$

$$- \underbrace{\frac{1}{2}(\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\nu\mu})}_{\Gamma^{\rho}{}_{\mu\nu} - \frac{1}{2}(\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu})} g_{\lambda\rho} - \frac{1}{2}(\Gamma^{\rho}{}_{\lambda\nu} - \Gamma^{\rho}{}_{\nu\lambda}) g_{\mu\rho} - \frac{1}{2}(\Gamma^{\rho}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\mu\lambda}) g_{\nu\rho}$$

$$= 0$$

$$\boxed{g^{\alpha\lambda} g_{\lambda\rho} = \delta^{\alpha}_{\rho}}$$

$$\therefore \quad \Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$$

$$+ \frac{1}{2} C^{\alpha}{}_{\mu\nu} + \frac{1}{2} C^{\rho}{}_{\nu\lambda} g_{\mu\rho} g^{\alpha\lambda} + \frac{1}{2} C^{\rho}{}_{\mu\lambda} g_{\nu\rho} g^{\alpha\lambda}$$

with $\boxed{\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu} \equiv C^{\alpha}{}_{\mu\nu}}$ torsion

Note that $\phi_{;\mu\nu} - \phi_{;\nu\mu} = (\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu})\phi_{;\lambda}$

$$= C^{\lambda}{}_{\mu\nu}\, \phi_{;\lambda}$$

$\therefore$ $C^{\lambda}{}_{\mu\nu}$: tensor

$$\boxed{\Gamma^{\lambda}{}_{\mu\nu} = \underbrace{\left\{ {\lambda \atop \mu\nu} \right\}}_{\text{Christoffel}} + \frac{1}{2}(C^{\lambda}{}_{\mu\nu} + C_{\mu\nu}{}^{\lambda} + C_{\nu\mu}{}^{\lambda})}$$

Riemannian : $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$

$$\therefore\ C^\lambda{}_{\mu\nu} = 0$$

$$\Gamma^\lambda{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} \text{ Christoffel}$$

$$\omega^{ab}{}_\mu = e^a{}_\lambda e^{b\nu} \left\{ {\lambda \atop \nu\mu} \right\} - e^{b\nu} e^a{}_{\nu,\mu}$$

$$\chi^a{}_{\lambda;\mu} = \chi^a{}_{\lambda,\mu} + \omega^a{}_{b\mu} \chi^b{}_\lambda - \Gamma^\rho{}_{\lambda\mu} \chi^a{}_\rho$$

$$\chi^a{}_{\lambda;\mu\nu} - \chi^a{}_{\lambda;\nu\mu} = \cdots\cdots$$

$$= (\omega^a{}_{b\mu,\nu} - \omega^a{}_{b\nu,\mu} + \omega^a{}_{c\nu} \omega^c{}_{b\mu} - \omega^a{}_{c\mu} \omega^c{}_{b\nu}) \chi^b{}_\lambda$$

$$- (\Gamma^\rho{}_{\lambda\mu,\nu} - \Gamma^\rho{}_{\lambda\nu,\mu} - \Gamma^\sigma{}_{\lambda\nu} \Gamma^\rho{}_{\sigma\mu} + \Gamma^\sigma{}_{\lambda\mu} \Gamma^\rho{}_{\sigma\nu}) \chi^a{}_\rho$$

$$- (\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}) \chi^a{}_{\lambda;\rho}$$

$$\equiv R^a{}_{b\mu\nu} \chi^b{}_\lambda - R^\rho{}_{\lambda\mu\nu} \chi^a{}_\rho - C^\rho{}_{\mu\nu} \chi^a{}_{\lambda;\rho}$$

$$\boxed{\begin{aligned} &\chi^a{}_{\lambda;\mu\nu} - \chi^a{}_{\lambda;\nu\mu} = R^a{}_{b\mu\nu} \chi^b{}_\lambda - R^\rho{}_{\lambda\mu\nu} \chi^a{}_\rho - C^\rho{}_{\mu\nu} \chi^a{}_{\lambda;\rho} \\ &R^a{}_{b\mu\nu} = \omega^a{}_{b\mu,\nu} - \omega^a{}_{b\nu,\mu} + \omega^a{}_{c\nu} \omega^c{}_{b\mu} - \omega^a{}_{c\mu} \omega^c{}_{b\nu} \\ &R^\rho{}_{\lambda\mu\nu} = \Gamma^\rho{}_{\lambda\mu,\nu} - \Gamma^\rho{}_{\lambda\nu,\mu} - \Gamma^\sigma{}_{\lambda\nu} \Gamma^\rho{}_{\sigma\mu} + \Gamma^\sigma{}_{\lambda\mu} \Gamma^\rho{}_{\sigma\nu} \end{aligned}}$$

$$\boxed{e_a{}^\lambda e_b{}^\rho R^{ab}{}_{\mu\nu} = \cdots = g^{\rho\sigma} R^\lambda{}_{\sigma\mu\nu}}$$

T. W. B. Kibble, JMP $\underline{2}$, 212 (1961)
D. W. Sciama, "Recent Development in G.R., 1962."

$$R = e_a{}^\mu e_b{}^\nu R^{ab}{}_{\mu\nu}$$

$$h = \det(e^a{}_\mu)$$

$$W = -\frac{c^3}{16\pi G}\int d^4x\, h\, R + W_{Dirac}$$

$$W_{Dirac} = \int d^4x\, h\left[\frac{1}{2}(\bar\psi i\gamma^a e_a{}^\mu D_\mu\psi - \bar\psi \bar D_\mu i\gamma^a e_a{}^\mu \psi) - m\bar\psi\psi\right]$$

(i) Invariance under general coord. transformations.
(ii) " " local Lorentz " .
(iii) Treat field variables in the theory

$$e^a{}_\mu,\ \omega^{ab}{}_\mu,\ \psi,\ \bar\psi$$

$e^a{}_\mu$: vierbein
$\omega^{ab}{}_\mu$: Lorentz connecti

as <u>independent</u>. They are determined by the resulting field equations.

A. T. Nieh and M. L. Yan, Ann. Phys. $\underline{138}$, 237 (1982).

Field Equations

(1) $\delta\psi$, $\delta\bar\psi$: Dirac equation

$$\delta\bar\psi : \quad \tfrac{1}{2} h i \gamma^a e_a{}^\mu D_\mu \psi + \tfrac{1}{2}\left(\partial_\mu - \tfrac{i}{4}\sigma_{cd}\omega^{cd}{}_\mu\right)(h i \gamma^a e_a{}^\mu \psi) - h m \psi =$$

$$\begin{aligned}
\partial_\mu (h e_a{}^\mu) &= h_{,\mu} e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= \left[h\, \partial_\mu \ln(\det e^a{}_\lambda)\right] e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= (h e_b{}^\lambda e^b{}_{\lambda,\mu}) e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= h \Gamma^\lambda{}_{\lambda\mu} e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= h(\Gamma^\lambda{}_{\mu\lambda} + C^\lambda{}_{\lambda\mu}) e_a{}^\mu + h e_a{}^\mu{}_{,\mu} \\
&= h\left(C^\lambda{}_{\lambda\mu} e_a{}^\mu + e_a{}^\mu{}_{;\mu} + \omega^b{}_{a\mu} e_b{}^\mu\right) \\
&= h\left(C^\lambda{}_{\lambda\mu} e_a{}^\mu + \omega^b{}_{a\mu} e_b{}^\mu\right)
\end{aligned}$$

$$[\sigma_{cd}\omega^{cd}{}_\mu, \gamma_a e^{a\mu}] = 4 i \omega^{ab}{}_\mu \gamma_a e_b{}^\mu$$

$$\boxed{\begin{gathered} i\gamma^a e_a{}^\mu \left(D_\mu + \tfrac{1}{2} C^\lambda{}_{\lambda\mu}\right)\psi - m\psi = 0 \\ D_\mu = \partial_\mu - \tfrac{i}{4}\sigma_{bc}\omega^{bc}{}_\mu \end{gathered}}$$

Dirac equation

(2) $\delta e_a{}^\mu$: Einstein equation

$$-\frac{c^3}{16\pi G}\left[-h\, e^a{}_\mu R + 2h\, e_b{}^\nu R^{ab}{}_{\mu\nu}\right]$$

$$+ \quad \cdots\cdots \qquad = 0$$

$$\Big\Downarrow \; e_{a\nu}\cdot \quad \text{Dirac eq.}$$

$$\boxed{\begin{aligned} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R &= \frac{8\pi G}{c^3} T_{\nu\mu} \\ T_{\nu\mu} &= \tfrac{1}{2}\left(\bar\psi i\gamma_\nu D_\mu \psi - \bar\psi \bar{D}_\mu i\gamma_\nu \psi\right) \end{aligned}} \qquad \gamma_\nu = e_{a\nu}\gamma^a$$

Note : $\begin{cases} R_{\mu\nu} \neq R_{\nu\mu} \\ T_{\nu\mu} \neq T_{\mu\nu} \end{cases}$ when torsion $\neq 0$.

(3) $\delta\omega^{ab}{}_\mu$: equation for Lorentz connection $\omega^{ab}{}_\mu$.

$$\frac{c^3}{8\pi G}\Big\{[h(e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)]_{,\nu} - \omega^d{}_{a\nu}\, h\,(e_d{}^\mu e_b{}^\nu - e_d{}^\nu e_b{}^\mu)$$
$$- \omega^d{}_{b\nu}\, h\,(e_a{}^\mu e_d{}^\nu - e_a{}^\nu e_d{}^\mu)\Big\}$$
$$+ h\,\frac{1}{4}\,\bar\psi\,(\gamma^\mu\sigma_{ab} + \sigma_{ab}\gamma^\mu)\psi = 0.$$

Let $\{[h(e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)]_{,\nu} - \cdots\cdots - \cdots\cdots\} \equiv h\, S_{ab}{}^\mu$.

$$\gamma^\mu\sigma_{ab} + \sigma_{ab}\gamma^\mu = 2\, e^{c\mu}\,\eta_{abcd}\,\gamma_5\gamma^d. \quad (\eta_{0123} = -1)$$

$$e_{c\mu}\, S_{ab}{}^\mu \equiv S_{abc}$$

$$\boxed{\therefore\ S_{abc} = -\frac{4\pi G}{c^3}\,\eta_{abcd}\,\bar\psi\gamma_5\gamma^d\psi}$$

On the other hand,

$$h\,(S_{abc} - S_{bca} - S_{cab} - \eta_{ac}\, S^d{}_{bd} - \eta_{bc}\, S_a{}^d{}_d)$$

$$= \cdots\cdots\cdots \qquad \boxed{e_a{}^\lambda\, e^a{}_{\lambda,\mu} = \frac{\partial}{\partial x^\mu}\ln\det e_a{}^\lambda = h^{-1} h_{,\mu}}$$

$$= 2h\,\omega_{abc}$$
$$+ h\big[-(e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)\, e_{c\mu,\nu}$$
$$+ (e_b{}^\mu e_c{}^\nu - e_b{}^\nu e_c{}^\mu)\, e_{a\mu,\nu}$$
$$+ (e_c{}^\mu e_a{}^\nu - e_c{}^\nu e_a{}^\mu)\, e_{b\mu,\nu}\big]$$

$$\boxed{\begin{aligned} \omega_{abc} &= \frac{1}{2}\left(c_{abc} - c_{bca} - c_{cab}\right) \\ &\quad + \frac{1}{2}\left(S_{abc} - S_{bca} - S_{cab} - \eta_{ac} S^{d}{}_{bd} - \eta_{bc} S_{a}{}^{d}{}_{d}\right) \\ c_{abc} &\equiv \left(e_a{}^{\mu} e_b{}^{\nu} - e_a{}^{\nu} e_b{}^{\mu}\right) e_{c\mu,\nu} \\ S_{abc} &= -\frac{4\pi G}{c^3}\, \eta_{abcd}\, \bar{\psi}\gamma_5\gamma^d\psi \end{aligned}}$$

$$\boxed{\begin{aligned} \omega_{abc} &= \overset{\circ}{\omega}_{abc} + \frac{2\pi G}{c^3}\, \eta_{abcd}\, \bar{\psi}\gamma_5\gamma^d\psi \\ \overset{\circ}{\omega}_{abc} &= \frac{1}{2}\left(c_{abc} - c_{bca} - c_{cab}\right) \end{aligned}}$$

Recall : $\Gamma^{\lambda}{}_{\mu\nu} = e_a{}^{\lambda}\left(e^{a}{}_{\mu,\nu} + \omega^{a}{}_{b\nu}\, e^{b}{}_{\mu}\right)$

$$\begin{aligned} \overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\nu} &= e_a{}^{\lambda}\left(e^{a}{}_{\mu,\nu} + \overset{\circ}{\omega}{}^{a}{}_{b\nu}\, e^{b}{}_{\mu}\right) \\ &= e_a{}^{\lambda}\left(e^{a}{}_{\mu,\nu} + \overset{\circ}{\omega}{}^{a}{}_{bc}\, e^{c}{}_{\nu}\, e^{b}{}_{\mu}\right) \\ &= \;\cdots\cdots \\ &= \frac{1}{2} g^{\lambda\rho}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right) = \left\{ {\lambda \atop \mu\nu} \right\} \end{aligned}$$

Christoffel

$$\boxed{\Gamma^{\lambda}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\nu} + \frac{2\pi G}{c^3}\, \varepsilon^{\lambda}{}_{\mu\nu\rho}\, \bar{\psi}\gamma_5\gamma^{\rho}\psi}$$

$$C^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} = \frac{4\pi G}{c^3}\, \varepsilon^{\lambda}{}_{\mu\nu\rho}\, \bar{\psi}\gamma_5\gamma^{\rho}\psi$$

$$R = g_\lambda{}^\mu g^{\rho\nu} R^\lambda{}_{\rho\mu\nu}$$

$$= g_\lambda{}^\mu g^{\rho\nu}\left[\Gamma^\lambda{}_{\rho\mu,\nu} - \Gamma^\lambda{}_{\rho\nu,\mu} - \Gamma^\lambda{}_{\sigma\mu}\Gamma^\sigma{}_{\rho\nu} + \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\rho\mu}\right]$$

$$\Gamma^\lambda{}_{\mu\nu} = \mathring{\Gamma}^\lambda{}_{\mu\nu} - \frac{1}{2}S^\lambda{}_{\mu\nu}, \qquad \mathring{\Gamma}^\lambda{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$$

$$R = g_\lambda{}^\mu g^{\rho\nu}\left[(\mathring{\Gamma}^\lambda{}_{\rho\mu} - \frac{1}{2}S^\lambda{}_{\rho\mu})_{,\nu} - (\mathring{\Gamma}^\lambda{}_{\rho\nu} - \frac{1}{2}S^\lambda{}_{\rho\nu})_{,\mu} - \cdots\cdots - \cdots\right]$$

$$= \cdots\cdots$$

$$= \mathring{R} + \frac{1}{4}S^{\lambda\mu\nu}S_{\lambda\mu\nu}, \qquad S_{\lambda\mu\nu} = -\frac{4\pi G}{c^3}\varepsilon_{\lambda\mu\nu\rho}\bar{\Psi}\gamma_5\gamma^\rho\psi$$

$$= \mathring{R} + \frac{1}{4}\left(-\frac{4\pi G}{c^3}\right)^2(-6)(\bar{\Psi}\gamma_5\gamma^a\psi)(\bar{\Psi}\gamma_5\gamma_a\psi)$$

$$\frac{1}{2}\left(\bar{\Psi} i\gamma^a e_a{}^\mu D_\mu\psi - \bar{\Psi}\bar{D}_\mu i\gamma^a e_a{}^\mu\psi\right)$$

$$D_\mu = \partial_\mu - \frac{i}{4}\omega^{ab}{}_\mu\sigma_{ab}$$

$$= \partial_\mu - \frac{i}{4}(\mathring{\omega}^{ab}{}_\mu - \frac{1}{2}S^{ab}{}_\mu)\sigma_{ab} = \mathring{D}_\mu + \frac{i}{8}S^{ab}{}_\mu\sigma_{ab}$$

$$\bar{D}_\mu = \cdots\cdots = \bar{\mathring{D}}_\mu - \frac{i}{8}S^{ab}{}_\mu\sigma_{ab}$$

$$\therefore \quad \frac{1}{2}\left(-\frac{1}{8}\right)e_a{}^\mu S_{bc\mu}(\gamma^a\sigma^{bc} + \sigma^{bc}\gamma^a)$$

$$= \left(-\frac{1}{8}\right)\left(-\frac{4\pi G}{c^3}\right)(-6)(\bar{\Psi}\gamma_5\gamma^a\psi)(\bar{\Psi}\gamma_5\gamma_a\psi)$$

$$W = -\frac{c^3}{16\pi G}\int d^4x\, h\, R$$

$$+\int d^4x\, h\left[\tfrac{1}{2}\left(\bar\psi i\gamma^a e_a{}^\mu D_\mu\psi - \bar\psi \bar D_\mu i\gamma^a e_a{}^\mu\psi\right) - m\bar\psi\psi\right]$$

$$= -\frac{c^3}{16\pi G}\int d^4x\, h\left[\mathring{R} - \frac{3}{2}\left(-\frac{4\pi G}{c^3}\right)^2(\bar\psi\gamma_5\gamma^a\psi)(\bar\psi\gamma_5\gamma^a\psi)\right]$$

$$+\int d^4x\, h\Big[\tfrac{1}{2}\left(\bar\psi i\gamma^a e_a{}^\mu \mathring{D}_\mu\psi - \bar\psi \mathring{\bar D}_\mu i\gamma^a e_a{}^\mu\psi\right) - m\bar\psi\psi$$

$$-3\frac{\pi G}{c^3}(\bar\psi\gamma_5\gamma^a\psi)(\bar\psi\gamma_5\gamma_a\psi)\Big]$$

$$\boxed{\begin{aligned} W = &-\frac{c^3}{16\pi G}\int d^4x\, h\, \mathring{R} \\ &+\int d^4x\, h\left[\tfrac{1}{2}\left(\bar\psi i\gamma^a e_a{}^\mu \mathring{D}_\mu\psi - \bar\psi \mathring{\bar D}_\mu i\gamma^a e_a{}^\mu\psi\right) - m\bar\psi\psi\right] \\ &-\frac{3}{2}\frac{\pi G}{c^3}\int d^4x\, h\,(\bar\psi\gamma_5\gamma^a\psi)(\bar\psi\gamma_5\gamma_a\psi) \end{aligned}}$$

Einstein eq. becomes :

$$\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} - g_{\mu\nu}\,12\left(\frac{\pi G}{c^3}\right)^2(\bar\psi\gamma_5\gamma^a\psi)(\bar\psi\gamma_5\gamma_a\psi)$$

$$= \mathring{T}_{\nu\mu}$$

cosmological
term ?

N.J. Poplawski (2011)

Tortional Topological Invariants

In Riemannian geometry, there are two well-known topological invariants:

Euler invariant:

$$\sqrt{-g}\, \varepsilon^{\mu\nu\lambda\rho}\, \varepsilon_{\alpha\beta\gamma\delta}\, R^{\alpha\beta}{}_{\mu\nu}\, R^{\gamma\delta}{}_{\lambda\rho} = \partial_\mu (\cdots\cdots)$$

Pontryagen invariant:

$$\sqrt{-g}\;\; \varepsilon^{\mu\nu\lambda\rho}\; R^{\alpha\beta}{}_{\mu\nu}\, R_{\alpha\beta\lambda\rho} = \partial_\mu (\cdots\cdots)$$

$\Longrightarrow$ Are they valid in Riemann - Cartan geometry?

$$\begin{cases} e^a{}_\mu \\ \omega^{ab}{}_\mu \end{cases} ; \qquad \begin{cases} g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu \\ \Gamma^\lambda{}_{\mu\nu} = e_a{}^\lambda (e^a{}_{\mu,\nu} + \omega^a{}_{b\nu}\, e^b{}_\mu) \end{cases}$$

$$R^{ab}{}_{\mu\nu} = \omega^{ab}{}_{\mu,\nu} - \omega^{ab}{}_{\nu,\mu} - \omega^a{}_{c\mu}\,\omega^{cb}{}_\nu + \omega^a{}_{c\nu}\,\omega^{cb}{}_\mu$$

$$\begin{aligned} e_a{}^\lambda e_b{}^\rho R^{ab}{}_{\mu\nu} &= \cdots\cdots \\ &= g^{\rho\sigma} (\Gamma^\lambda{}_{\sigma\mu,\nu} - \Gamma^\lambda{}_{\sigma\nu,\mu} - \Gamma^\lambda{}_{\alpha\mu}\Gamma^\alpha{}_{\sigma\nu} + \Gamma^\lambda{}_{\alpha\nu}\Gamma^\alpha{}_{\sigma\mu}) \\ &= g^{\rho\sigma} R^\lambda{}_{\sigma\mu\nu} \end{aligned}$$

$\frac{\sigma_{ab}}{2}$: generators for Lorentz transf. :

$$\boxed{\left[\frac{\sigma_{ab}}{2}, \frac{\sigma_{cd}}{2}\right] = i\left(\eta_{bc}\frac{\sigma_{ad}}{2} - \eta_{ac}\frac{\sigma_{bd}}{2} + \eta_{ad}\frac{\sigma_{bc}}{2} - \eta_{bd}\frac{\sigma_{ac}}{2}\right).}$$

Let $\quad \varpi_\mu = \frac{1}{4}\sigma_{ab}\,\omega^{ab}{}_\mu$.

Then.

$$\begin{aligned}
&\varpi_{\mu,\nu} - \varpi_{\nu,\mu} + i\,[\varpi_\mu, \varpi_\nu] \\
&= \frac{1}{4}\sigma_{ab}\left(\omega^{ab}{}_{\mu,\nu} - \omega^{ab}{}_{\nu,\mu} - \omega^{a}{}_{c\mu}\,\omega^{cb}{}_\nu + \omega^{a}{}_{c\nu}\,\omega^{cb}{}_\mu\right) \\
&= \frac{1}{4}\sigma_{ab}\,R^{ab}{}_{\mu\nu} \\
&\equiv \mathcal{R}_{\mu\nu}
\end{aligned}$$

Consider

$$\begin{aligned}
&\mathrm{Tr}\left(\gamma_5\,\mathcal{R}_{\mu\nu}\,\mathcal{R}_{\lambda\rho}\right) \\
&= \left(\frac{1}{4}\right)^2 \underbrace{\mathrm{Tr}\left(\gamma_5\,\sigma_{ab}\,\sigma_{cd}\right)}_{=\,4i\,\eta_{abcd}} R^{ab}{}_{\mu\nu}\,R^{cd}{}_{\lambda\rho} \\
&= \frac{i}{4}\,\eta_{abcd}\,R^{ab}{}_{\mu\nu}\,R^{cd}{}_{\lambda\rho}
\end{aligned}$$

$$\sigma_{ab}\sigma_{cd} = \frac{1}{2}[\sigma_{ab}, \sigma_{cd}] + \frac{1}{2}\{\sigma_{ab}, \sigma_{cd}\}$$

$$\frac{1}{2}[\sigma_{ab}, \sigma_{cd}] = i(\eta_{bc}\sigma_{ad} - \eta_{ac}\sigma_{bd} + \eta_{ad}\sigma_{bc} - \eta_{bd}\sigma^{ac})$$

$$\frac{1}{2}\{\sigma_{ab}, \sigma_{cd}\} = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} + i\eta_{abcd}\gamma_5$$

$$\eta_{0123} = -1$$

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\alpha\beta\gamma\delta}\,R^{\alpha\beta}{}_{\mu\nu}\,R^{\gamma\delta}{}_{\lambda\rho}$$

$$\downarrow \quad \varepsilon_{\alpha\beta\gamma\delta} = e^a{}_\alpha e^b{}_\beta e^c{}_\gamma e^d{}_\delta\,\eta_{abcd}$$

$$= \sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\eta_{abcd}\,R^{ab}{}_{\mu\nu}\,R^{cd}{}_{\lambda\rho}$$

$$\downarrow \quad \mathrm{Tr}\,(\gamma_5\,\sigma_{ab}\,\sigma_{cd}) = 4i\,\eta_{abcd}\ .\ \eta_{0123} = -1.$$

$$= -4i\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\,(\gamma_5\,\mathcal{R}_{\mu\nu}\,\mathcal{R}_{\lambda\rho})$$

$$= -16i\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\Big\{\gamma_5\big(\varpi_{\nu,\mu}\varpi_{\rho,\lambda} - i\,\varpi_{\nu,\mu}\varpi_\lambda\varpi_\rho - i\,\varpi_\mu\varpi_\nu\varpi_{\rho,\lambda} - \varpi_\mu\varpi_\nu\varpi_\lambda\varpi_\rho\big)\Big\}$$

$$\downarrow \quad \begin{aligned} &[\gamma_5, \varpi_\mu] = 0 \\ &\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}(\gamma_5\,\varpi_\mu\varpi_\nu\varpi_\lambda\varpi_\rho) = \cdots\cdots = 0 \\ &\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\ \partial_\mu(\varpi_\nu\varpi_\lambda\varpi_\rho) = 3\,\varepsilon^{\mu\nu\lambda\rho}\,\varpi_{\nu,\mu}\varpi_\lambda\varpi_\rho \end{aligned}$$

$$= -16i\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\Big\{\gamma_5\Big[(\partial_\mu\varpi_\nu)(\partial_\lambda\varpi_\rho) - i\,\frac{2}{3}\,\partial_\mu(\varpi_\nu\varpi_\lambda\varpi_\rho)\Big]\Big\}$$

$$\downarrow \quad \sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho} : \text{constant}$$

$$= -16i\,\partial_\mu\Big\{\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\Big[\gamma_5\Big(\varpi_\nu\,\partial_\lambda\varpi_\rho - i\,\frac{2}{3}\,\varpi_\nu\varpi_\lambda\varpi_\rho\Big)\Big]\Big\}$$

Euler

Similarly,

$$\sqrt{-g}\ \varepsilon^{\mu\nu\lambda\rho}\ R^{\alpha\beta}{}_{\mu\nu}\ R_{\alpha\beta\lambda\rho}$$

$$\Big\downarrow\quad \mathrm{Tr}(\sigma_{ab}\sigma_{cd}) = 4(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})$$

$$= 2\sqrt{-g}\ \varepsilon^{\mu\nu\lambda\rho}\ \mathrm{Tr}(\mathcal{R}_{\mu\nu}\mathcal{R}_{\lambda\rho})$$

$$= \quad \cdots\cdots$$

$$= \partial_\mu\left\{ 8\sqrt{-g}\ \varepsilon^{\mu\nu\lambda\rho}\ \mathrm{Tr}\left[\varpi_\nu\partial_\lambda\varpi_\rho - i\frac{2}{3}\varpi_\nu\varpi_\lambda\varpi_\rho\right]\right\}$$

Pontryagin

MacDowell and Mansouri
PRL $\underline{38}$, 739 (1977).

de Sitter : $SO(1,3,\pm 1)$, $\frac{5\times 4}{2} = 10$ generators

$A, B = 0,1,2,3,5$, $\eta_{AB} = (1,-1,-1,-1,\pm 1)$

de Sitter algebra :

$$\frac{i}{2}[\sigma_{AB}, \sigma_{CD}] = \eta_{AC}\sigma_{BD} - \eta_{AD}\sigma_{BC} + \eta_{BD}\sigma_{AC} - \eta_{BC}\sigma_{AD}$$

Now, $\begin{cases} \omega^{ab}{}_\mu : 6 \\ e^a{}_\mu : 4 \end{cases}$ $\quad 6+4=10$

Let

$$\omega^{AB}{}_\mu = \begin{cases} \omega^{ab}{}_\mu , & \text{for } A,B = 0,1,2,3 \\ \omega^{a5}{}_\mu = \frac{1}{l} e^a{}_\mu , & \text{for } A, \cancel{B} = 0,1,2,3 \\ & \quad B = 5 \end{cases}$$

l: constant, dim. of length.

$\gamma^0, \gamma^1, \gamma^2, \gamma^3$: $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$

$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$: $(\gamma_5)^2 = 1$, $\{\gamma^5, \gamma^a\} = 0$.

$$\gamma_A = \begin{cases} (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5) & : \{\gamma_A, \gamma_B\} = 2\eta_{AB} = (0,-1,-1,-1,1) \\ (\gamma_0, \gamma_1, \gamma_2, \gamma_3, i\gamma_5) & : \{\gamma_A, \gamma_B\} = 2\eta_{AB} = (0,-1,-1,-1,-1) \end{cases}$$

In either case, $\sigma_{AB} \equiv \frac{i}{2}[\gamma_A, \gamma_B]$ satisfy the algebra.

Define $\Omega_\mu \equiv \frac{1}{4} \sigma_{AB} \omega^{AB}{}_\mu$

$$\Omega_{\mu,\nu} - \Omega_{\nu,\mu} + i[\Omega_\mu, \Omega_\nu] \equiv \frac{1}{4} \sigma_{AB} F^{AB}{}_{\mu\nu}$$
$$= \mathcal{F}_{\mu\nu}$$

$$F^{AB}{}_{\mu\nu} = \omega^{AB}{}_{\mu,\nu} - \omega^{AB}{}_{\nu,\mu} - \eta_{CD}\, \omega^{AC}{}_\mu \omega^{DB}{}_\nu + \eta_{CD}\, \omega^{AC}{}_\nu \omega^{DB}{}_\mu$$

$$F^{AB}{}_{\mu\nu} = \begin{cases} F^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} + \frac{1}{\ell^2} \eta_{55} (e^a{}_\mu e^b{}_\nu - e^a{}_\nu e^b{}_\mu) \\ F^{a5}{}_{\mu\nu} = \frac{1}{\ell} e^a{}_\lambda C^\lambda{}_{\mu\nu} \end{cases}$$

$$\sqrt{-g}\, \varepsilon^{\mu\nu\lambda\rho} F^{AB}{}_{\mu\nu} F_{AB\lambda\rho}$$
$$= 2\sqrt{-g}\, \varepsilon^{\mu\nu\lambda\rho}\, \mathrm{Tr}(\mathcal{F}_{\mu\nu} \mathcal{F}_{\lambda\rho})$$
$$= \cdots\cdots$$
$$= \partial_\mu \left\{ 8\sqrt{-g}\, \varepsilon^{\mu\nu\lambda\rho}\, \mathrm{Tr}\left(\Omega_\nu \partial_\lambda \Omega_\rho - i\tfrac{2}{3} \Omega_\nu \Omega_\lambda \Omega_\rho\right)\right\}$$

We had, for Lorentz $SO(1,3)$:

$$\sqrt{-g}\, \varepsilon^{\mu\nu\lambda\rho} R^{ab}{}_{\mu\nu} R_{ab\lambda\rho}$$
$$= \partial_\mu \left\{ 8\sqrt{-g}\, \varepsilon^{\mu\nu\lambda\rho}\, \mathrm{Tr}\left(\varpi_\nu \partial_\lambda \varpi_\rho - i\tfrac{2}{3} \varpi_\nu \varpi_\lambda \varpi_\rho\right)\right\}$$

Difference :

$$\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho} F^{AB}{}_{\mu\nu} F_{AB\lambda\rho} \qquad SO(1,3,\pm1) \text{ Pontryagin inv.}$$

$$-\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho} R^{ab}{}_{\mu\nu} R_{ab\lambda\rho} \qquad SO(1,3) \quad " \quad "$$

$$= \frac{1}{\ell^2}\eta_{55}\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\left(4R_{\mu\nu\lambda\rho} + 2C^{\alpha}{}_{\mu\nu}C_{\alpha\lambda\rho}\right).$$

$$\partial_\mu\{\cdots\cdots\} - \partial_\mu\{\cdots\cdots\} = \cdots\cdots$$

$$= \frac{1}{\ell^2}\eta_{55}\,\partial_\mu\left\{-4\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho} C_{\nu\lambda\rho}\right\}$$

$$\boxed{\begin{aligned} &\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\left(R_{\mu\nu\lambda\rho} + \tfrac{1}{2}C^{\alpha}{}_{\mu\nu}C_{\alpha\lambda\rho}\right) \\ &\qquad = \partial_\mu\left(-\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho} C_{\nu\lambda\rho}\right) \\ &\text{or,} \\ &-R_{ab}\wedge e^a\wedge e^b + C_a\wedge C^a = d(C_a\wedge e^a) \end{aligned}}$$

H.T. Nieh and M.L. Yan.
J. Math. Phys. 23, 373 (1982).

Additional Topological Invariants

H.T. Nieh, PRD 98, 104045 (2018)

$C^\lambda{}_{\mu\nu}$: torsion tensor, antisym. in μ + ν.

Define $\quad C_\mu{}^{ab} = e^a{}_\lambda e^b{}_\rho C_\mu{}^{\lambda\rho}$

$$\omega'^{ab}{}_\mu \equiv \omega^{ab}{}_\mu + \xi C_\mu{}^{ab}, \quad \xi: \text{arbitrary numerical coeff.}$$

Let $\quad \varpi_\mu = \frac{1}{4}\sigma_{ab}\omega^{ab}{}_\mu, \quad C_\mu = \frac{1}{4}\sigma_{ab} C_\mu{}^{ab}$

$$\varpi'_\mu = \frac{1}{4}\sigma_{ab}\,\omega'^{ab}{}_\mu$$

$$= \varpi_\mu + \xi C_\mu$$

Define

$$\mathcal{R}'_{\mu\nu} = \varpi'_{\mu,\nu} - \varpi'_{\nu,\mu} + i[\varpi'_\mu, \varpi'_\nu]$$

$$= \frac{1}{4}\sigma_{ab} R'^{ab}{}_{\mu\nu}$$

$$R'^{ab}{}_{\mu\nu} = \omega'^{ab}{}_{\mu,\nu} - \omega'^{ab}{}_{\nu,\mu} - \omega'^{ac}{}_\mu \omega'_c{}^b{}_\nu + \omega'^{ac}{}_\nu \omega'_c{}^b{}_\mu$$

$$= R^{ab}{}_{\mu\nu} + \xi\,(C^{ab}{}_{\mu\nu} + C_\lambda{}^{ab} C^\lambda{}_{\mu\nu})$$

$$+ \xi^2(-C_\mu{}^{ac} C_{\nu c}{}^b + C_\nu{}^{ac} C_{\mu c}{}^b)$$

where $C^{ab}{}_{\mu\nu} = C_\mu{}^{ab}{}_{,\nu} - C_\nu{}^{ab}{}_{,\mu}$.

Pontryagin-type identity:

$$\sqrt{-g}\;\varepsilon^{\mu\nu\lambda\rho}\,R'^{\alpha\beta}{}_{\mu\nu}\,R'_{\alpha\beta\lambda\rho}$$

$$= 2\sqrt{-g}\;\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(\mathcal{R}'_{\mu\nu}\,\mathcal{R}'_{\lambda\rho}\right)$$

$$= \cdots\cdots$$

$$= \partial_\mu\left\{8\sqrt{-g}\;\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(\bar{\varpi}'_\nu\,\partial_\lambda\bar{\varpi}'_\rho - i\frac{2}{3}\bar{\varpi}'_\nu\bar{\varpi}'_\lambda\bar{\varpi}'_\rho\right)\right\}$$

Expand both sides as series in ξ:

$$\sqrt{-g}\;\varepsilon^{\mu\nu\lambda\rho}\,R'^{\alpha\beta}{}_{\mu\nu}\,R'_{\alpha\beta\lambda\rho}$$

$$= \sqrt{-g}\;\varepsilon^{\mu\nu\lambda\rho}\Big\{R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta\lambda\rho} + \xi\,2R^{\alpha\beta}{}_{\mu\nu}\left(C_{\alpha\beta\lambda\rho} + C_{\sigma\alpha\beta}C^{\sigma}{}_{\lambda\rho}\right)$$

$$+\xi^2\Big[4R^{\alpha\beta}{}_{\mu\nu}C_{\lambda\alpha}{}^{\sigma}C_{\rho\beta\sigma} + \left(C^{\alpha\beta}{}_{\mu\nu} + C_{\sigma}{}^{\alpha\beta}C^{\sigma}{}_{\mu\nu}\right)\left(C_{\alpha\beta\lambda\rho} + C_{\eta\alpha\beta}C^{\eta}{}_{\lambda\rho}\right.$$

$$\left.+\xi^3\,4\left(C^{\alpha\beta}{}_{\mu\nu} + C_{\sigma}{}^{\alpha\beta}C^{\sigma}{}_{\mu\nu}\right)C_{\lambda\alpha}{}^{\gamma}C_{\rho\beta\gamma}\quad\right\}$$

$$\partial_\mu\left\{8\sqrt{-g}\;\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(\bar{\varpi}'_\nu\,\partial_\lambda\bar{\varpi}'_\rho - i\frac{2}{3}\bar{\varpi}'_\nu\bar{\varpi}'_\lambda\bar{\varpi}'_\rho\right)\right\}$$

$$= \partial_\mu\Big\{8\sqrt{-g}\;\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\Big[\left(\bar{\varpi}_\nu\,\partial_\lambda\bar{\varpi}_\rho - \frac{i\,2}{3}\bar{\varpi}_\nu\bar{\varpi}_\lambda\bar{\varpi}_\rho\right)$$

$$+\xi\left(\bar{\varpi}_\nu\,\partial_\lambda C_\rho + C_\nu\,\partial_\lambda\bar{\varpi}_\rho - 2i\,C_\nu\bar{\varpi}_\lambda\bar{\varpi}_\rho\right)$$

$$+\xi^2\left(C_\nu\partial_\lambda C_\rho - 2iC_\nu C_\lambda\bar{\varpi}_\rho\right) + \xi^3\left(-i\frac{2}{3}C_\nu C_\lambda C_\rho\right)\Big]\Big\}$$

Equate terms of the same ξ power:

$$* \quad \sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho} R^{\alpha\beta}{}_{\mu\nu}\left(C_{\alpha\beta\mu\nu} + C_{\sigma\alpha\beta} C^{\sigma}{}_{\mu\nu}\right)$$

$$= \partial_\mu \left\{ 4\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\, \mathrm{Tr}\left(\varpi_\nu \partial_\lambda C_\rho + C_\nu \partial_\lambda \varpi_\rho - 2i\, C_\nu \varpi_\lambda \varpi_\rho\right)\right\}$$

$$* \quad \sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\left[4R^{\alpha\beta}{}_{\mu\nu} C_{\lambda\alpha}{}^{\sigma} C_{\rho\beta\sigma} + \left(C^{\alpha\beta}{}_{\mu\nu} + C_{\gamma}{}^{\alpha\beta} C^{\gamma}{}_{\mu\nu}\right)\left(C_{\alpha\beta\lambda\rho} + C_{\delta\alpha\beta} C^{\delta}{}_{\lambda\rho}\right)\right]$$

$$= \partial_\mu \left\{ 8\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\, \mathrm{Tr}\left(C_\nu \partial_\lambda C_\rho - 2i\, C_\nu C_\lambda \varpi_\rho\right)\right\}$$

$$* \quad \sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\left(C^{\alpha\beta}{}_{\mu\nu} + C_{\sigma}{}^{\alpha\beta} C^{\sigma}{}_{\mu\nu}\right) C_{\lambda\alpha}{}^{\gamma} C_{\rho\beta\gamma}$$

$$= \partial_\mu \left\{ -\frac{4i}{3}\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\, \mathrm{Tr}\left(C_\nu C_\lambda C_\rho\right)\right\}$$

$$= \partial_\mu \left\{ -\frac{2}{3}\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\, C_{\nu\alpha}{}^{\beta} C_{\lambda\beta\gamma} C_\rho{}^{\gamma\alpha}\right\}$$

Euler-type identity:

$$\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\alpha\beta\gamma\delta}\,R'^{\alpha\beta}{}_{\mu\nu}\,R'_{\alpha\beta\lambda\rho}$$
$$= \partial_\mu\left\{-16i\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left[\gamma_5\left(\varpi'_\nu\partial_\lambda\varpi'_\rho - i\tfrac{2}{3}\varpi'_\nu\varpi'_\lambda\varpi'_\rho\right)\right]\right\}$$

Similarly, expand both sides as power series in ξ and equating terms of the ξ power, giving rise to 3 additional identities:

* $$\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\alpha\beta\gamma\delta}\,R^{\alpha\beta}{}_{\mu\nu}\left(C^{\gamma\delta}{}_{\lambda\rho} + C_\sigma{}^{\gamma\delta}C^\sigma{}_{\lambda\rho}\right)$$
$$= \partial_\mu\left\{-8i\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left[\gamma_5\left(\varpi_\nu\partial_\lambda C_\rho + C_\nu\partial_\lambda\varpi_\rho - 2i\,C_\nu\varpi_\lambda\varpi_\rho\right)\right]\right\}$$

* $$\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\alpha\beta\gamma\delta}\left[4R^{\alpha\beta}{}_{\mu\nu}C_\lambda{}^{\gamma\sigma}C_\rho{}^{\delta}{}_{\sigma} + \left(C^{\alpha\beta}{}_{\mu\nu} + C_\sigma{}^{\alpha\beta}C^\sigma{}_{\mu\nu}\right)\left(C^{\gamma\delta}{}_{\lambda\rho} + C_\eta{}^{\gamma\delta}C^\eta{}_{\lambda\rho}\right)\right]$$
$$= \partial_\mu\left\{-16i\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left[\gamma_5\left(C_\nu\partial_\lambda C_\rho - 2i\,C_\nu C_\lambda\varpi_\rho\right)\right]\right\}$$

* $$\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\alpha\beta\gamma\delta}\left(C^{\alpha\beta}{}_{\mu\nu} + C_\sigma{}^{\alpha\beta}C^\sigma{}_{\mu\nu}\right)C_\lambda{}^{\gamma\eta}C_\rho{}^{\delta}{}_{\eta}$$
$$= \partial_\mu\left[-\tfrac{8}{3}\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}\left(\gamma_5\,C_\nu C_\lambda C_\rho\right)\right]$$
$$= \partial_\mu\left[-\tfrac{2}{3}\sqrt{-g}\,\varepsilon^{\mu\nu\lambda\rho}\,C_{\nu\alpha}{}^{\beta}\,C_{\lambda\beta\gamma}\,{}^*C_\rho{}^{\gamma\alpha}\right]$$

where ${}^*C_\rho{}^{\gamma\alpha} = \varepsilon^{\gamma\alpha\beta\delta}\,C_{\rho\beta\delta}$