On Primary Relations at Tree-Level in String Theory and Field Theory

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Introduction

- Color decomposition

  Gauge field amplitudes at tree-level have the color decomposition form

$$M(1^{a_1}, 2^{a_2}, ..., N^{a_N}) = \sum_{\sigma \in S_N} Tr(T^{a_{\sigma_1}} ... T^{a_{\sigma_N}}) A(\sigma_1, ..., \sigma_N). \quad (1)$$

  - In string theory this formula can be obtained by adding Chan-Paton factors to the ends of open strings.
  - In field theory field theory, one can derive this formula from Feynman diagrams(Fig. 1).

  The color decomposition is useful in studying multi-gluon amplitudes.

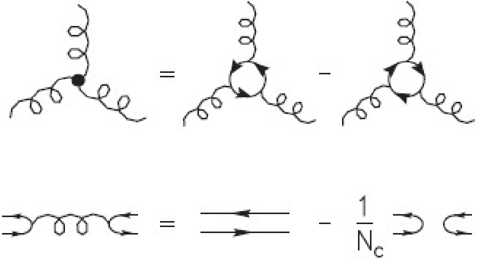

Figure 1: Diagrammatic equations for simplifying $SU(N_c)$ color algebra. Curly lines ("gluon propagators") represent adjoint indices, oriented solid lines ("quark propagators") represent fundamental indices, and "quark-gluon vertices" represent the generator matrices $(T^a)_i^{\,\bar{\jmath}}$.

Figure: 1. Color decomposition from Feynman diagrams

- Cyclic symmetry, KK relation and BCJ relation
  - Cyclic symmetry

$$A(1,2,...,N)=A(N,1,...,N-1). \tag{2}$$

  - KK-BCJ relation for open string tree-amplitudes(String theory proof: Bjerrum-Bohr,Damgaard and Vanhove'09; Stieberger'09)

$$\begin{aligned}\mathcal{V}(\beta;\alpha|N) &\equiv A_o(\beta_s,...,\beta_1,\alpha_1,...,\alpha_r,N)+(-1)^{s-1}\\ &\times \sum_{\sigma\in P(O\{\alpha\}\cup O\{\beta\}),\sigma_1=\alpha_1} \mathcal{P}_{\{\beta^T,\alpha,N\},\{\sigma,N\}}A_o(\sigma,N)=0.\end{aligned} \tag{3}$$

    Here we define the momentum kernel

$$\mathcal{P}_{\{\sigma\},\{\tau\}}=e^{-2i\pi\alpha'\sum\limits_{i,j}k_i\cdot k_j\left[\theta(\sigma^{-1}(i)-\sigma^{-1}(j))\theta(\tau^{-1}(j)-\tau^{-1}(i))\right]}. \tag{4}$$

    - Real part⇒ KK relation(Kleiss, Kuijf'89)
    - Imaginary part ⇒ BCJ relation(Bern,Carrasco,Johansson'08)
    - $\alpha'\to 0$ ⇒ KK and BCJ relations in field theory(field theory proofs: Duca,Dixon,Maltoni'99; Feng, Huang, Jia'10; Chen, Du, Feng'11)
  - The reduction of the number of independent amplitudes:
    - Cyclic symmetry $N!\to(N-1)!$
    - KK relation $(N-1)!\to(N-2)!$
    - BCJ relation $(N-2)!\to(N-3)!$

Generating KK and BCJ relations by two primary relations

- From primary relations to $U(1)$-like decoupling identity
  If $s = 1$, (3) becomes $U(1)$-like decoupling identity.
  - Real part $\Rightarrow$ fundamental KK relation

$$
\begin{aligned}
& \mathcal{K}(\beta_1; \alpha_1, ..., \alpha_r | N) \equiv A_o(\beta_1, \alpha_1, \alpha_2, ..., \alpha_r, N) \\
+ \quad & \cos[2\pi\alpha' k_{\beta_1} k_{\alpha_1}] A_o(\alpha_1, \beta_1, \alpha_2, ..., \alpha_r, N) \\
+ \quad & ... \\
+ \quad & \cos[2\pi\alpha'(k_{\beta_1} \cdot k_{\alpha_1} + k_{\beta_1} \cdot k_{\alpha_2} + ... + k_{\beta_1} \cdot k_{\alpha_r})] \\
& \times A_o(\alpha_1, \alpha_2, ..., \alpha_r, \beta_1, N) \\
= \quad & 0.
\end{aligned} \tag{5}
$$

  - Imaginary part $\Rightarrow$ fundamental BCJ relation

$$
\begin{aligned}
& \mathcal{B}(\beta_1; \alpha_1, ..., \alpha_r | N) \equiv \sin[2\pi\alpha' k_{\beta_1} k_{\alpha_1}] A_o(\alpha_1, \beta_1, \alpha_2, ..., \alpha_r, N) \\
+ \quad & \sin[2\pi\alpha'(k_{\beta_1} \cdot k_{\alpha_1} + k_{\beta_1} \cdot k_{\alpha_2})] A_o(\alpha_1, \alpha_2, \beta_1, ..., \alpha_r, N) \\
+ \quad & ... + \sin[2\pi\alpha'(k_{\beta_1} \cdot k_{\alpha_1} + k_{\beta_1} \cdot k_{\alpha_2} + ... + k_{\beta_1} \cdot k_{\alpha_r})] \\
& \times A_o(\alpha_1, \alpha_2, ..., \alpha_r, \beta_1, N) \\
= \quad & 0.
\end{aligned} \tag{6}
$$

If we consider the fundamental BCJ relation and the cyclic symmetry as the primary relations, the fundamental KK relation can be generated

$$\mathcal{K}(\beta_1;\alpha_1,...,\alpha_r|N) = -\frac{1}{\sin(\pi\alpha' s_{\beta_1\alpha_1})}\mathcal{B}(\beta_1;\alpha_2,...,\alpha_r,N|\alpha_1) + \cot(\pi\alpha' s_{\beta_1\alpha_1})\mathcal{B}(\beta_1;\alpha_1,...,\alpha_r|N). \quad (7)$$

If we consider the fundamental KK relation and the cyclic symmetry as the primary relations, the fundamental BCJ relation can be generated

$$\mathcal{B}(\beta_1;\alpha_1,...,\alpha_r|N) = \frac{1}{\sin(\pi\alpha' s_{\beta_1\alpha_1})}\mathcal{K}(\beta_1;\alpha_2,...,\alpha_r,N|\alpha_1) - \cot(\pi\alpha' s_{\beta_1\alpha_1})\mathcal{K}(\beta_1;\alpha_1,...,\alpha_r|N). \quad (8)$$

- From $U(1)$-like decoupling identity to generalized $U(1)$-like decoupling identity

  Generalized $U(1)$-like decoupling identity is

$$\mathcal{U}(\beta;\alpha|N) \equiv \sum_{\sigma\in P(O\{\beta_1,...,\beta_s\}\cup O\{\alpha_1,...,\alpha_r\})} \mathcal{P}_{\{\beta,\alpha,N\},\{\sigma,N\}} A_o(\sigma,N) = 0.$$

  It can be generated by $U(1)$-like decoupling identity. For example, if there are two βs. To see this we consider a linear combination of amplitude

$$\begin{aligned}&\mathcal{U}(\gamma_1,...,\gamma_t;\beta_1,...,\beta_s;\alpha_1,...,\alpha_r|N)\\ \equiv &\sum_{\tau\in P(O\{\alpha\}\cup O\{\beta\})} \mathcal{P}_{\{\gamma,\beta,\alpha,N\},\{\gamma,\tau,N\}} \mathcal{U}(\gamma_1,...,\gamma_t;\tau_1,...,\tau_{s+r}|N).\end{aligned} \tag{10}$$

  It can also be expressed as

$$\begin{aligned}&\mathcal{U}(\gamma_1,...,\gamma_t;\beta_1,...,\beta_s;\alpha_1,...,\alpha_r|N)\\ = &\sum_{\sigma\in P(O\{\gamma\}\cup O\{\beta\}\cup O\{\alpha\})} \mathcal{P}_{\{\gamma,\beta,\alpha,N\},\{\sigma,N\}} A(\sigma,N)\end{aligned}$$

and

$$
\begin{aligned}
& \mathcal{U}(\gamma_1, ..., \gamma_t; \beta_1, ..., \beta_s; \alpha_1, ..., \alpha_r|N) \\
= & \sum_{\tau \in P(O\{\gamma\} \cup O\{\beta\})} \mathcal{P}_{\{\gamma,\beta,\alpha,N\},\{\tau,\alpha,N\}} \mathcal{U}(\tau_1, ..., \tau_{t+s}; \alpha_1, ..., \alpha_r|N).
\end{aligned}
$$

Boundary condition is

$$
\mathcal{U}(\emptyset; \beta; \alpha|N) = \mathcal{U}(\beta; \emptyset; \alpha|N) = \mathcal{U}(\beta; \alpha|N). \tag{11}
$$

Using the properties of momentum kernel, we have

$$
\begin{aligned}
& e^{-2i\pi\alpha' k_{\beta_1} \cdot k_{\beta_2}} \mathcal{U}(\beta_1, \beta_2; \alpha_1, ..., \alpha_r|N) - \mathcal{U}(\beta_2; \beta_1; \alpha_1, ..., \alpha_r|N) \\
+ & \mathcal{U}(\beta_2, \beta_1; \alpha_1, ..., \alpha_r|N) = 0.
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
& e^{-2i\pi\alpha' k_{\beta_2} \cdot k_{\beta_1}} \mathcal{U}(\beta_2, \beta_1; \alpha_1, ..., \alpha_r|N) - \mathcal{U}(\beta_1; \beta_2; \alpha_1, ..., \alpha_r|N) \\
+ & \mathcal{U}(\beta_1, \beta_2; \alpha_1, ..., \alpha_r|N) = 0.
\end{aligned} \tag{13}
$$

From the equations above and the boundary condition, we get

$$\begin{aligned}
&\mathcal{U}(\beta_1,\beta_2;\alpha_1,...,\alpha_r|N)\\
=\ &\frac{1}{2i\sin(2\pi\alpha' k_{\beta_1}\cdot k_{\beta_2})}\Bigg[e^{2i\pi\alpha' k_{\beta_1}\cdot k_{\beta_2}}\mathcal{U}(\beta_1;\beta_2;\alpha_1,...,\alpha_r|N)\\
&-\mathcal{U}(\beta_2;\beta_1;\alpha_1,...,\alpha_r|N)\Bigg]. \qquad (14)
\end{aligned}$$

Since we have

$$\begin{aligned}
&\mathcal{U}(\beta_1;\beta_2;\alpha_1,...,\alpha_r|N)\\
=\ &\sum_{\tau\in P(O\{\alpha_1,...,\alpha_r\}\cup\{\beta_2\})}\mathcal{P}_{\{\beta_1,\beta_2,\alpha_1,...,\alpha_r,N\},\{\beta_1,\tau_1,...,\tau_{1+r},N\}}\\
&\times\mathcal{U}(\beta_1;\tau_1,...,\tau_{r+1}|N), \qquad (15)
\end{aligned}$$

The generalized $U(1)$-like decoupling identity with two βs are expressed as linear combinations of $U(1)$-like decoupling identity.

- From generalized $U(1)$-like decoupling identity to KK-BCJ relation

  All the KK-BCJ relations can be expressed by linear combinations of generalized $U(1)$-like decoupling identity. For the case with only one β,

$$\mathcal{U}(\beta_1;\alpha_1,...,\alpha_r|N)=\mathcal{V}(\beta_1;\alpha_1,...,\alpha_r|N)=0. \tag{16}$$

  We take the case with two βs as an example. $\mathcal{V}$ and $\mathcal{U}$ satisfy the relation

$$\begin{aligned} & \mathcal{V}(\beta_1,\beta_2;\alpha_1,...,\alpha_r|N)-\mathcal{V}(\beta_2;\beta_1,\alpha_1,...,\alpha_r|N) \\ = & -e^{-2i\pi\alpha' k_1\cdot k_2}\mathcal{U}(\beta_1,\beta_2;\alpha_1,...,\alpha_r|N), \end{aligned} \tag{17}$$

  thus they can be solved from each other.

From this relation, we express the $\mathcal{V}$ with two βs by the $\mathcal{U}$s with βs no more than two

$$\begin{aligned} & \mathcal{V}(\beta_1, \beta_2; \alpha_1, ..., \alpha_r|N) \\ = \ & \mathcal{U}(\beta_2; \beta_1, \alpha_1, ..., \alpha_r|N) - e^{-2i\pi\alpha' k_{\beta_1}\cdot k_{\beta_2}}\mathcal{U}(\beta_1, \beta_2; \alpha_1, ..., \alpha_r|N). \end{aligned} \tag{18}$$

Since the KK-BCJ relations are generated by generalized $U(1)$-like decoupling identities, the generalized $U(1)$-like decoupling identities are generated by $U(1)$-like decoupling identities and the $U(1)$-like decoupling identities are generated by the primary relations, all the KK-BCJ relations can be generated by the primary relations.

- Field theory limits
  The discussion can be extended to field theory by taking $\alpha' \to 0$. However, in field theory there are no kinematic factors in $U(1)$-decoupling identity(fundamental KK relation), we cannot use the $U(1)$-decoupling identity to generate the fundamental BCJ relation. Therefore, only the fundamental BCJ relation and cyclic symmetry can be considered as primary relations. We take five-point relation with two βs as an example.

The field theory limit of real part is given as

$$\begin{aligned}
& \mathcal{K}^f(3,4;1,2|5) \\
= & -\frac{1}{2s_{43}}\mathcal{B}^f(4;1,2,5|3) + \frac{1}{2s_{34}}\mathcal{B}^f(3;4,1,2|5) \\
& + \frac{1}{2s_{34}}\frac{s_{31}+s_{41}}{s_{31}}\mathcal{B}^f(3;1,4,2|5) + \frac{1}{2s_{34}}\frac{s_{31}+s_{41}+s_{42}}{s_{31}}\mathcal{B}^f(3;1,2,4|5) \\
& - \frac{1}{2s_{34}}\frac{s_{41}+s_{34}+s_{31}}{s_{41}}\mathcal{B}^f(4;1,3,2|5) - \frac{1}{2s_{34}}\frac{s_{41}+s_{34}+s_{31}+s_{32}}{s_{41}}\mathcal{B}^f(4;1,2,3|5) \\
& - \frac{1}{2s_{34}}\frac{s_{41}}{s_{31}}\mathcal{B}^f(3;4,2,5|1) - \frac{1}{2s_{34}}\frac{1}{2s_{34}}\frac{s_{41}+s_{42}}{s_{31}}\mathcal{B}^f(3;2,4,5|1) \\
& + \frac{1}{2s_{34}}\frac{s_{34}+s_{31}}{s_{41}}\mathcal{B}^f(4;3,2,5|1) + \frac{1}{2s_{34}}\frac{s_{34}+s_{31}+s_{32}}{s_{41}}\mathcal{B}^f(4;2,3,5|1). \qquad (19)
\end{aligned}$$

The field theory limit of imaginary part is given as

$$\begin{aligned}
& \mathcal{B}^f(3,4;1,2|5) \\
= & -\frac{s_{41}}{s_{34}}\mathcal{B}^f(3;1,4,2|5) - \frac{s_{41}+s_{42}}{s_{34}}\mathcal{B}^f(3;1,2,4|5) \\
& + \frac{s_{34}+s_{31}}{s_{34}}\mathcal{B}^f(4;1,3,2|5) + \frac{s_{34}+s_{31}+s_{32}}{s_{34}}\mathcal{B}^f(4;1,2,3|5). \qquad (20)
\end{aligned}$$

A general monodromy relation

One can extend the discussions on monodromy to get a more general relation.

$$
\begin{aligned}
& \mathcal{W}(\gamma_1, ..., \gamma_t; \beta_1, ..., \beta_s; \alpha_1, ..., \alpha_r | N) \\
\equiv & \sum_{\tau \in P(O\{\gamma\} \cup O\{\beta^T\})} \mathcal{P}^*_{\{\gamma, \beta^T, \alpha, N\}, \{\tau, \alpha, N\}} A_o(\tau, \alpha, N) \\
+ & (-1)^{s-1} \sum_{\sigma \in P(O\{\alpha\} \cup O\{\beta\}) | \sigma_1 = \alpha_1} \mathcal{P}_{\{\gamma, \beta^T, \alpha, N\}, \{\gamma, \sigma, N\}} A_o(\gamma, \sigma, N),
\end{aligned}
\tag{21}
$$

This relation can be explained by the contour of worldsheet integral(Fig. 3). The KK-BCJ relation(Fig. 2) is the special case with no γ.

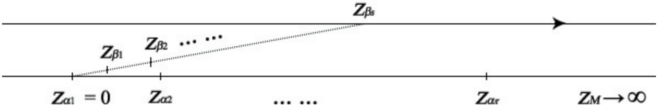

Figure: 2. Contour for KK-BCJ relation

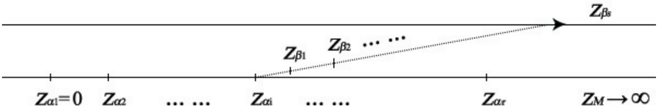

Figure: 3. Contour for general monodromy relation

This general monodromy relation can also be generated by primary relations. For example, for the case with only one β, by using the properties of momentum kernel, we have

$$\begin{aligned} & \mathcal{W}(\gamma_1, ..., \gamma_t; \beta_1; \alpha_2, ..., \alpha_r|N) \\ = & \; e^{2i\pi\alpha' k_{\gamma_t}\cdot k_{\beta_1}} \mathcal{W}(\gamma_1, ..., \gamma_{t-1}; \beta_1; \gamma_t, \alpha_1, ..., \alpha_r|N). \end{aligned} \quad (22)$$

Thus $\mathcal{W}(\gamma_1, ..., \gamma_t; \beta_1; \alpha_2, ..., \alpha_r|N)$ can be expressed by the $\mathcal{W}$ with no γ

$$\begin{aligned} & \mathcal{W}(\gamma_1, ..., \gamma_t; \beta_1; \alpha_2, ..., \alpha_r|N) \\ = & \; e^{2i\pi\alpha' \sum_{i=1}^{t} k_{\gamma_i}\cdot k_{\beta_1}} \mathcal{W}(\emptyset; \beta_1; \gamma_1, ..., \gamma_t, \alpha_1, ..., \alpha_r|N). \end{aligned} \quad (23)$$

In a similar way, All the general monodromy relation can be given as linear combinations of KK-BCJ relations, thus they are generated by the primary relations.

The field theory limit of general monodromy relation can be given by taking $\alpha' \to 0$. For example

$$
\begin{aligned}
&\mathcal{W}^f(1,2;3,4;5|6) \\
&= A_g(1,2,4,3,5,6) + [1 + i\pi\alpha'(s_{42}+s_{32})]A_g(1,4,3,2,5,6) \\
&+[1+i\pi\alpha'(s_{42})]A_g(1,4,2,3,5,6) + [1+i\pi\alpha'(s_{41}+s_{31}+s_{42}+s_{32})]A_g(4,3,1,2,5,6) \\
&+[1+i\pi\alpha'(s_{41}+s_{42}+s_{32})]A_g(4,1,3,2,5,6) + [1+i\pi\alpha'(s_{41}+s_{42})]A_g(4,1,2,3,5,6) \\
&-[1-i\pi\alpha'(s_{45}+s_{34}+s_{35})]A_g(1,2,5,3,4,6) \\
&= 0.
\end{aligned}
\tag{24}
$$

The real part gives

$$
\begin{aligned}
&A_g(1,2,4,3,5,6) + A_g(1,4,3,2,5,6) + A_g(1,4,2,3,5,6) \\
&+A_g(4,3,1,2,5,6) + A_g(4,1,3,2,5,6) + A_g(4,1,2,3,5,6) - A_g(1,2,5,4,3,6) \\
&= 0,
\end{aligned}
\tag{25}
$$

The imaginary part gives

$$
\begin{aligned}
&(s_{42}+s_{32})A_g(1,4,3,2,5,6) + s_{42}A_g(1,4,2,3,5,6) \\
&+(s_{41}+s_{31}+s_{42}+s_{32})A_g(4,3,1,2,5,6) + (s_{41}+s_{42}+s_{32})A_g(4,1,3,2,5,6) \\
&+(s_{41}+s_{42})A_g(4,1,2,3,5,6) + (s_{45}+s_{34}+s_{35})A_g(1,2,5,3,4,6) \\
&= 0.
\end{aligned}
\tag{26}
$$

Minimal-basis expansion in string theory

The reduction of the independent amplitudes in string theory from KK-basis to BCJ-basis is not apparent. However, one can use the BCJ relations to solve this minimal-basis expansion out. the general formula of minimal-basis expansion is

$$
\begin{aligned}
& A_o(1,\beta_1,...,\beta_s,2,\alpha_1,...,\alpha_{N-s-3},N) \\
= & \sum_{\sigma\in P(O\{\alpha\}\cup\{\beta\})} \sum_{\text{All divisions} O\{\beta\}\to O\{\beta_1,\dots,\beta_{i_1}\}O\{\beta_{i_1+1},\dots,\beta_{i_2}\},\dots,O\{\beta_{i_{n-1}+1},\dots,\beta_{i_n}\}} \\
& \prod_{j=0}^{n-1}\left[-\frac{\mathcal{S}_{\{\beta_{i_{j+1}},\dots,\beta_1,1,2,\sigma/\{\beta_{i_{j+1}},\dots,\beta_1\},N\},\{1,\beta_1,\dots,\beta_{i_j},2,\sigma/\{\beta_1,\dots,\beta_{i_j}\},N\}}}{\sin[\pi\alpha' s_{1\beta_1,\dots,\beta_{i_j}}]}\Theta_{j+1}\right] \\
& \times A_o(1,2,\sigma,N). \qquad (27)
\end{aligned}
$$

where $\Theta_j=\prod_{k=i_{j-1}+1}^{i_j}\theta(\sigma^{-1}(\beta_{k+1})-\sigma^{-1}(\beta_k))$ for $i_j>i_{i-1}+1$ and $\Theta_j=1$ for $(i_j=i_{j-1}+1)$.

If the minimal-basis expansion is right for the cases with $s \leq 2$, let us consider the minimal-basis expansion with three βs.The BCJ relation at level-3 is given as

$$\begin{aligned}
& A_o(1,\beta_1,\beta_2,\beta_3,2,\alpha_1,...,\alpha_{N-6},N) \\
= \quad & -\frac{1}{\sin[\pi\alpha' s_{1\beta_1\beta_2\beta_3}]} \\
& \times\Bigg[\sum_{\sigma\in P(O\{\beta_3\}\cup O\{\alpha\})}\mathcal{S}_{\{\beta_3,\beta_2,\beta_1,1,2,\alpha,N\},\{1,\beta_1,\beta_2,2,\sigma,N\}}A_o(1,\beta_1,\beta_2,2,\sigma,N) \\
& + \sum_{\sigma'\in P(O\{\beta_2,\beta_3\}\cup O\{\alpha\})}\mathcal{S}_{\{\beta_3,\beta_2,\beta_1,1,2,\alpha,N\},\{1,\beta_1,2,\sigma',N\}}A_o(1,\beta_1,2,\sigma',N) \\
& + \sum_{\sigma''\in P(O\{\beta_1,\beta_2,\beta_3\}\cup O\{\alpha\})}\mathcal{S}_{\{\beta_3,\beta_2,\beta_1,1,2,\alpha,N\},\{1,2,\sigma'',N\}}A_o(1,2,\sigma'',N)\Bigg]. \quad (28)
\end{aligned}$$

The minimal-basis expansions with one β is

$$\begin{aligned} & A_o(1,\beta_1,2,\alpha_1,...,\alpha_{N-4},N) \\ = \quad & -\frac{1}{\sin[2\pi\alpha' k_1\cdot k_{\beta_1}]}\sum_{\sigma\in P(O\{\beta_1\}\cup O\{\alpha\})}\mathcal{S}_{\{\beta_1,1,2,\alpha_1,\ldots,\alpha_{N-4},N\},\{1,2,\sigma,N\}}A_o(1,2,\sigma,N), \end{aligned} \tag{29}$$

and that with two βs is

$$\begin{aligned} & A_o(1,\beta_1,\beta_2,2,\alpha_1,...,\alpha_{N-5},N) \\ = \quad & \sum_{\sigma\in P(O\{\alpha\}\cup\{\beta_1,\beta_2\})}\left[\frac{\mathcal{S}_{\{\beta_2,\beta_1,1,2,\alpha,N\},\{1,\beta_1,2,\sigma/\{\beta_1\},N\}}}{\sin(\pi\alpha' s_{1\beta_1\beta_2})}\frac{\mathcal{S}_{\{\beta_1,1,2,\sigma/\{\beta_1\},N\},\{1,2,\sigma,N\}}}{\sin(\pi\alpha' s_{1\beta_1})}\right. \\ & \left.-\frac{\mathcal{S}_{\{\beta_2,\beta_1,1,2,\alpha,N\},\{1,2,\sigma,N\}}}{\sin(\pi\alpha' s_{1\beta_1\beta_2})}\theta(\sigma^{-1}(\beta_2)-\sigma^{-1}(\beta_1))\right]A_o(1,2,\sigma,N). \end{aligned} \tag{30}$$

Substituting these two expansions into the BCJ relation with three βs, we can easily obtain the minimal-basis expansion with three βs

$$
\begin{aligned}
&A_o(1,\beta_1,\beta_2,\beta_3,2,\alpha_1,\ldots,\alpha_{N-6},N)=\sum_{\sigma\in P(O\{\alpha\}\cup\{\beta_1,\beta_2,\beta_3\})}\\
&\left[(-1)^3\frac{\mathcal{S}_{\{\beta_3,\beta_2,\beta_1,1,2,\alpha,N\},\{1,\beta_1,\beta_2,2,\sigma/\{\beta_1,\beta_2\},N\}}}{\sin[i\pi\alpha' s_{1\beta_1\beta_2\beta_3}]}\frac{\mathcal{S}_{\{\beta_2,\beta_1,1,2,\sigma/\{\beta_1,\beta_2\},N\},\{1,\beta_1,2,\sigma/\{\beta_1\},N\}}}{\sin(\pi\alpha' s_{1\beta_1\beta_2})}\right.\\
&\times\frac{\mathcal{S}_{\{\beta_1,1,2,\sigma/\{\beta_1\},N\},\{1,2,\sigma,N\}}}{\sin(\pi\alpha' s_{1\beta_1})}\\
&+\frac{\mathcal{S}_{\{\beta_3,\beta_2,\beta_1,1,2,\alpha,N\},\{1,\beta_1,\beta_2,2,\sigma/\{\beta_1,\beta_2\},N\}}}{\sin[\pi\alpha' s_{1\beta_1\beta_2\beta_3}]}\frac{\mathcal{S}_{\{\beta_2,\beta_1,1,2,\sigma/\{\beta_1,\beta_2\},N\},\{1,2,\sigma,N\}}}{\sin(\pi\alpha' s_{1\beta_1\beta_2})}\\
&\times\theta(\sigma^{-1}(\beta_2)-\sigma^{-1}(\beta_1))\\
&+\frac{\mathcal{S}_{\{\beta_3,\beta_2,\beta_1,1,2,\alpha,N\},\{1,\beta_1,2,\sigma/\{\beta_1\},N\}}}{\sin[\pi\alpha' s_{1\beta_1\beta_2\beta_3}]}\theta(\sigma^{-1}(\beta_3)-\sigma^{-1}(\beta_2))\frac{\mathcal{S}_{\{\beta_1,1,2,\sigma/\{\beta_1\},N\},\{1,2,\sigma,N\}}}{\sin(\pi\alpha' s_{1\beta_1})}\\
&\left.-\frac{\mathcal{S}_{\{\beta_3,\beta_2,\beta_1,1,2,\alpha,N\},\{1,2,\sigma,N\}}}{\sin[\pi\alpha' s_{1\beta_1\beta_2\beta_3}]}\theta(\sigma^{-1}(\beta_3)-\sigma^{-1}(\beta_2))\theta(\sigma^{-1}(\beta_2)-\sigma^{-1}(\beta_1))\right]A_o(1,2,\sigma,N).
\end{aligned}
\tag{31}
$$

Conclusion

- All the KK and BCJ relations can be generated by two primary relations. The primary relations in string theory can be chosen as cyclic symmetry as well as either one of fundamental KK relation and fundamental BCJ relation. In field theory, the $U(1)$-decoupling identity(fundamental KK relation) cannot be chosen as primary relation.
- The general monodromy relation can also be generated by primary relations.
- One can derive and prove the minimal-basis expansion recursively.