

# Modular Invariance and Higher Loop Amplitudes in String Perturbation Theory

Chuan-Jie Zhu

Physics Department, Renmin University of China  
Beijing, P. R. China

Hangzhou, April, 15th, 2012

---

# Contents

1. String Perturbation Theory
2. Superstring Theory: The 3 Different Formalisms
3. The Pure Spinor Formalism
4. Known Results at Tree, One-Loop and Two-Loops
5. Modular Invariance and Higher Loop Amplitudes

---

# 1. String Perturbation Theory

## String theory by path integral

**The string action:**

$$S[g, X] = -\frac{T}{2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu.$$

**could be quantized by the (Wick rotated) path integral:**

$$Z = \int \mathcal{D}g \mathcal{D}X e^{-S[g, X]}.$$

---

In order to do this let us first find what gauge fixing conditions are permissible. For an arbitrary Riemann surface, it is not always possible to take the conformal gauge globally.

Under an arbitrary local coordinate transformation

$$\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha = \tilde{\sigma}^\alpha(\sigma),$$

the metric  $g_{\alpha\beta}(\sigma)$  transform as follows:

$$g_{\alpha\beta}(\sigma) \rightarrow \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \sigma^\gamma}{\partial \tilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\beta} g_{\gamma\delta}(\sigma).$$

---

The possibility of taking the conformal gauge is that any metric  $g_{\alpha\beta}(\sigma)$  can be changed to the conformal gauge, giving rise to the following equation:

$$\frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha} \frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta} g_{\gamma\delta}(\sigma) = e^{\varphi(\sigma)} \delta_{\alpha\beta}.$$

By a simple counting of the freedoms, the number of independent functions matches. However this is not enough. We need to show that the required transformation is nonsingular, i.e. the Jacobian for passing from  $g_{\alpha\beta}$  to  $(\varphi, \tilde{\sigma}^\alpha)$  is non-zero. To show this

---

we consider an infinitesimal variation of the metric:

$$\delta g_{\alpha\beta} = \delta\varphi g_{\alpha\beta} + \nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha},$$

where  $\xi_{\alpha} = g_{\alpha\beta}\xi^{\beta}$  and  $\xi^{\beta} = \tilde{\sigma}^{\beta} - \sigma^{\beta}$ . The nonsingular nature of the transformation is proved if for any symmetric  $\delta g_{\alpha\beta}$  we can find  $\delta\varphi$  and  $\xi_{\alpha}$  such that the above equation will hold. In other words, we must be able to solve the equation

$$\delta\varphi g_{\alpha\beta} + \nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha} = \delta g_{\alpha\beta} \equiv \gamma_{\alpha\beta},$$

---

or

$$(P \xi)_{\alpha\beta} = \nabla_{\alpha} \xi_{\beta} + \delta g_{\alpha\beta} \nabla_{\beta} \xi_{\alpha} - g_{\alpha\beta} \gamma_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \gamma_{\delta}^{\delta}.$$

The question, whether the conformal gauge is always permissible, is reduced now to the possibility of solving these equations which we shall write symbolically as:

$$P \xi = \gamma.$$

Here we have denoted by  $P$  the differential operator which takes vector fields into traceless tensors (notice that the number of independent components is the

---

same). There exist a conjugate operator which acts in the opposite direction—transforming tensors into vectors. It is easy to see that this equation will be solvable if and only if the conjugate operator  $P^+$  doesn't have zero modes. On the other hand, the solution is not unique if  $P$  has zero modes.

So our conclusion is that zero modes of the operator  $P^+$  mean that the conformal gauge is not permissible, and zero modes of  $P$  mean that the conformal gauge is not unique (and one should further fix the remaining gauge freedom).



---

The number of zero modes is give by index theorem. We will not go into the details of these mathematics and only recall the results. We have

$$N_0(P) - N_0(P^+) = 3\chi = 6 - 6g,$$

where  $N_0$  denotes the number of zero modes,  $\chi$  is the Euler number of the Riemann surface and  $g$  is the genus. In particular, we have the following more precise results:

$$N_0(P) = 6, \quad N_0(P^+) = 0, \quad \text{for } g = 0 \text{ (sphere),}$$

---

$$N_0(P) = 2, \quad N_0(P^+) = 2, \quad \text{for } g = 1 \text{ (torus),}$$
$$N_0(P) = 0, \quad N_0(P^+) = 6g - 6, \quad \text{for } g \geq 2.$$

So we found that on a sphere we can always take a conformal gauge which is defined modulo  $SL(2, C)$  transformations (with **6** ( $= N_0(P)$ ) real parameters) which requires further gauge fixing, like the fixing of three out of four complex  $z_i$ ,  $i = 1, 2, 3, 4$  (the location of the inserted vertices) in the case of four-particle string amplitude at tree level. In the case of Riemann surface with higher genus we have topological

---

obstructions for the conformal gauge. The best thing which can be done is the following choice of gauge

$$g_{\alpha\beta}(\sigma) = g_{\alpha\beta}^{(0)}(\sigma; \tau_1, \tau_2, \dots, \tau_{6g-6}),$$

where  $g_{\alpha\beta}^{(0)}$  is a metric which depends on  $6g - 6$  (real) parameters and, e.g. which can be chosen to have constant negative curvature. Now the integration over all metrics (i.e. geometry) then reduces to integration over gauge orbit and also a  $6g - 6$  dimensional integration over  $\{\tau_i, i = 1, 2, \dots, 6g - 6\}$ —the moduli space of inequivalent metrics under coordinate and

---

conformal transformations. Our next task is to derive the explicit measure for such integration.

Before doing this, we mention an important mathematical result. Roughly speaking, this moduli space is a complex space. It is quite useful to use complex coordinates for this moduli space and also for the Riemann surface. In complex coordinates, the metric tensor on a Riemann surface is given by the components  $g_{z\bar{z}}$ ,  $g_{zz}$  and  $g_{\bar{z}\bar{z}}$ . We cover the whole Riemann surface by several patches and on each patch we can take the components  $g_{zz}$  and  $g_{\bar{z}\bar{z}}$  to **0**. Then

---

we have:

$$\delta g_{zz} = \nabla_z \xi_z,$$

$$\delta g_{\bar{z}\bar{z}} = \nabla_{\bar{z}} \xi_{\bar{z}},$$

$$\delta g_{z\bar{z}} = \delta\varphi g_{z\bar{z}} + g_{z\bar{z}} (\nabla^z \xi_z + \nabla^{\bar{z}} \xi_{\bar{z}}).$$

Here  $\nabla_z$  and  $\nabla_{\bar{z}}$  are covariant derivatives. Notice that the only non-vanishing components of the Christoffel symbol are

$$\Gamma_{zz}^z = g^{z\bar{z}} \partial_z g_{z\bar{z}} \quad \text{and} \quad \Gamma_{\bar{z}\bar{z}}^{\bar{z}} = g^{z\bar{z}} \partial_{\bar{z}} g_{z\bar{z}},$$

---

we have

$$\begin{aligned}\nabla_z \xi_z &= g_{z\bar{z}} \partial_z (g^{z\bar{z}} \xi_z), \\ \nabla^z \xi_z &= g^{z\bar{z}} \nabla_{\bar{z}} \xi_z = g^{z\bar{z}} \partial_{\bar{z}} \xi_z,\end{aligned}$$

and

$$\begin{aligned}\nabla_z \xi_z &= g_{z\bar{z}} \partial_z (g^{z\bar{z}} \xi_z) = \gamma_{zz}, \\ \nabla_{\bar{z}} \xi_{\bar{z}} &= g_{z\bar{z}} \partial_{\bar{z}} (g^{z\bar{z}} \xi_{\bar{z}}) = \gamma_{\bar{z}\bar{z}}.\end{aligned}$$

From these equations we see that the zero modes of the operator  $P$  are holomorphic or anti-holomorphic

---

vectors. On the other hand, the operator  $P^+$  reduces to

$$(P V)_{zz} = -\nabla^z V_{zz} = g^{z\bar{z}} \nabla_{\bar{z}} V_{zz} = g^{z\bar{z}} \partial_{\bar{z}} V_{zz},$$

$$(P V)_{\bar{z}\bar{z}} = -\nabla^{\bar{z}} V_{\bar{z}\bar{z}} = g^{z\bar{z}} \nabla_z V_{\bar{z}\bar{z}} = g^{z\bar{z}} \partial_z V_{\bar{z}\bar{z}},$$

and its zero modes are holomorphic or anti-holomorphic tensors. We will denote these tensors as  $\phi_{zz}^i$  and  $\bar{\phi}_{\bar{z}\bar{z}}^i$ ,  $i = 1, 2, \dots, 3g - 3$ .  $\{\phi_{zz}^i, i = 1, 2, \dots, 3g - 3\}$  constitutes a basis of holomorphic 2-differentials.

From the above discussions we see that an arbitrary

---

variation of  $\delta g$  can be written in the following form:

$$\delta g_{zz} = \delta_z \xi_z + \sum_{i=1}^{3g-3} \delta \tau^i \phi_{zz}^i,$$

$$\delta g_{\bar{z}\bar{z}} = \nabla_{\bar{z}} \xi_{\bar{z}} + \sum_{i=1}^{3g-3} \delta \bar{\tau}^i \bar{\tau} \bar{\phi}_{\bar{z}\bar{z}}^i,$$

$$\delta g_{z\bar{z}} = \delta_\varphi g_{z\bar{z}} + g_{z\bar{z}} (\nabla^z \xi_z + \nabla^{\bar{z}} \xi_{\bar{z}}) \equiv \delta \tilde{\varphi} g_{z\bar{z}}.$$

The path integral measure  $\mathcal{D}g$  is defined by specifying an inner product in the tangent space of all metrics.



---

**A natural inner product is**

$$\|\delta g\|^2 = \frac{1}{2} \int d^2\sigma \sqrt{g} g^{\alpha\gamma} g^{\beta\delta} \delta g_{\alpha\beta} \delta g_{\gamma\delta}.$$

**By using the decomposition of the variation of the metric we have**

$$\begin{aligned} \|\delta g\|^2 &= \int d^2z g_{z\bar{z}} \left[ (g^{z\bar{z}} \delta g_{z\bar{z}})^2 + (g^{z\bar{z}})^2 \delta g_{zz} \delta g_{\bar{z}\bar{z}} \right] \\ &= \int d^2z \left[ g_{z\bar{z}} (\delta\tilde{\varphi})^2 + g^{z\bar{z}} \nabla_z \xi_z \nabla_{\bar{z}} \xi_{\bar{z}} \right] \end{aligned}$$

---


$$\begin{aligned}
& + \sum_{i,j=1}^{3g-3} \delta\tau^i \delta\bar{\tau}^j g^{z\bar{z}} \phi_{zz}^i \bar{\phi}_{\bar{z}\bar{z}}^j \Big] \\
= & \int d^2z g_{z\bar{z}} (\delta\tilde{\varphi})^2 + \int d^2z g_{z\bar{z}} \partial_{\bar{z}} \xi^z \partial_z \xi^{\bar{z}} \\
& + \sum_{i,j=1}^{3g-3} \delta\tau^i \delta\bar{\tau}^j \langle \phi^j, \phi^i \rangle.
\end{aligned}$$

**From this expression we get**

$$\mathcal{D}g = \mathcal{D}\tilde{\varphi} \mathcal{D}\xi^z \mathcal{D}\xi^{\bar{z}} d^{6g-6} \tau |\text{Det}' \partial_{\bar{z}}|^2 \det \langle \phi^i, \phi^j \rangle,$$

---

where prime in  $\text{Det}'\partial_{\bar{z}}$  denotes the omission of zero modes. If there is no anomaly for the conformal and coordinate transformations which is the case for  $D = 26$ , the integration over  $\tilde{\varphi}$ ,  $\xi^z$  and  $\xi^{\bar{z}}$  can be factorized out of the path integral and we have the partition function at genus  $g$ :

$$Z_g = \int_{M_g} d^{6g-6}\tau \int \mathcal{D}X |\text{Det}'\partial_{\bar{z}}|^2 \det\langle\phi^i, \phi^j\rangle e^{-S[g^{(0)}(\tau), X]}.$$

Notice that the appearance of the factor  $\det\langle\phi^i, \phi^j\rangle$  is associated with the decomposition of  $\delta g_{zz}$ . In other

---

words, we have chosen a particular gauge slice. We can also choose a different gauge slice, e.g.

$$\delta g_{zz} = \delta_z \xi'_z + \sum_{i=1}^{3g-3} \delta y^i \mu_{zz}^i,$$

where  $\mu_{zz}^i = g_{z\bar{z}} \mu_z^{i\bar{z}}$  and  $\mu_z^{i\bar{z}}$ 's are called Beltrami differentials. We have

$$\delta_z \xi'_z + \sum_{i=1}^{3g-3} \delta y^i \mu_{zz}^i = \delta_z \xi_z + \sum_{i=1}^{3g-3} \delta \tau^i \phi_{zz}^i.$$

---

Multiplying both sides with  $\bar{\phi}_{z\bar{z}}^j g^{z\bar{z}}$  and integrating over the whole Riemann surface we get

$$\delta\tau^i \langle \phi^j, \phi^i \rangle = \delta y^i \langle \phi^j, \mu^i \rangle.$$

This gives the following transformation rule for integration over (holomorphic)  $\tau$  and  $y$ :

$$d^{3g-3}\tau \det \langle \phi^i, \phi^j \rangle = d^{3g-3}y \det \langle \phi^i, \mu^j \rangle.$$

---

In terms of  $y$  we have:

$$Z_g = \int_{M_g} d^{6g-6}y \int \mathcal{D}X |\text{Det}' \partial_{\bar{z}}|^2 \frac{|\det \langle \phi^i, \mu^j \rangle|^2}{\det \langle \phi^i, \phi^j \rangle} e^{-S[g^{(0)}(y), X]}.$$

Following the standard Faddeev-Popov procedure and introducing anti-commuting fields  $c^z$  and  $c^{\bar{z}}$  for the gauge parameters  $\xi^z$  and  $\xi^{\bar{z}}$  of the general coordinate transformations,  $Z_g$  can be written as the following

---

form:

$$Z_g = \int_{M_g} d^{6g-6}y \int \mathcal{D}[c^z c^{\bar{z}} b_{zz} b_{\bar{z}\bar{z}} X] \\ \times \prod_{i=1}^{3g-3} |\langle \mu^i, b_{zz} \rangle|^2 e^{-S[g^{(0)}(y), X] - S_{gh}}.$$

**Amplitudes can be easily constructed. We have**

---


$$\begin{aligned}
A_{j_1, \dots, j_n}^g(k_1, \dots, k_n) &= \int_{M_g} d^{6g-6}y \int \mathcal{D}[c^z c^{\bar{z}} b_{zz} b_{\bar{z}\bar{z}} X] \\
&\times \prod_{i=1}^{3g-3} |\langle \mu^i, b_{zz} \rangle|^2 e^{-S[g^{(0)}(y), X] - S_{gh}} \\
&\times \prod_{i=1}^n \int d^2\sigma_i (\det g(\sigma_i))^{1/2} V_{j_i}(k_i, \sigma_i)
\end{aligned}$$

$V_j(k, \sigma)$  is a vertex operator describing a specific particle.



---

## 2. Superstring Theory: The 3 Different Formalisms

### 1) Green-Schwarz formalism:

- quantization only in light-cone gauge
- space-time super-symmetric string theory
- computation of tree amplitude is quite easy
- It has never been used for computing multi-particle and higher-loop amplitudes (dependence on insertion points)

---

## 2) Ramond-Neveu-Schwarz (RNS) formalism

- Spacetime supersymmetric only after GSO proj.
- Higher loops: summation over spin structure and modular invariance.
- Applied to multi-particle, higher-loop (2-loop, see below) and topological string theory amplitudes.

---

### 3) Berkovits' pure spinor formalism

- Lorentz covariant and manifestly spacetime supersymmetric (no summation over spin structures).
- All integer dimensional free fields on (ordinary) Riemann surface.
- Shortcoming: pure spinor constraints and very complicated composite  $\tilde{b}$  fields.
- A non-minimal pure spinor formalism.

---

## 3. The Pure Spinor Formalism

In Berkovits' pure spinor formalism:

- **Basic variables:**  $X^\mu(z, \bar{z})$ ,  $\theta^\alpha(z)$  and  $p_\alpha(z)$  (conjugate to  $\theta(z)$ ). (For NRS:  $X^\mu(z, \bar{z})$  **and**  $\psi^\mu(z)$ .)
- **Also introducing bosonic pure spinor ghost variable  $\lambda^\alpha$  (and their conjugates  $w_\alpha$ ):**

$$\lambda^\alpha \gamma_{\alpha\beta}^\mu \lambda^\beta = 0$$

**For NRS, the ghosts are  $b(z)$ ,  $c(z)$  and  $\beta(z)$ ,  $\gamma(z)$ .**

---

## The BRST operator

$$Q = \oint \lambda^\alpha d_\alpha$$

is used to impose the fermionic constraints:

$$d_\alpha = p_\alpha - \frac{1}{2}(\gamma^\mu \theta)_\alpha \left[ \partial X_\mu + \frac{1}{4} \partial \theta \gamma_\mu \theta \right] = 0$$

To insure Lorentz covariance,  $w_\alpha$  only appears in:

$$J = w_\alpha \lambda^\alpha, \quad N_{mn} = \frac{1}{2} w_\alpha (\gamma_{mn})^\alpha_\beta \lambda^\beta$$

---

## Loop amplitudes (Berkovits, hep-th/0406055)

- vanishing of the  $m = 0$   $N$ -particle amplitudes  $N \leq 3$
- 1- and 2-loop 4-particle amplitudes
- vanishing of the multi-loop 4-particle leading contribution (absence of the  $R^4$  term)
- a computable multi-loop 4-particle amplitude?

---

## Key points

- **Picture changing operators:**

- “picture-lowering” operator:  $Y_C = C_\alpha \theta^\alpha \delta(C_\beta \lambda^\beta)$
- “picture-raising” operator:

$$Z_B = \frac{1}{2} B_{mn} \lambda \gamma^{mn} d \delta(B^{pq} N_{pq}), \quad Z_J = \lambda^\alpha d_\alpha \delta(J)$$

- **A construction of the “b-ghost fields”:**

$$\{Q, b(z)\} = T(z)$$

---


$$\{Q, \tilde{b}_B(z, w)\} = T(z)Z_B(w)$$

$$\tilde{b}_B(z, w) = b_B(z) + T(z) \int_z^w du B_{pq} \partial N^{pq}(u) \delta(BN(u))$$

**The construction of the very complicated  $b_B(z)$  field is as follows:**

$$b_B = G^\alpha Z_\alpha + H^{\alpha\beta} Z_{\alpha\beta} - K^{\alpha\beta\gamma} Z_{\alpha\beta\gamma} - L^{\alpha\beta\gamma\delta} Z_{\alpha\beta\gamma\delta} + b_4^{(b)}$$

$$b_4^{(b)} = B_{ab} \left[ -T N^{ab} - \frac{1}{4} J \partial N^{ab} + \frac{1}{4} N^{ab} \partial J \right]$$



---


$$\begin{aligned}
& \left. + \frac{1}{2} N^a{}_c \partial N^{bc} \right] \delta(BN) \\
& = (T N + J \partial N + N \partial J + N \partial N) \delta(BN)
\end{aligned}$$

**Schematically the other terms are:**

$$\begin{aligned}
b_1 &= G^\alpha Z_\alpha = G^\alpha d \delta(BN), \\
b_2 &= H^{\alpha\beta} Z_{\alpha\beta} = H^{\alpha\beta} (\Pi \delta(BN) + dd \delta'(BN)) \\
b_3 &= K^{\alpha\beta\gamma} (\partial \theta \delta(BN) + \Pi d \delta'(BN) + ddd \delta''(BN)), \\
b_4^{(a)} &= L^{\alpha\beta\gamma\delta} (\partial \theta d \delta'(BN) + \Pi \Pi \delta'(BN) \\
& \quad + \Pi d d \delta''(BN) + d d d d \delta^{(3)}(BN))
\end{aligned}$$

---


$$\begin{aligned}
G^\alpha &= \Pi d + N \partial \theta + J \partial \theta + \partial^2 \theta \\
H^{\alpha\beta} &= \Pi N + \Pi J + \partial \Pi + dd, \\
K^{\alpha\beta\gamma} &= Nd + Jd + \partial d, \\
L^{\alpha\beta\gamma\delta} &= NN + NJ + JJ + \partial N + \partial J.
\end{aligned}$$

**The most important property of the  $b$ -field is that every term has engineering dimension 4 or less, except the term  $TN\delta(BN)$ .  $\{\lambda, \theta, \Pi, d, w\}$  have engineering dimension  $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ . ( $N$  has 2 and  $T$  has 4.)**

- 
- Integration over  $\theta^\alpha$  and  $p_\alpha$  requires a  $16 + 16g$  zero modes to give a non-vanishing result.

$$\begin{aligned}
\mathcal{A} = & \int d^2\tau_1 \dots d^2\tau_{3g-3} \langle \mid \prod_{P=1}^{3g-3} \int d^2u_P \mu_P(u_P) \tilde{b}_{B_P}(u_P, z_P) \\
& \times \prod_{P=3g-2}^{10g} Z_{B_P}(z_P) \\
& \times \prod_{R=1}^g Z_J(v_R) \prod_{I=1}^{11} Y_{C_I}(y_I) \mid^2 \prod_{T=1}^N \int d^2t_T U_T(t_T) \rangle
\end{aligned}$$

---

$$U = e^{ik \cdot X} (\partial \theta^\alpha A_\alpha(\theta) + \Pi^m A_m(\theta) + d_\alpha W^\alpha(\theta) + \frac{1}{2} N^{mn} \mathcal{F}_{mn}(\theta))$$

- Counting of the  $d_\alpha$  zero modes:
  - The massless vertex operator can give at most 1 zero modes of  $p_\alpha$ .
  - Each picture-raising operator  $Z_B$  gives at most 1 zero modes of  $p_\alpha$ .
  - Each “b-ghost fields” can give at most 4 zero modes of  $p_\alpha$ , but there are other restrictions.

---

**Picture changing operators contribute  $8g + 3$   $d$ -field. The engineering dimensions of  $3g - 3 \tilde{b}_B$ -fields is  $12(g - 1)$ . If there are also  $M \delta'(BN)$ , the total is:**

$$12(g - 1) + 2M.$$

**These can be used to obtain**

$$8(g - 1) + \frac{4M}{3}$$

**$d$ -fields. For  $M = 3$ , this is  $8g - 4$ .**

**For  $M = 4$ , this is  $8g - 3 + \frac{1}{3}$ . So there is an extra  $\frac{1}{2}$**

---

engineering dimension. This can be only the  $\theta$  field.

Only  $M = 3, 4$  contributes.

$$b = c_0 \delta(BN) + \sum_{i=1}^3 c_i \delta^{(i)}(BN),$$

$$\begin{aligned} c_0 = & \Pi dd + N d\partial\theta + J d\partial\theta + d\partial^2\theta \\ & + (N + J + \partial) \Pi\Pi + \Pi dd \\ & + N d\partial\theta + J d\partial\theta + \partial d\partial\theta \\ & (TN + J\partial N + N\partial J + N\partial N), \end{aligned}$$

---


$$\begin{aligned}
c_1 &= (N\Pi + J\Pi + \partial\Pi + dd) dd \\
&\quad (N dd + J dd + \partial dd)\Pi \\
&\quad (NN + JN + JJ + \partial N + \partial J)(d\partial\theta + \Pi\Pi), \\
c_2 &= (N + J + \partial) dddd \\
&\quad + (NN + JN + JJ + \partial N + \partial J) \Pi dd, \\
c_3 &= (NN + JN + JJ + \partial N + \partial J) dddd.
\end{aligned}$$

---

## 4. Known Results at Tree, 1-Loop and 2-Loops

The  $n$   $(NS, \widetilde{NS})$  particle amplitudes:

$$\begin{aligned} i\mathcal{A}_n(k_i, \epsilon_i) &= \int \prod_{i=4}^n d^2 z_i \langle [c\mathcal{V}_B^{(-1)}](z_1, k_1, \epsilon_1) [c\mathcal{V}_B^{(0)}](z_2, k_2, \epsilon_2) \\ &\quad \times [c\mathcal{V}_B^{(-1)}](z_3, k_3, \epsilon_3) \prod_{i=4}^n [c\mathcal{V}_B^{(0)}](z_i, k_i, \epsilon_i) \\ &\quad \times (\text{right-moving part}) \rangle, \end{aligned}$$

To compute the 4-particle we do need the right-moving part to get the full amplitude.



---

## One-loop amplitudes: the massless 4-particle case

$$\mathcal{A}_4^{1\text{-loop}} = g_4^{1\text{-loop}} K(k_i, \epsilon_i) \int_F \frac{d^2\tau}{(\text{Im}\tau)^2} \int \prod_{i=1}^4 \frac{d^2z_i}{\text{Im}\tau} \\ \times \prod_{r<s} \left| \frac{\Theta_1(z_{rs}|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\text{Im}\tau}(\text{Im}z_{rs})^2\right) \right|^{\alpha' k_r \cdot k_s}$$

You may fix one  $z_i$  to an arbitrary point.  $K(k_i, \epsilon_i)$  is a kinematic factor and  $s, t, u$  are Mandelstam variables:  
 $s = -(k_1 + k_2)^2, \dots$

---

## The 2-loop 4-particle amplitude:

$$\begin{aligned}
 \mathcal{A}_{II} &\sim \int \frac{1}{T^5} \frac{\prod_{i=1}^6 d^2 a_i}{dV_{pr} |\prod_{i<j} a_{ij}|^2} \prod_{i=1}^4 \frac{d^2 z_i}{|y(z_i)|^2} \prod_{i<j} e^{-k_i \cdot k_j \langle X(z_i) X(z_j) \rangle} \\
 &\quad \times |s(z_1 z_2 + z_3 z_4) + t(z_1 z_4 + z_2 z_3) + u(z_1 z_3 + z_2 z_4)|^2 \\
 dV_{pr} &= \frac{d^2 a_i d^2 a_j d^2 a_k}{|a_{ij} a_{ik} a_{jk}|^2}, \quad T = \int \frac{d^2 z_1 d^2 z_2 |z_1 - z_2|^2}{|y(z_1) y(z_2)|^2}, \\
 \langle X(z_i) X(z_j) \rangle &\equiv G(z_i, z_j) = -\ln |E(z_i, z_j)|^2 \\
 &\quad + 2\pi (\text{Im } \Omega)_{IJ}^{-1} \left( \text{Im} \int_{z_i}^{z_j} \omega_I \right) \left( \text{Im} \int_{z_i}^{z_j} \omega_J \right)
 \end{aligned}$$

---

A better but equivalent form derived by **D' Hoker and Phong (hep-th/0501197)**:

$$\begin{aligned}
 \mathbf{A}_{II}(\epsilon_i, k_i) &= \frac{K \bar{K}}{2^{12} \pi^4} \int \frac{|\prod_{I \leq J} d\Omega_{IJ}|^2}{(\det \text{Im } \Omega)^5} \\
 &\quad \times \int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp\left(-\sum_{i < j} k_i \cdot k_j G(z_i, z_j)\right) \\
 \mathcal{Y}_S &= +(k_1 - k_2) \cdot (k_3 - k_4) \Delta(z_1, z_2) \Delta(z_3, z_4) + \dots \\
 &\quad \propto s(z_1 z_2 + z_3 z_4) + t(z_1 z_4 + z_2 z_3) + u(z_1 z_3 + z_2 z_4) \\
 \Delta(z, w) &\equiv \omega_1(z) \omega_2(w) - \omega_1(w) \omega_2(z)
 \end{aligned}$$

---

## 5. Modular Invar. and Higher Loop Amplitudes

**Modular transformation:**

$$\omega_i \rightarrow \tilde{\omega}_i = \omega_j (C\Omega + D)_{ji}^{-1},$$

$$\Omega_{ij} \rightarrow \tilde{\Omega}_{ij} = ((A\Omega + B)(C\Omega + D))_{ij}^{-1},$$

**The scalar Green function:**

$$G(z, w) = -\ln |E(z, w)|^2 + 2\pi\Im \int_z^w \omega_i (\Im\Omega)_{ij}^{-1} \Im \int_z^w \omega_j$$

---

is modular invariant.

**Some other modular invariant combinations:**

$$\int \mu_I(u, \bar{u}) \omega_j(u) \langle \partial X(u) \prod_{i=1}^4 e^{ik_i \cdot X(z_i)} \rangle \prod_{i=1}^4 \omega(z_i)$$

**Only  $\det(\omega_i(z_j))$  is a covariant object under modular transformation. This is antisymmetric under  $z_i \leftrightarrow z_j$ .**