On "Time Crystals"

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Refs: Alfred Shapere and Frank Wilczek: 1202.2537

暗能量↔ 能量“案”?

Introduction

What is a crystal?

- A state of the relavent quantum fields which corresponds to a local minimum of energy (i.e. a local ground state);
- Space translation and/or rotation groups are spontaneously broken to some discrete subgroups;
- time translation is preserved;

Introduction

What is a time crystal?

- Time crystal is analogous to usual crystal in that it is still a local ground state of the relavent dynamical degrees of freedoms;
- Unlike the usual crystals, time translation symmetry is spontaneously broken';

Introduction

- In terms of infinitesimal transformations, a dynamical degree of freedom ϕ transforms under space and time translations as $\delta\phi = \epsilon\,\partial_x\phi$ and $\delta\phi = \epsilon\,\partial_t\phi$ respectively. So spontaneous breaking of space translation implies $\partial_x\phi \neq 0$ and spontaneous breaking of time translation implies $\partial_t\phi \neq 0$ in the ground state;

Introduction

- It is quite easy to construct models of time-independent, conservative dynamical systems with local ground states in which $\partial_x \phi \neq 0$. Examples:

$$V_1(\phi) = -\kappa_1 \frac{d\phi}{dx} + \frac{\lambda_1}{2}\left(\frac{d\phi}{dx}\right)^2,$$

$$V_2(\phi) = -\frac{\kappa_1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\lambda_2}{4}\left(\frac{d\phi}{dx}\right)^4,$$

where all Greek coefficients are positive. The ground states are respectively $\frac{d\phi_1}{dx} = \frac{\kappa_1}{\lambda_1}$ and $\frac{d\phi_2}{dx} = \pm\sqrt{\frac{\kappa_2}{\lambda_2}}$. In either cases the space translation is spontaneously broken;

Introduction

- Note that these are only potential energies. Kinetic energy terms are omitted because in the ground states these naturally vanish;
- The question is: can we construct models of time-independent, conservative dynamical systems with ground states which spontaneously breaks the time translation symmetry?
- Shapere and Wilczek have pronounced a positive answer:

ScienceNews
MAGAZINE OF THE SOCIETY FOR SCIENCE & THE PUBLIC

Crystals may be possible in time as well as space

Theory proposes objects in their lowest energy state can loop in the fourth dimension forever

By Alexandra Witze
March 24th, 2012; Vol.181 #6 (p. 8)

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What sounds like the title of a bad fantasy movie — time crystals — could be the next big thing in theoretical physics.

In two new papers, Nobel Prize–winning physicist Frank Wilczek lays out the mathematics of how an object moving in its lowest energy state could experience a sort of structure in time. Such a "time crystal" would be the temporal equivalent of an everyday crystal, in which atoms occupy positions that repeat periodically in space.

The work, done partly with physicist Alfred Shapere of the University of Kentucky, appeared February 12 on arXiv.org.

"We don't know whether such things do exist in nature, but the surprise is that they can exist," says Maulik Parikh, a physicist at Arizona State University in Tempe.

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"Time Crystals" Could Be a Legitimate Form of Perpetual Motion

Physicists explore the concept that cold states of matter can form repeated patterns in time

By Ron Cowen | February 27, 2012 | 14

Wilczek describes his work in this article and in this one coauthored by Alfred Shapere of the University of Kentucky, that he posted on the physics preprint server, arXiv.org, on February 12.

"The papers themselves are perfectly respectable, undoubtedly correct, and interesting," says cosmologist Sean Carroll of the California Institute of Technology.

Known for his pioneering work in developing quantum chromodynamics, the theory that explains how the particles inside atomic nuclei stick together, Wilczek, a professor at the Massachusetts Institute of Technology, says he got his latest idea two years ago while teaching a course on group theory. That branch of mathematics, which uses matrices to describe the symmetries inherent in families of elementary particles, also describes and classifies the structure of crystals. Materials such as a liquid or a gas in equilibrium, made of uniformly distributed particles, exhibit perfect spatial symmetry—they look the same everywhere and in every direction.

Simple model building and analysis

- But things look subtler than it appears in their paper. Let's see.
- The first, most naive models which may possibly yield spontaneous breaking of time translation symmetry in the ground state are given by changing the spacial derivatives in V_1 and V_2 given above into time derivatives, i.e.

$$L_1(\phi) = -\kappa_1\dot{\phi} + \frac{\lambda_1}{2}\dot{\phi}^2,$$
$$L_2(\phi) = -\frac{\kappa_1}{2}\dot{\phi}^2 + \frac{\lambda_2}{4}\dot{\phi}^4.$$

These are assumed to be the Lagrangians of certain dynamical systems;

Simple model building and analysis

- The associated energy functions are respectively

$$E_1(\phi, \dot{\phi}) = \frac{\lambda_1}{2}\dot{\phi}^2,$$

$$E_2(\phi, \dot{\phi}) = -\frac{\kappa_1}{2}\dot{\phi}^2 + \frac{3\lambda_2}{4}\dot{\phi}^4.$$

- E_1 is minimized at $\dot{\phi} = 0$, hence there is no spontaneous breaking of time translation in the ground state; E_2 is minimized at $\dot{\phi} = \pm\sqrt{\kappa_2/3\lambda_2}$, so it seems that there is indeed spontaneous breaking of time translation in Model 2;
- The condition $\dot{\phi} \neq 0$ at the ground state seems to imply that the system undergoes perpetual motion in its lowest energy state;
- There are a number of related conceptual issues .....

Simple model building and analysis

- Consider a generic Hamiltonian system. The energy $H = H(q, p)$ is a function in phase space. The ground state condition (i.e. the condition for H to take its extremal value) is

$$\frac{\partial H}{\partial q} = \frac{\partial H}{\partial p} = 0.$$

This, combined with the standard (canonical) Hamiltonian equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}$$

indicates that the ground state necessarily satisfies $\dot{q} = \dot{p} = 0$, so perpetual motion in the ground state looks impossible;

Simple model building and analysis

- However, Model 2 is NOT canonical. Let $\kappa_2 = \kappa, \lambda_2 = 1$, we have

$$p \equiv \frac{\partial L_2}{\partial \dot{\phi}} = \dot{\phi}^3 - \kappa\dot{\phi},$$

$$H_2(\phi, p) \equiv p\dot{\phi} - L_2 = p\dot{\phi} + \frac{\kappa}{2}\dot{\phi}^2 - \dot{\phi}^4.$$

  The Hamiltonian H_2 is not written in terms of the phase space variable (ϕ, p) and indeed it cannot be, because $\dot{\phi}$ regarded as function of p is *multivalued*;

- At the time translation symmetry breaking ground states, the energy function has turning points where it is not differentiable with respects to p;

Simple model building and analysis

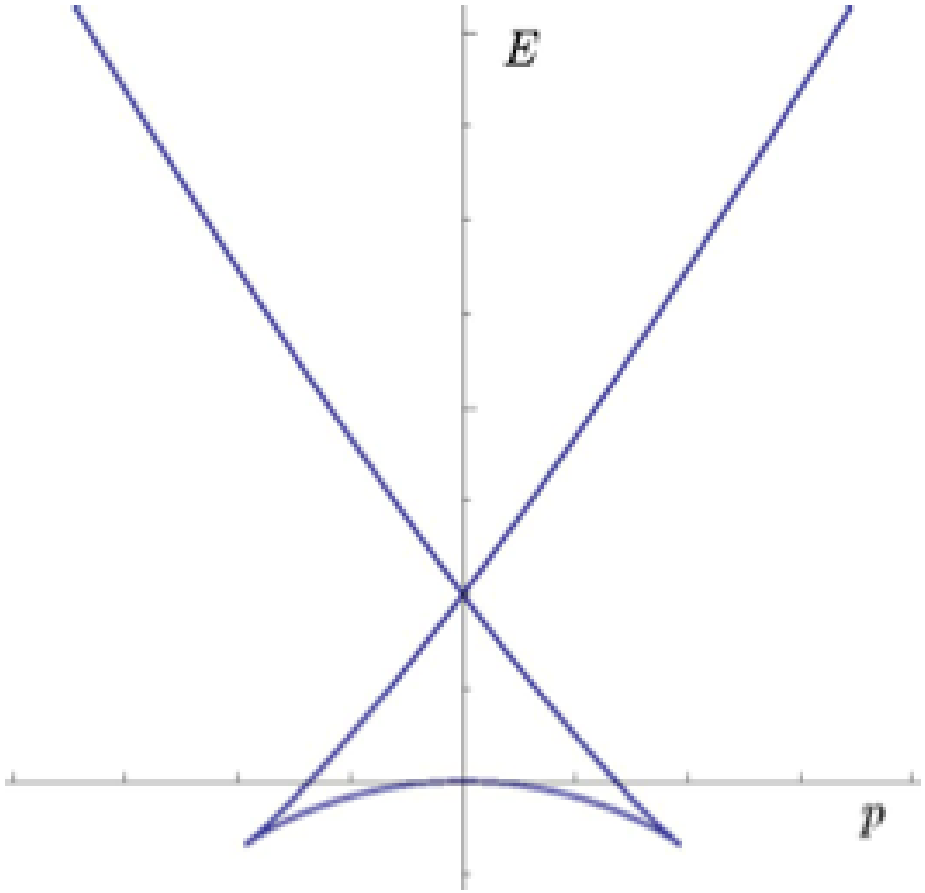

FIG. 1: Energy is a multivalued function of momentum.

Simple model building and analysis

- Moreover, for Hamiltonian systems written in non-canonical variables, the correct Hamiltonian equation of motion should NOT be written in the form $\dot{p} = -\frac{\partial H}{\partial q}, \dot{q} = \frac{\partial H}{\partial p}$ but rather $\dot{q} = \{q, H\}, \dot{p} = \{p, H\}$. If H involves variables other than q, p, the evaluation of the Poisson bracket becomes a problem. — I'll come back to this point later;

Simple model building and analysis

- There is another issue even if we stick to Lagrangian formalism. Consider a generic Lagrangian system in which energy is minimized at nonzero velocity $\dot{\phi}_0^k \neq 0$. Then

$$0 = \left.\frac{\partial E}{\partial \dot{\phi}^j}\right|_{\dot{\phi}_0^k} = \left.\frac{\partial}{\partial \dot{\phi}^j}\left(\frac{\partial L}{\partial \dot{\phi}^k} - L\right)\right|_{\dot{\phi}_0^k}$$
$$= \left.\left(\frac{\partial^2 L}{\partial \dot{\phi}^k \partial \dot{\phi}^j}\right)\right|_{\dot{\phi}_0^k} \dot{\phi}_0^j.$$

- Comparing with the equations of motion

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}^k}\right) - \frac{\partial L}{\partial \phi^k} = \left(\frac{\partial^2 L}{\partial \dot{\phi}^k \partial \dot{\phi}^j}\right)\ddot{\phi}^j + \ldots$$

This leads to a difficulty if there are forces that require acceleration in the direction ϕ^j.

Simple model building and analysis

- Returning to the concrete models, we will encounter even severe problems. Suppose we will to add a potential $V(\phi)$ to L_2. Then minimizing V requires $\phi = \phi_0 = constant$, while minimizing the kinetic energy requires perpetual motion;
- However, this apparent contradiction can be resolved in certain cases, as will be shown below.

Simple model building and analysis

A model with nontrivial potential: fgh model

- Lagrangian:

$$L = f\dot{\phi}^4 + g\dot{\phi}^2 + h,$$

f, g, h are all functions of ϕ with $f > 0$ and $g < 0$;

- energy function:

$$\begin{aligned} E &= 3f\dot{\phi}^4 + g\dot{\phi}^2 - h \\ &= 3f\left(\dot{\phi}^2 + \frac{g}{6f}\right)^2 - \frac{g^2}{12f} - h; \end{aligned}$$

- Consistency condition:

$$\frac{g^2}{12f} + h = c = constant.$$

Simple model building and analysis

- The energy is minimized by

$$\dot{\phi} = \pm\sqrt{-g/6f},$$

without requiring ϕ to be constant, thus avoiding the contradiction described above;

- Setting $c = 0$ we can rewrite the energy function equation as

$$\dot{\phi}^2 + \frac{g}{6f} = \pm\sqrt{\frac{E}{3f}}.$$

Simple model building and analysis

- This can be regarded as the conservation of pseudo energy of an effective particle with mass $1/2$ and E-dependent energy

$$\tilde{E} = \dot{\phi}^2 + V(E, \phi),$$

$$V(E, \phi) = \frac{g}{6f} \mp \sqrt{\frac{E}{3f}}.$$

Simple model building and analysis

- At $\dot{\phi} = 0$, the potential $V(E, \phi)$ attains its maximum,

$$V_{max} = \frac{g}{6f} \mp \sqrt{\frac{E}{3f}}\Big|_{\phi_t},$$

  where ϕ_t is the coordinate at the turning point(s);

- At the ground state, $\dot{\phi} \neq 0$, so the motion is confined in a regime $V \leq V_{max}$.

Models with higher symmetry

- Consider the following "double sombrero" model:

$$L = \frac{1}{4}(\dot{\psi}_1^2 + \dot{\psi}_2^2 - \kappa)^2 + V(\psi_1, \psi_2).$$

The matrix

$$\frac{\partial^2 L}{\partial \dot{\psi}_1 \partial \dot{\psi}_2} = \begin{pmatrix} 3\dot{\psi}_1^2 + \dot{\psi}_2^2 - \kappa & 2\dot{\psi}_1\dot{\psi}_2 \\ 2\dot{\psi}_1\dot{\psi}_2 & \dot{\psi}_1^2 + 3\dot{\psi}_2^2 - \kappa \end{pmatrix}$$

has an eigenvector $\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix}$ of zero eigenvalue iff

$$v^2 = \dot{\psi}_1^2 + \dot{\psi}_2^2 - \kappa/3.$$

Models with higher symmetry

- If the potential V has a one-parameter family of degenerate minima, the minimum-energy solution will move along the trough of V at constant speed $\kappa/3$. Example:

$$V = -\frac{\mu}{2}(\dot{\psi}_1^2 + \dot{\psi}_2^2) + \frac{\lambda}{4}(\dot{\psi}_1^2 + \dot{\psi}_2^2)^2.$$

- Introducing $\psi_1 + i\psi_2 = \rho e^{i\phi} \equiv \varphi$, the model Lagrangian can be written as

$$L = \frac{1}{4}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 - \kappa)^2 + \frac{\mu}{2}\rho^2 - \frac{\lambda}{4}\rho^4.$$

Models with higher symmetry

- The energy function minimizes at constant $\rho \neq 0$ and $\dot{\phi} \neq 0$;
- Generally speaking, any Lagrangian with a kinetic term that is a polynomial in $\dot{\phi}, \dot{\rho}$ and ρ, and a potential energy depending only on ρ, will preserve the symmetry $\phi \to \phi + \eta$;

Models with higher symmetry

- The charge associated with the original symmetry is $Q = -i\int(\varphi^*\pi_{\varphi^*} - \varphi\pi_\varphi)$ depends only on $\dot{\phi}$ and ρ. Thus for states with constant non vanishing $\dot{\phi}$ and ρ, we have a constant charge density;
- If our system is embedded in a larger symmetry-conserving bath and undergoes a transition to the symmetry-breaking state, then the transition will necessarily be accompanied by radiation of an appropriate balancing charge;
- Though the time translation $\phi(t) \to \phi(t+\epsilon)$ and phase translation $\phi \to \phi + \eta$ are both broken by the constant $\dot{\phi}$ solution $\phi = \omega t + \beta$, the combined transformation with $\omega\epsilon + \eta = 0$ remains to be a symmetry.

Models with higher symmetry

Relativistic considerations:

- The energy function associated with the Lagrangian

$$L = ((\partial_0\phi)^2 - (\nabla\phi)^2)^n$$

reads

$$E = ((2n-1)(\partial_0\phi)^2 + (\nabla\phi)^2)((\partial_0\phi)^2 - (\nabla\phi)^2)^{n-1}.$$

For even n, this is unbounded from below. For odd n, the energy is semi-positive definite and has a zero at $(\partial_0\phi)^2 = (\nabla\phi)^2$, unless $n = 1$.

Models with higher symmetry

- Bounded energy require the leading term has odd n and positive coefficient;
- In relativistic context, a typical physical model involving higher powers of $\partial_0\phi$ is the k-inflation model (which might have some connection with the problem of dark energy), e.g. as presented in hep-th/9904075 with

$$\mathcal{L} = \frac{1}{2}K(\varphi)(\nabla\varphi)^2 + \frac{1}{4}L(\varphi)(\nabla\varphi)^4.$$

- Other examples involve higher derivative models of gravity, such as those discussed here previously.

Hamiltonian analysis of the simplest model

- In all models mentioned above, a common feature (or defect) is that the energy function could not be written as a single valued function in the phase space, at least near the ground states of interests. This is a signature that the coordinates used for presenting the Lagrangians might not be good phase space local coordinates near the ground states — needs careful Hamiltonian analysis!

Hamiltonian analysis of the simplest model

- Recall the Lagrangian of the simplest model – the Model 2:

$$L_2(\phi) = -\frac{\kappa}{2}\dot{\phi}^2 + \frac{\lambda}{4}\dot{\phi}^4.$$

- Such models might also be thought of as containing higher time derivatives, because the physically significant quantity is the action

$$I = \int L_2 dt,$$

which, after a partial integration, will contain time derivatives of ϕ of order up to 2;

Hamiltonian analysis of the simplest model

- Usually, a dynamical system containing higher order time derivatives is considered to be involving certain constraints. But in the original form of the Lagrangian, we see no constraints at all;
- As mentioned previously, the conjugate momentum

$$p = \frac{\partial L}{\partial \dot{\phi}} = \lambda \dot{\phi}^3 - \kappa \dot{\phi}$$

  is not a good variable for writing out the corresponding Hamiltonian uniquely;
- The energy as a function of $\phi, \dot{\phi}$ is not a useful quantity in Hamiltonian description of the system;

Hamiltonian analysis of the simplest model

- To provide a remedy to the above subtle situations, we rewrite the Lagrangian in a different form,

$$L = -\frac{\kappa\rho^2}{2} + \frac{\lambda}{4}\rho^4 + \gamma(\rho - \dot{\phi}).$$

  New coordinates ρ, γ are introduced so that γ plays the role of a Lagrangian multiplier and solving ρ in terms of $\dot{\phi}$ will give us the original Lagrangian L_2;

- This Lagrangian contains only the time derivative of ϕ but not of the other two coordinates. The time derivative of ϕ appears only linearly. So this is a constrained system;

Hamiltonian analysis of the simplest model

- The apparent 6D phase space spanned by $(\rho, \gamma, \phi, \pi_\rho, \pi_\gamma, \pi_\phi)$ (which we relabel as $(q_1, q_2, q_3, p_1, p_2, p_3)$ in the following) does not possess a well defined symplectic structure;
- To remedy this, we employ the Dirac procedure. The set of primary constraints is presented by the definitions of the canonical conjugate momenta,

$$G_1 = p_3 + q_2 \simeq 0,$$
$$G_2 = p_2 \simeq 0,$$
$$G_3 = p_1 \simeq 0.$$

Hamiltonian analysis of the simplest model

- The time evolution of these with respect to the total Hamiltonian

$$H_{total} \equiv H + \sum_{i=1}^{3} \mu_i G_i$$
$$= \frac{\kappa {q_1}^2}{2} - \frac{\lambda}{4} {q_1}^4 - q_2 q_1 + \mu_1 p_1 + \mu_2 p_2 + \mu_3 (p_3 + q_2)$$

gives rise to a new secondary constraint

$$G_4 = -\kappa q_1 + \lambda q_1^3 + q_2 \simeq 0,$$

and it turns out that there is no further constraints and all the constraints $G_1 \sim G_4$ are of the second class.

Hamiltonian analysis of the simplest model

- Standard procedure yields the final Dirac Poisson bracket. The only remaining degrees of freedom are (q_1, q_3) (and the other phase space variables are either constrained to be zero or related to $q_1 and q_3$ thanks to the constraints) with

$$\{q_1, q_3\}_{DB} = \frac{1}{\kappa - 3\lambda {q_1}^2}$$

and of course $\{q_1, q_1\}_{DB} = \{q_3, q_3\}_{DB} = 0$.

Hamiltonian analysis of the simplest model

- The final Hamiltonian on the reduced phase space spanned by (q_1, q_3) is given by

$$H_{fin} = -\frac{1}{2}\kappa {q_1}^2 + \frac{3}{4}\lambda {q_1}^4,$$

and the Hamiltonian equations of motion are

$$\dot{q}_3 = q_1, \qquad \dot{q}_1 = 0.$$

- The minima of the final Hamiltonian are located at

$$q_1 = \pm\sqrt{\kappa/3\lambda},$$

which undergoes no perpetual motion since $\dot{q}_1 = 0$.

Hamiltonian analysis of the simplest model

- A remaining subtle point is that (q_1, q_3) are NOT the Darboux coordinates on the reduced phase space. Recall that Darboux theorem states that the symplectic structure of any symplectic manifold can be locally written as $\omega = dq \wedge dp$, with (q, p) being the Darboux coordinates;
- The correct Darboux coordinates in the present case are

$$q \equiv \kappa q_1 - \lambda q_1^3, \qquad p \equiv q_3.$$

Note that p is NOT associated with the velocity $\dot{q}$ since the latter is always zero.

Hamiltonian analysis of the simplest model

- The final Hamiltonian is independent of p, but it is multivalued again in q! Anyway the ground state will no longer suffer from spontaneous breaking of time translation symmetry;
- Breaking of time translation symmetry is conceptually uncomfortable because of the conservation of energy. Recall that Noether theorem states that conserved charges are associated with continuous symmetry of dynamical systems rather than with discrete symmetry of the system.

Discussion

- We have carried out similar analysis of several of the previously mentioned models, all lead to similar conclusion that in the reduced phase space with correct local coordinates, the ground state undergoes no perpetual notion;
- There is an accompany paper by Wilczek on Quantum Time Crystals (arXiv:1202.2539). Due to the problems we encountered in the classical case, I'll give no comment on the quantum considerations.
- Scientists combined with media sometimes produce stunning news, but we should always be cautious!

Discussion

- One may, of course, make a change of coordinates $q_1 \to \tilde{q}_1 = q_3, q_3 \to \tilde{q}_3 = -q_1$ and reinterpret $\tilde{q} = \tilde{q}_1$ and $\tilde{p} = -\kappa\tilde{q}_3 + \lambda\tilde{q}_3^3$ with $\{\tilde{q}, \tilde{p}\}_{DB} = 1$ as the Darboux coordinate on the reduced phase space. Doing so the final Hamiltonian will read

$$H_{fin} = -\frac{1}{2}\kappa\tilde{q}_3^2 + \frac{3}{4}\lambda\tilde{q}_3^4,$$

which is independent of $\tilde{q}$ and is multivalued in terms of $\tilde{p}$. The Hamiltonian equations of motion now read

$$\dot{\tilde{q}}_3 = 0, \qquad \dot{\tilde{q}}_1 = -\tilde{q}_3.$$

Discussion

- At the ground state, $\dot{\tilde{q}} = -\tilde{q}_3 = \pm\sqrt{\kappa/3\lambda} \neq 0$, so it seems perpetual motion reappears again!
- However, the ground state value of $\tilde{p}$ is exactly zero! So, the paradox becomes: what is the appropriate description of motion? Velocity or momentum? From the very structure of Hamiltonian dynamics, I would prefer that nontrivial motion should be encoded in non-vanishing momentum, rather than nonzero apparent velocity.
- Anyway, the ground state conditions $\frac{\partial H_{fin}}{\partial \tilde{q}_1} = \frac{\partial H_{fin}}{\partial \tilde{q}_3} = 0$ hold without encountering the problem of turning point singularity, if we stick to the non-Darboux coordinates $(\tilde{q}_1, \tilde{q}_3)$.