Scattering Amplitude

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based on review with Prof. Luo, arXiv:1111.5759

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Scattering amplitude is a fundamental object in QFT:

- It is the primary object used comparing with experiment data.
- It contains many important information about the quantum field theory in general as well as special information for individual QFT
- For most cases, we can only do the perturbative calculaton and the general method is Feynman diagrams, which requires explicit Lagrangian formula

Difficulties by Feynman diagrams:

- First, for some theories, the Feynman rule is not so simple. For example, for pure gravity in de Donder gauge, there are about 100 terms for the three vertex and infinite number of other higher vertices.
- Secondly, even with simple Feynman rules, the number of diagrams will increase dramatically with more and more external particles. For example, for pure QCD at tree-level with n external legs

$n =$	4	5	6	7	8	9	10
	4	25	220	2485	34300	559, 405	10, 525, 900

For seven-gluon at one loop, there are 227, 585 Feynman diagrams.

Gravity

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Propagator in de Donder gauge:

$$P_{\mu\nu;\alpha\beta}(k) = \frac{1}{2}\Big[\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\alpha}\eta_{\nu\beta}\Big]\frac{i}{k^2+i\epsilon}$$

Three vertex:

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\
&\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$

About 100 terms in three vertex
An infinite number of other messy vertices.
Naïve conclusion: Gravity is a nasty mess.

[From Bern's talk]

- Thirdly, the Feynman diagram does not respect the symmetry: only sum up some subset of diagrams, gauge symmetry is recovered. In other word, the calculation is not efficient and there are huge cancelations in middle steps.
- Another point is not so obvious is that results presented by Feynman diagrams may not be the simplest. For example, amplitude of tree-level five gluons is given by next page

$$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$$

- Using Feynman diagrams, we can calculate both off-shell and on-shell amplitudes. However, simplification happens most times for on-shell amplitudes, which are what we will focus.
- For on-shell amplitudes with massless particles, instead of (p_μ, ϵ_μ), we can use spinor variables $(\lambda, \widetilde{\lambda})$. One of most important advantage using spinor notation is the factorization between λ and $\widetilde{\lambda}$ explicitly.

[Xu, Zhang, Chang, 1987]

- Using this new notation, we get extremely compact expression. For example, tree-level MHV amplitude for color ordered

$$A_n(1^+, ..., i^-, ..., j^-, ...n^+) = \frac{\langle i|j\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle ... \langle n|1\rangle}$$

[Parke, Taylor, 1986; Berends, Giele, 1988]

Spinor Notations

- Massless Dirac equation is $\not{p}u = 0$. There are two fundamental solutions: negative chirality spinor λ_α with $(1/2, 0)$ and positive chirality spinor $\widetilde{\lambda}_{\dot{\alpha}}$ with $(0, 1/2)$.
- Using $p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}}$, momentum can be written as two by two matrix.
- A special property is that for massless momentum $p^2 = 0$, two by two matrix can be written as $p_{\alpha\dot{\alpha}} = \lambda_\alpha \widetilde{\lambda}_{\dot{\alpha}}$. The factorization property is the key of much simple expression of amplitudes when using spinor notation.

Spinor Notation

- First, we have following mapping

$$u_{\pm} = \frac{1 \pm \gamma_5}{2} u(k), \quad v_{\mp} = \frac{1 \pm \gamma_5}{2} u(k),$$
$$\overline{u_{\pm}} = \overline{u(k)} \frac{1 \mp \gamma_5}{2}, \quad \overline{v_{\mp}} = \overline{v(k)} \frac{1 \mp \gamma_5}{2}$$

- Using this we have

$$\langle i|j\rangle = \overline{u_-}(k_i) u_+(k_j), \quad \langle i|P|j] = \overline{u_-}(k_i) \not{P} u_-(k_j)$$

- One most important fact is the polarization vector can be written as

$$\epsilon^+_\nu(k|\mu) = \frac{+\langle \mu|\gamma_\nu|k]}{\sqrt{2}\langle \mu|k\rangle}, \quad \epsilon^-_\nu(k|\mu) = \frac{-[\mu|\gamma_\nu|k\rangle}{\sqrt{2}[\mu|k]},$$

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Twistor space

Witten's twistor program: [Witten, 2003]

- MHV-amplitude is

$$A_n(1^+, ..., i^-, ..., j^-, ...n^+) = \delta^4(\sum_i p_i)\frac{\langle i|j\rangle^4}{\langle 1|2\rangle \langle 2|3\rangle ... \langle n|1\rangle}$$

- Using the spinor notation, momentum conservation can be written as

$$\int d^4x \exp\left(ix_{a\dot{a}}\sum_{i=1}^{n}\lambda_i^a\widetilde{\lambda}_i^{\dot{a}}\right)$$

- Fourier transformation $\widetilde{\lambda}$

$$\widetilde{f}(\mu) = \int \frac{d^2\widetilde{\lambda}}{(2\pi)^2}e^{i\mu^{\dot{a}}\widetilde{\lambda}_{\dot{a}}}f(\lambda, \widetilde{\lambda})$$

Twistor space

- Because MHV-amplitude does not include $\widetilde{\lambda}$ variables, those transformation leads to amplitude in the twistor space $\widetilde{A}(la_i, \mu_i)$

$$\int d^4x \prod_{i=1}^{n} \int \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2} \exp\left(i x_{a\dot{a}} \lambda_i^a \widetilde{\lambda}_i^{\dot{a}} + i \mu^{i\dot{a}} \widetilde{\lambda}_i^{\dot{a}} \right)$$

$$\int d^4x \prod_{i=1}^{n} \delta^2(\mu^{i\dot{a}} + x_{a\dot{a}} \lambda_i^a) f(\lambda_i)$$

- Given point $x_{a\dot{a}}$, the pair of equations

$$\mu^{i\dot{a}} + x_{a\dot{a}} \lambda_i^a = 0, \quad \dot{a} = 1, 2$$

defines a real algebraic curve C in the $RP^3 = (\lambda_a, \mu_{\dot{a}})$.

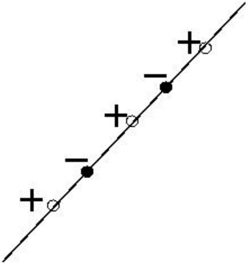
Twistor space

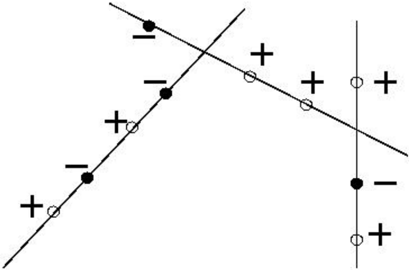
- The integral $\int d^4x$ is the integral over moduli space of a real, degree one, genus zero curve C in the $RP^3 = (\lambda_a, \mu_{\dot{a}})$.
- Geometrical meaning: MHV amplitude locates at a straight line in twistor space.
- How about the NMHV-amplitude (with three negative helicity)? It can be shown to be **genus zero, degree two curve**.

Twistor space

Summary of geometric picture of tree-level amplitude:

- MHV amplitude is a straight line in twistor space
- Move negative helicity case can be constructed by line intersections. At intersection point, a pair $(+,-)$-helicity is assigned to two interactive lines to make each line has two and only two negative helicities.





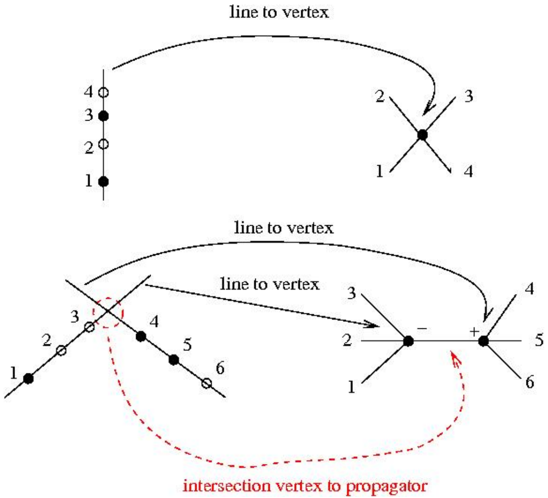
Intersection of lines with 5 negative

Twistor space

Motivated by geometric picture, CSW-diagram is proposed [Cachazo, Svrcek, Witten, 2004]

- Straight line in twistor space $\Longrightarrow$ **vertex**
- Point in twistor space $\Longrightarrow$ **a line connected to the vertex**
- Intersection points (a pair of point) in twistor space $\Longrightarrow$ **expanded into a propagator**
- Intersected diagram in twistor space $\Longrightarrow$ **Feynman line CSW-diagram**

Picture of translation:

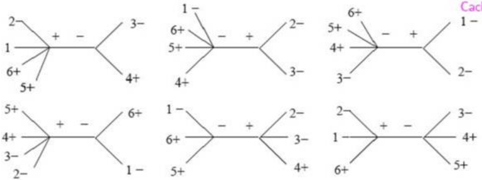


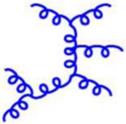
220 Feynman diagrams reduced to 6 CSW diagram:

Cachazo, Svrcek and Witten

QCD gluon scattering amplitude

$- - - + + +$





$$A_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{\langle 1\,2\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|\,5+6+1\,|q\rangle\,\langle 5|\,6+1+2\,|q\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|\,4\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,3\,|q\rangle}$$
$$+ \frac{\langle 1|\,4+5+6\,|q\rangle^3}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 4|\,5+6+1\,|q\rangle} \times \frac{1}{s_{23}} \times \frac{\langle 2\,3\rangle^3}{\langle 3|\,2\,|q\rangle\,\langle 2|\,3\,|q\rangle}$$
$$+ \frac{\langle 3|\,4+5+6\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6|\,3+4+5\,|q\rangle} \times \frac{1}{s_{12}} \times \frac{\langle 1\,2\rangle^3}{\langle 2|\,1\,|q\rangle\,\langle 1|\,2\,|q\rangle}$$
$$+ \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,2+3+4\,|q\rangle\,\langle 2|\,3+4+5\,|q\rangle} \times \frac{1}{s_{61}} \times \frac{\langle 1|\,6\,|q\rangle^3}{\langle 6\,1\rangle\,\langle 6|\,1\,|q\rangle}$$
$$+ \frac{\langle 1|\,5+6\,|q\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 5|\,6+1\,|q\rangle} \times \frac{1}{s_{561}} \times \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,2+3\,|q\rangle\,\langle 2|\,3+4\,|q\rangle}$$
$$+ \frac{\langle 1\,2\rangle^3}{\langle 6\,1\rangle\,\langle 2|\,6+1\,|q\rangle\,\langle 6|\,1+2\,|q\rangle} \times \frac{1}{s_{612}} \times \frac{\langle 3|\,4+5\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,3+4\,|q\rangle}$$

Twistor space

- The MHV-diagram has following properties: (1) There are infinite number of MHV vertexes. (2) The helicity of off-shell propagator should be properly defined.
- It has been generalized to include fermion, scalar, massive fermion, massive scalars, loop amplitude, etc.
- It can be understood for Lagrangian by non-linear field redefinition. [Ettle, Morris]

Lagrangian derivation of MHV

Lagrangian derivation of MHV-rule:

[Ettle, Morris, 2006]

- Take light-cone gauge $\widehat{A} = A_0 + A_3 = 0$. Longitudinal field $A_0 - A_3$ becomes non-dynamical and can be integrated out.
- Now we left with $A^+ = A_1 + iA_2$ and $A^- = A_1 - iA_2$ as two physical helicity states.
- YM Lagrangian becomes

$$L = L^{-+} + L^{++-} + L^{--+} + L^{--++}$$

where symmetry between $+, -$ helicities is still manifest.

Lagrangian derivation of MHV

- Make field redefinition from A (we use A for A^+ and $\overline{A}$ for A^- now) to B such that

$$L^{-+}[A] + L^{++-}[A] = L^{-+}[B]$$

- Thus we have

$$A = \sum_{i=1}^{n} c_i B, \quad \overline{A} = \overline{B}(1 + \sum_{i=1}^{n} c_i B)$$

- Then it is easy to see Lagrangian has the form

$$L = L^{-+}[B] + L^{--+}[B] + L^{--++}[B] + ...$$

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Structure for Tree-level amplitudes: $\Longleftarrow$ It will be used late:

- Only singularity is poles. From Feynman diagrams, it appears when propagators are on-shell.
- Factorization property: When one propagator goes to on-shell, i.e., $P^2 - m^2 \to 0$, we have

$$A^{tree}(1,..,n) \quad \to \quad \sum_{\lambda} A_{m+1}(1,..,m,P^{\lambda})\frac{1}{P_{1m}^2 - m^2} A_{n-m+1}(-P^{-\lambda}, m+1,...,n)$$

In fact, this point gives the residue at the pole.

BCFW deformation

- One basic assumption: Tree-level amplitude $\mathcal{M}$ can be considered as a rational function of complex momenta.
- BCFW deformation: Let us consider following deformation. Picking two external momenta p_1, p_2 and auxiliary momentum q, we do following deformation:

$$p_1(z) = p_1 + zq, \qquad p_2(z) = p_2 - zq$$

and impose following conditions:

$$q^2 = q \cdot p_1 = q \cdot p_2 = 0$$

[Britto, Cachazo, Feng , 2004] [Britto, Cachazo, Feng , Witten, 2004]

BCFW recursion relation

Two good points of BCFW deformation:

- It keeps the momentum conservation conditions: $p_1 + p_2 = p_1(z) + p_2(z)$
- It keeps on-shell conditions $p_1^2 = p_1(z)^2$, $p_2^2 = p_2(z)^2$;
- Amplitude becomes the meromorphic function of single complex variable z. $(P + zq)^2 = P^2 + z(2P \cdot q)$. $\Longleftarrow$ Much easy to study.

BCFW-derivation

- Considering the contour integration $I = \oint dz A(z)/z$ by two ways:
  - Doing it along the point $z = \infty$, we get the "boundary contribution" $I = B$.
  - Doing it for big cycle around $z = 0$, we have $I = A(0) + \sum_\alpha \text{Res}(A(z)/z)|_{z_\alpha}$.
- Combining above we have

$$A(z = 0) = B - \sum_{poles\ z_\alpha} \text{Res}\left(\frac{A(z)}{z}\right)_{z=z_\alpha}$$

Pole part

- Location: Pole happens when one propagator goes to on-shell, i.e., $P^2 + z(2P \cdot q) = 0$. From it we find the location of pole $z_\alpha = \frac{P_\alpha^2}{-2P \cdot q}$.
- Reside: Given by Factorization property:

$$\left(\frac{A(z)}{z}\right)_{z=z_\alpha} = \sum_\lambda A^L_{m+1}(1, .., m, P^\lambda(z_\alpha))$$

$$\frac{1}{P^2} A^R_{n-m+1}(-P^{-\lambda}(z_\alpha), m+1, ..., n)$$

Boundary part

- It has following three cases:
  - When $z \to \infty$, $A(z) \to \sum_{i=0}^{k} c_i z^i + \mathcal{O}(1/z)$ with $c_0 \neq 0 \Longrightarrow$ nonzero boundary contribution
  - When $z \to \infty$, $A(z) \sim \frac{1}{z} \Longrightarrow$ zero boundary contribution
  - When $z \to \infty$, $A(z) \sim \frac{1}{z^k}$, $k \geq 2 \Longrightarrow$ zero boundary contribution and bonus relations
- Boundary behavior is a very nontrivial problem. Fortunately, for some theories under right choice of p_1, p_2, we have $\mathcal{M}(z) \to 0$ when $z \to \infty$. These include gauge and gravity theory.

  [Britto, Cachazo, Feng , Witten, 2004] [Arkani-Hamed, Kaplan 2008]

BCFW recursion

BCFW recursion relation for gluons:

[Britto, Cachazo, Feng , 2004]

- The formula is

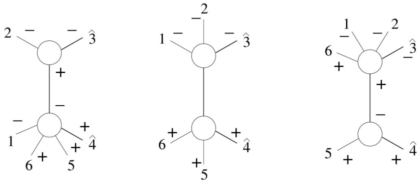
$$A_n(1,2,\ldots,(n-1)^-,n^+) = \sum_{i=1}^{n-3}\sum_{h=+,-} A_{i+2}(\hat{n},1,2,\ldots i,-\hat{P}_{n,i}^h)$$

$$\frac{1}{P_{n,i}^2} A_{n-i}(+\hat{P}_{n,i}^{-h}, i+1,\ldots,n-2,n\hat{-}1)$$

BCFW

Example: Six gluon amplitude

- The contributed terms are given by



(a) (b) (c)

- The result is given by

$$A(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{1}{\langle 5|3+4|2]} \left(\frac{\langle 1|2+3|4]^3}{[2\,3][3\,4]\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,t_2^{[3]}} + \frac{\langle 3|4+5|6]^3}{[6\,1][1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,t_3^{[3]}} \right)$$

Generalization One—Massive theory

- The solution of q exists for $D \geq 4$. Thus it can be applied to massive theory and higher dimension quantum field theories
- For the case $p_j^2 \neq 0$, we first construct two null momenta by linear combinations $\eta_\pm = (p_i + x_\pm p_j)$ with $x_\pm = \left(-2p_i \cdot p_i \pm \sqrt{(2p_i \cdot p_j)^2 - 4p_i^2 p_j^2}\right) / 2p_j^2$. The solution can be

$$q = \lambda_{\eta_+} \widetilde{\lambda}_{\eta_-}, \qquad \text{or} \qquad q = \lambda_{\eta_-} \widetilde{\lambda}_{\eta_+} \ .$$

[Badger, Glover, Khoze and Svrcek, 2005]

Generalization Two— SUSY theory

- For $\mathcal{N}=4$ theory, super-wave-function is given by Grassmann variables η^A $(A=1,2,3,4)$

$$\begin{aligned}\Phi(p,\eta) &= G^+(p)+\eta^A\Gamma_A(p)+\frac{1}{2}\eta^A\eta^B S_{AB}(p)\\ &\quad+\frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\Gamma}^D(p)+\frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p)\,,\end{aligned}$$

- The generalized BCFW-deformation

$$\lambda_i(z)=\lambda_i+z\lambda_j,\quad \widetilde{\lambda}_j(z)=\widetilde{\lambda}_j-z\widetilde{\lambda}_j,\quad \eta_j(z)=\eta_j-z\eta_i,$$

so both momentum $\delta^4(\sum_i\lambda_i\widetilde{\lambda}_i)$ and super-momentum $\delta^{(8)}(\sum_{i=1}^n\ \lambda_i^\alpha\,\eta_i^A)$ conservations are kept

Generalization Two— SUSY theory

- Now we need to sum over super-multiplet

$$\mathcal{A}=\sum_{\text{split }\alpha}\int d^4\eta_{P_i}\mathcal{A}_L(p_i(z_\alpha),p_\alpha(z_\alpha))\frac{1}{p_\alpha^2}\mathcal{A}_R(p_j(z_\alpha),-P_\alpha(z_\alpha))\,.$$

[Arkani-Hamed, Cachazo and J. Kaplan, 2008; Brandhuber, Heslop and Travaglini, 2008]

Generalization Three– Off-shell current

- The famous Berends-Giele off-shell recursion relation is given by

$$J^{\mu}(1,2,...,k) = \frac{-i}{p_{1,k}^2}\left[\sum_{i=1}^{k-1} V_3^{\mu\nu\rho}(p_{1,i},p_{i+1,k})\,J_\nu(1,...,i)\,J_\rho(i+1,...,k) + \sum_{j=i+1}^{k-1}\sum_{i=1}^{k-2} V_4^{\mu\nu\rho\sigma} J_\nu(1,...,i)\,J_\rho(i+1,...,j)\,J_\sigma(j+1,...,k)\right]$$

[Berends, Giele, 1988]

- Off-shell current is gauge dependent. First place is gauge choice of polarization vector

$$\epsilon_{i\mu}^{+} = \frac{\langle r_i|\gamma_\mu|p_i]}{\sqrt{2}\,\langle r_i|p_i\rangle}, \qquad \epsilon_{i\mu}^{-} = \frac{[r_i|\gamma_\mu|p_i\rangle}{\sqrt{2}\,[r_i|p_i]}$$

Generalization Three– Off-shell current

- The second gauge choice is propagator. We will use Feynman gauge
- To deal with the gauge dependence, we need to define two more polarization vectors

$$\epsilon_\mu^L = p_i, \quad \epsilon_\mu^T = \frac{\langle r_i|\gamma_\mu|r_i]}{2p_i \cdot r_i}$$

so we have

$$\begin{aligned} 0 &= \epsilon^+ \cdot \epsilon^+ = \epsilon^+ \cdot \epsilon^L = \epsilon^+ \cdot \epsilon^T = \epsilon^- \cdot \epsilon^- \\ &= \epsilon^- \cdot \epsilon^L = \epsilon^- \cdot \epsilon^T = \epsilon^T \cdot \epsilon^T = \epsilon^L \cdot \epsilon^L \\ 1 &= \epsilon^+ \cdot \epsilon^- = \epsilon^L \cdot \epsilon^T \end{aligned}$$

- The key observation is that now we have

$$g_{\mu\nu} = \epsilon_\mu^+ \epsilon_\nu^- + \epsilon_\mu^- \epsilon_\nu^+ + \epsilon_\mu^L \epsilon_\nu^T + \epsilon_\mu^T \epsilon_\nu^L$$

Generalization Three– Off-shell current

- Taking $(i,j) = (1,k)$, the recursion relation is given by

$$
\begin{aligned}
& J^{\mu}(1,2,...,k) \\
= & \sum_{i=2}^{k-1} \sum_{h,\tilde{h}} \left[A\left(\hat{1},...,i,\hat{p}^{h}\right) \cdot \frac{1}{p_{1,i}^{2}} \cdot J^{\mu}\left(-\hat{p}^{\tilde{h}}, i+1,...,\hat{k}\right) \right. \\
& \left. + J^{\mu}\left(\hat{1},...,i,\hat{p}^{h}\right) \cdot \frac{1}{p_{i+1,k}^{2}} \cdot A\left(-\hat{p}^{\tilde{h}}, i+1,...,\hat{k}\right) \right],
\end{aligned}
$$

where the sum is over $(h,\tilde{h}) = (+,-),(-,+),(L,T),(T,L)$.

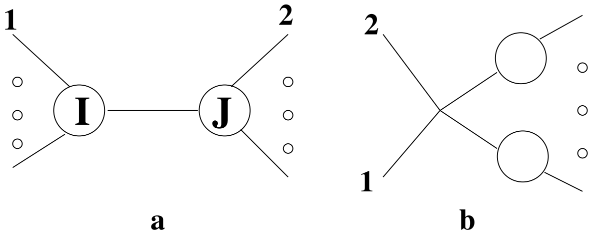
[Feng, Zhang, 2011]

Generalization Four– Nonzero boundary contribution

- Pole structure and their residues are universal for local theory, but the boundary contributions are not
- It is a (quasi)-global phenominon, i.e., depending the chosen pair and whole helicity configuration
- There are three ways to deal with boundary contributions:
    - Using auxiliary fields to make contributions in new QFT zero.
      [Benincasa, Cachazo, 2007; Boels, 2010]
    - Analyze Feynman diagrams directly
      [Feng, Wang, Wang, Zhang, 2009; Feng, Liu, 2010; Feng, Zhang, 2011]
    - Transfer to the discussion of roots of amplitude
      [Benincasa, Conde, 2011; Feng, Jia, Luo, Luo, 2011]

Feynman diagram for $\lambda\phi^4$ theory

- With $(1,2)$-pair deformation, Feynman diagrams will be following two types:

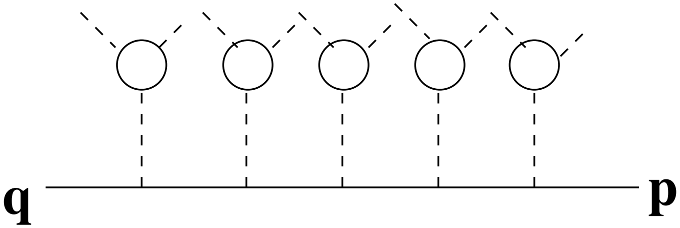


- Boundary contribution is

$$A_b = (-i\lambda) \sum_{\mathcal{I}' \bigcup \mathcal{J}' = \{n\} \backslash \{i,j\}} A_{\mathcal{I}'}(\{K_{\mathcal{I}'}\}) \frac{1}{p^2_{\mathcal{I}'}} \frac{1}{p^2_{\mathcal{J}'}} A_{\mathcal{J}'}(\{K_{\mathcal{J}'}\})$$

Feynman diagram for Yukawa theory

- Same analysis for typical Feynman diagram



- Only a few types of Feynman diagrams give boundary contributions and they can be evaluated directly

[Feng, Wang, Wang, Zhang, 2009; Feng, Liu, 2010; Feng, Zhang, 2011]

Roots of amplitude

Another angel for boundary contributions:

- $$\begin{aligned} M_n(z) &= \sum_{k\in\mathcal{P}^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{p_k^2(z)} + C_0 + \sum_{l=1}^{v} C_l z^l \\ &= c\frac{\prod_s (z-w_s)^{m_s}}{\prod_{k=1}^{N_p} p_k^2(z)} \end{aligned}$$
- Split all roots into two groups $\mathcal{I}, \mathcal{J}$. For $n_{\mathcal{I}} < N_p$

$$c\frac{\prod_{s=1}^{n_{\mathcal{I}}}(z-w_s)}{\prod_{k=1}^{N_p} p_k^2(z)} = \sum_{k\in\mathcal{P}^{(i,j)}} \frac{c_k}{p_k^2(z)}$$

$$M_n(z) = \sum_{k\in\mathcal{P}^{(i,j)}} \frac{c_k}{p_k^2(z)} \prod_{t=1}^{n_{\mathcal{J}}}(z-w_t)$$

Roots of amplitude

- Perform a contour integration around the pole z_k and obtain

$$\frac{M_L(z_k)M_R(z_k)}{(-2p_k\cdot q)} = \frac{c_k}{(-2p_k\cdot q)}\prod_{t=1}^{n_{\mathcal{J}}}(z_k-w_t),$$

so

$$c_k = \frac{M_L(z_k)M_R(z_k)}{\prod_{t=1}^{n_{\mathcal{J}}}(z_k-w_t)}$$

and finally

$$M_n(z) = \sum_{k\in\mathcal{P}^{(i,j)}}\frac{M_L(z_k)M_R(z_k)}{p_k^2(z)}\prod_{t=1}^{v+1}\frac{(z-w_t)}{z_k-w_t}$$

by setting $n_{\mathcal{I}} = N_p - 1$.

[Benincasa, Conde, 2011; Feng, Jia, Luo, Luo, 2011]

Comments for boundary BCFW-relation

- Root method is very general and useful for theoretical discussions. However, it is very hard to find root recursively, especially roots are in general not rational function
- Feynman diagram method is practical, but not general since we need to do analysis for each different theory
- Both methods are not completely satisfied and better method is desired

Generalization five–Bonus relation

- Bonus relations can be derived from the observation

$$0 = \oint \frac{dz}{z} z^b A(z), \quad b = 1, 1, ..., a-1, \quad if, \;\; A(z) \to \frac{1}{z^a} \;.$$

Because the z^b factor, there is no pole at $z = 0$. Taking contributions from other poles, we have bonus relations

$$0 = \sum_{\alpha} \sum_{h} A_L(p^h(z_\alpha)) \frac{z_\alpha^b}{p^2} A_R(-p^{-h}(z_\alpha))$$

for $b = 1, ..., a-1$.

[Arkani-Hamed, Cachazo, Kaplan, 2008]

Generalization six– Rational part of one loop amplitude

The new features appeared in this generalization are:

- There are double poles like $\langle a|b\rangle \,/\, [a|b]^2$, thus we need to find way to reproduce double pole and single pole contained inside double pole
- Loop factorization formula is

$$A_n^{1-\text{loop}} \to A_L^{1-\text{loop}} A_R^{\text{tree}} + A_L^{\text{tree}} A_R^{1-\text{loop}} + A_L^{\text{tree}} \mathcal{S} A_R^{\text{tree}} \; .$$

[Bern, Dixon, Kosower, 2005]

Generalization six– Rational part of one loop amplitude

Solution for above two difficulties:

- Two collinear momenta provide following divergent expression

$$A_{3;1}(1^+,2^+,3^+) = \frac{[1|2]\,[2|3]\,[3|1]}{K_{12}^2}$$

Thus double pole structure can then be obtained as

$$A_L^{\rm tree}\frac{1}{K_{a,a+1}^2}A_{3;1}(-\widehat{K}_{a,a+1}^+,\widetilde{a}^+,(a+1)^+)$$
$$\to A_L^{\rm tree}\frac{1}{K_{a,a+1}^2}\frac{1}{K_{a,a+1}^2}\left[\widehat{K}_{a,a+1}|\widehat{a}\right]\left[\widehat{a}|a+1\right]\left[a+1|\widehat{K}_{a,a+1}\right]$$

[Bern, Dixon, Kosower, 2005]

Generalization six– Rational part of one loop amplitude

- Single pole inside double pole is solved by multiplying by a dimensionless function

$$K_{cd}^2 \mathcal{S}^{(0)}(a, s^+, b) \mathcal{S}^{(0)}(c, s^-, d)$$

, where the *soft factor* is given

$$\mathcal{S}^{(0)}(a, s^+, b) = \frac{\langle a|b\rangle}{\langle a|s\rangle \langle s|b\rangle}, \quad \mathcal{S}^{(0)}(c, s^-, d) = -\frac{[c|d]}{[c|s]\,[s|d]}\ .$$

[Bern, Dixon, Kosower, 2005]

Generalization seven– QFT in 3D

- The deformation null momentum q has solution when and only when $D \geq 4$.
- For 3D,

$$p^{\alpha\beta} = x^{\mu}(\sigma_{\mu})^{\alpha\beta} = \lambda^{\alpha}\lambda^{\beta}$$

thus on-shell BCFW-deformation can considered as matrix transformation over two spinors

$$\begin{pmatrix} \lambda_i(z) \\ \lambda_j(z) \end{pmatrix} = R(z) \begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix},$$

This transformation keeps on-shell condition automatically

[Gang, Huang, Koh, Lee, Lipstein, 2010]

Generalization seven– QFT in 3D

- Conservation of momenta leads to

$$\left(\lambda_i(z) \quad \lambda_j(z) \right) \left(\begin{array}{c} \lambda_i(z) \\ \lambda_j(z) \end{array} \right) = \left(\lambda_i \quad \lambda_j \right) \left(\begin{array}{c} \lambda_i \\ \lambda_j \end{array} \right)$$

or

$$R^T(z)R(z) = I, \quad R(z) \in SO(2, C)$$

- With parameterization

$$R(z) = \left(\begin{array}{cc} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{array} \right),$$

propagator is

$$\widehat{p}_f^2(z) = a_f z^{-2} + b_f + c_f z^2$$

Generalization seven– QFT in 3D

- Now the derivation is to start from contour integration

$$A(z=1) = \oint_{z=1} \frac{dz}{z-1} A(z)$$

where the contour is a small circle around $z=1$.

- Each on-shell propagator will gives four poles and we need to sum up their contributions.

Generalization eight– Different deformation

- Previous recursion relations based on the BCFW-deformation where two particles have been deformed
- However, there are other deformations we can consider. For example, for NMHV-amplitude, we do following holomorphic deformations

$$
|i(z)] = |i] + z\langle j|k\rangle|\eta]\,,\quad |j(z)] = |j] + z\langle k|i\rangle|\eta]\,,
$$
$$
|k(z)] = |k] + z\langle i|j\rangle|\eta]\,,
$$

  where i, j, k have negative helicities.
- This deformation keeps (1) on-shell conditions; (2) momentum conservation.

[Risager, 2005]

Generalization eight– Different deformation

- Using the new deformation we can derive recursion relation using $\oint (dz/z)A(z)$ as

$$A = \sum_{\alpha, i\in A_L} A_L(z_\alpha)\frac{1}{p_\alpha^2}A_R(z_\alpha)$$

It is nothing, but the MHV-decomposition for NMHV-amplitude.

[Cachazo, Svrcek, Witten, 2004]

- For general $\mathrm{N}^{n-1}\mathrm{MHV}$-amplitudes, we make the deformation

$$|m_i(z)] = m_i + z r_i\, |\eta]\,, \quad i = 1, ..., n+1,$$

for $n+1$ particles of negative helicity. Here $\sum_i r_i\, |m_i\rangle = 0$ to ensure momentum conservation.

[Risager, 2005]

Generalization nine– string theory

- For string tree-level amplitude, we still have general structures: single pole and factorization properties.
- Comparing to familiar QFT, string theory has a better convergent behavior. It has been shown for any helicity (state) configurations, there is at least a kinematic region with vanishing boundary contribution.
- Conclusion: BCFW recursion relation can be generalized to string theory.

[Boels, 2008; Cheung, D. O'Connell and B. Wecht, 2010]

Generalization nine– string theory

Difficulties:

- String has infinity number of middle states to be summed.
- It is very difficult to get the polarization tensor for high level physical states
- How to carry out the sum?

Generalization nine– string theory

Example of four tachyon in bosonic open string theory:

- Expand

$$A(1,2,3,4)=\int_0^1 dz_2\,(1-z_2)^{k_3\cdot k_2}\,z_2^{k_2\cdot k_1}$$

we get

$$A(1,2,3,4)\sum_{a=0}^{\infty}\begin{pmatrix} k_3\cdot k_2 \\ a \end{pmatrix}(-)^a\frac{2}{(k_1+k_2)^2+2(a-1)}$$

[Chan, Lee, Feng, Fu, Yang, Wang, to appear]

Generalization nine– string theory

Example of four tachyon in bosonic open string theory:

- Under the shifting (k_1, k_4) we have infinite single poles with locations

$$z_a = \frac{(k_1 + k_2)^2 + 2(a-1)}{-2q \cdot (k_1 + k_2)}, \quad a = 0, 1, ...,$$

and the sum over physical states at this pole gives residue

$$\sum_{states} \sum_{h} A_L(1, 2, P_a^h(z_a)) A_R(-P_a^{\widetilde{h}}(z_a), 3, 4) = (-1)^a \begin{pmatrix} k_3 \cdot k_2 \\ a \end{pmatrix}$$

- **The problem:** How to derive this result from BCFW recursion relation?

[Chan, Lee, Feng, Fu, Yang, Wang, to appear]

Generalization nine– string theory

- The key part is to enlarge the sum over all physical states at given level a to the sum over all Fock states
- Contribution from un-physical Fock states will decouple by Ward-Identity in string theory (or "non ghost theorem").
- Thus the sum is replaced with

$$\sum_{\{N_{\mu,n}\}} \left|\{N_{\mu,n}\};\widehat{P}\right\rangle \mathcal{T}_{\{N_{\mu,n}\}} \left\langle\{N_{\mu,n}\};\widehat{P}\right|$$

$$= \sum_{\sum_n nN_n=N} \left\{\prod_{n=1}^{\infty} \frac{(\alpha_{-n}^{\mu_{N_n,1}}\alpha_{-n}^{\mu_{N_n,2}}\ldots\alpha_{-n}^{\mu_{N_n,N_n}})}{\sqrt{N_n!n^{N_n}}}\right\}\left|0;\widehat{P}\right\rangle$$

$$\prod_{n=1}^{\infty}(g_{\mu_{N_n,1}\nu_{N_n,1}}g_{\mu_{N_n,2}\nu_{N_n,2}}\ldots g_{\mu_{N_n,N_n}\nu_{N_n,N_n}})$$

$$\left\langle 0;\widehat{P}\right|\left\{\prod_{n=1}^{\infty}\frac{(\alpha_{+n}^{\nu_{N_n,1}}\alpha_{+n}^{\nu_{N_n,2}}\ldots\alpha_{+n}^{\nu_{N_n,N_n}})}{\sqrt{N_n!n^{N_n}}}\right\}$$

Generalization nine– string theory

- The left three point amplitude is given by

$$\langle 0; -k_1 | V_0(k_2, z) | \{N_{\mu,n}\}; P\rangle = \delta(k_1 + k_2 + P) \prod_{\mu=0}^{D-1} \prod_{m=1}^{\infty} \frac{(-k_2^{\mu})^{N_{\mu,m}}}{\sqrt{m^{N_{\mu,m}} N_{\mu,m}!}}$$

and the right handed side is given by

$$\langle \{N_{\mu,n}\}; P | V_0(k_3, z) | 0; k_4\rangle = \delta(P - k_3 - k_4) \prod_{\mu=0}^{D-1} \prod_{m=1}^{\infty} \frac{(k_3^{\mu})^{N_{\mu,m}}}{\sqrt{N_{\mu,m}! m^{N_{\mu,m}}}}$$

Generalization nine– string theory

- The result for given N-level is

$$I_N = \sum_{\sum nN_n = N} \prod \frac{(-k_2 \cdot k_3)^{N_n}}{N_n! n^{N_n}}$$
$$= (-)^N \sum_{J=1}^{N} \frac{S(N,J)}{N!} (k_2 \cdot k_3)^J = (-)^N \binom{k_2 \cdot k_3}{N}$$

where $N = \sum_{n=1}^{\infty} nN_n$, $J = \sum_{n=1}^{\infty} N_n$ and $S(N,J)$ is the Stirling number of the first kind.

Contents

S-matrix reload?

- The derivation depends on the following observations: (1) for tree-level amplitudes, there are only single poles from propagators under BCFW-deformation; (2) the residues of single poles are determined by factorization properties; (3) with proper choice of deformation, the boundary contribution is zero.
- Among these observations, the first two are universal for all local quantum field theories. One naturally generalizes to other quantum field theories, by carefully taking care of boundary contributions.
- Result is a beautiful realization of S-matrix program

S-matrix reload?

- Proposed around 1960's. Initial purpose is for strong interaction
  [D.I. Olive, Phys. Rev. 135,B 745(1964); G.F. Chew, "The Analytic S-Matrix: A Basis for Nuclear Democracy", W.A.Benjamin, Inc., 1966; R.J. Eden, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, "The Analytic S-Matrix", Cambridge University Press, 1966.]
- Only general assumption of S-matrix program: Causality, Locality, Lorentz symmetry, gauge symmetry and analyticity etc.
- Traditional S-matrix program has following two characters: (1) It is multi-complex function, so it is much more difficult to study; (2) For general complex momenta, the amplitude is off-shell.

Applications of on-shell recursion relation can be divided into following two types:

- **Calculation of various amplitudes:** This is the initial motivation leading to the discovery of on-shell recursion relation. It is also one of most important practical applications for high energy experiments.
- **Understanding of various properties of QFT:** It has two distinguish features:
  - It keeps only on-shell information
  - It relies only on some general properties of QFT, so it opens new way to study QFT in the frame of S-matrix program

Example One: BCJ relation:

- Color ordering: The basic idea is to write amplitudes into gauge invariant subset, thus we can calculate these subsets one by one.

$$M^{tree}(1,2,...,n) = \sum_{permutation} \mathrm{Tr}(T_{a_1}...T_{a_n})A_n^{tree}(a_1,a_2,...,a_n)$$

- Color ordering separate the group information from the dynamical information.
- Naively there are $(n-1)!$ different dynamical basis, but there are some relations among them to reduce to independent basis $(n-3)!$.

Four relations for ordered gluon amplitudes:

- Color-order reversed relation:

$$A(n, \{\beta_1, ..., \beta_{n-2}\}, 1) = (-)^n A(1, \beta_{n-2}, \beta_{n-1}, ..., \beta_1, n)$$

- The $U(1)$-decoupling relation is given by

$$\sum_{\sigma \in cyclic} A_n(1, \sigma(2, 3, ..., n)) = 0$$

- KK-relation: [Kleiss, Kujif, 1989]

$$A_n(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in OP(\{\alpha\},\{\beta^T\})} A_n(1,\sigma,n)\ .$$

  where sum is over partial ordering.
- Example

$$A(1,\{2\},5,\{3,4\} = A(1,2,4,3,5)$$
$$+A((1,4,2,3,5) + A(1,4,3,2,5)$$

BCJ-relation: [Bern, Carraso, Johansson, 2008]

$$A_n(1,2,\{\alpha\},3,\{\beta\}) = \sum_{\sigma_i \in POP} A_n(1,2,3,\sigma_i)\mathcal{F},$$
$$\alpha = \{4,5,...,m\}$$
$$\beta = \{m+1, m+2, ..., n\}$$

- Beautiful proof from string theory

[Bjerrum-Bohr, Damgaard, Vanhove, 2009]

[Stieberger, 2009]

- Pure field theory proof

[Feng, Huang, Jia, 2010]

[Chen, Du, Feng, 2011]

- First we want to remark that all BCJ-relation can be derived from the one with length one at the set α. We call it the fundamental set.
- The form of fundamental one

$$
\begin{aligned}
0 &= I_4 = A(2,4,3,1)(s_{43}+s_{41}) + A(2,3,4,1)s_{41} \\
0 &= I_5 = A(2,4,3,5,1)(s_{43}+s_{45}+s_{41}) \\
&\quad + A(2,3,4,5,1)(s_{45}+s_{41}) + A(2,3,5,4,1)s_{41} \\
0 &= I_6 = A(2,4,3,5,6,1)(s_{43}+s_{45}+s_{46}+s_{41}) \\
&\quad + A(2,3,4,5,6,1)(s_{45}+s_{46}+s_{41}) \\
&\quad + A(2,3,5,4,6,1)(s_{46}+s_{41}) + A(2,3,5,6,4,1)s_{41}
\end{aligned}
$$

- The dual format by momentum conservation

$$\begin{aligned} 0 &= A(2,4,3,5,1)s_{24} + A(2,3,4,5,1)(s_{24}+s_{34}) \\ &\quad + A(2,3,5,4,1)(s_{24}+s_{34}+s_{54}) \end{aligned}$$

- A special case with $n=3$: $A(1,2,3)s_{23}=0$.

- The dual format by momentum conservation

$$\begin{aligned} 0 &= A(2,4,3,5,1)s_{24} + A(2,3,4,5,1)(s_{24}+s_{34}) \\ &\quad + A(2,3,5,4,1)(s_{24}+s_{34}+s_{54}) \end{aligned}$$

- A special case with $n=3$: $A(1,2,3)s_{23}=0$.

Now we want to show following four facts:

- (1) Color-order reversed relation for general n;
- (2) The $U(1)$-decoupling relation;
- (3) The KK-relation;
- (4) The BCJ relation;

The only assumption we will use: BCFW cut-constructibility of gluon amplitudes
Another fact from previous discussion is that for color-ordered three-point amplitude we have

$$A(1,2,3) = -A(3,2,1)$$

[Feng, Huang, Jia, 2010]

Color-order reversed relation

Color-order reversed relation:

$$\begin{aligned} & A(1,n,\{\beta_1,...,\beta_{n-2}\}) \\ = & \sum_{i=1}^{n-3} A(n,\beta_1,...,\beta_i,-P)\frac{1}{P^2}A(P,\beta_{i+1},...,\beta_{n-2},1) \\ = & \sum_{i=1}^{n-3}(-)^{n-i}A(1,\beta_{n-2},...,\beta_{i+1},P)\frac{1}{P^2}(-)^{i+2}A(-P,\beta_i,...,\beta_1,n) \\ = & (-)^n A(1,\beta_{n-2},\beta_{n-1},...,\beta_1,n) \end{aligned}$$

$U(1)$-decoupling identity

$U(1)$-decoupling identity. It can be done by induction for which we use $n=5$ to show the idea (by $(1,2)$-deformation)

$$\begin{array}{rcllllll}
A(1,2,3,4,5) & = & A(1,P_{23},4,5) & + & A(1,P_{234},5) & + & 0 \\
A(1,5,2,3,4) & = & A(1,5,P_{23},4) & + & A(1,5,P_{234}) & + & A(1,P_{52},3,4) \\
A(1,4,5,2,3) & = & A(1,4,5,P_{23}) & + & 0 & + & A(1,4,P_{52},3) \\
A(1,3,4,5,2) & = & 0 & + & 0 & + & A(1,3,4,P_{52})
\end{array}$$

$$\begin{array}{llll}
+ & 0 & + & 0 \\
+ & A(1,P_{523},4) & + & 0 \\
+ & A(1,4,P_{523}) & + & A(1,P_{452},3) \\
+ & 0 & + & A(1,3,P_{452})
\end{array}$$

where

$$A(1,P_{23},4,5) \equiv A(\widehat{1},\widehat{P}_{23},4,5)\frac{1}{s_{23}}A(-\widehat{P}_{23},\widehat{2},3)$$

KK-relation

- KK-relation: [Kleiss, Kujif, 1989]

$$A_n(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in OP(\{\alpha\},\{\beta^T\})} A_n(1,\sigma,n)\ .$$

where sum is over partial ordering.

- Example

$$A(1,\{2\},5,\{3,4\} = A(1,2,4,3,5)$$
$$+A((1,4,2,3,5) + A(1,4,3,2,5)$$

KK-relation

- We use $(1,5)$-shifting for $n=5$. First step we do BCFW expansion:

$$\begin{aligned} A(1,2,5,3,4) &= A(4,1,2,P_{35})\frac{1}{P_{35}^2}A(-P_{35},5,3) \\ &+A(3,4,1,P_{25})\frac{1}{P_{25}^2}A(-P_{25},2,5) \\ &+A(1,2,-P_{12})\frac{1}{P_{12}^2}A(P_{12},5,3,4) \\ &+A(4,1,-P_{41})\frac{1}{P_{41}^2}A(P_{41},2,5,3) \end{aligned}$$

Using Color-order reverse, $U(1)$ and KK for components:

$$\begin{aligned} & A(1,2,5,3,4) \\ = \; & (-A(1,2,4,P_{35}) - A(1,4,2,P_{35}))\frac{1}{P_{35}^2}(-A(-P_{35},3,5)) \\ & +A(1,4,3,P_{25})\frac{1}{P_{25}^2}A(-P_{25},2,5) \\ & +A(1,2,-P_{12})\frac{1}{P_{12}^2}A(P_{12},4,3,5) \\ & +(-A(1,4,-P_{41}))\frac{1}{P_{41}^2}(-A(P_{41},2,3,5) - A(P_{41},3,2,5)) \end{aligned}$$

$$T_1 + T_4 = A(1,2,4,3,5), \quad T_2 + T_5 = A(1,4,2,3,5),$$
$$T_3 + T_6 = A(1,4,3,2,5)$$

BCJ relation

- Take $(1,6)$ to do the deformation, consider combination

$$
\begin{aligned}
I_6(z) \quad = \quad & s_{2\hat{1}} A(\hat{1},2,3,4,5,\hat{6}) + (s_{2\hat{1}} + s_{32}) A(\hat{1},3,2,4,5,\hat{6}) \\
& + (s_{2\hat{1}} + s_{32} + s_{42}) A(\hat{1},3,4,2,5,\hat{6}) \\
& + (s_{2\hat{1}} + s_{32} + s_{42} + s_{52}) A(\hat{1},3,4,5,2,\hat{6})
\end{aligned}
$$

- Consider contour integration $\oint_{z=0} \frac{dz}{z} I_6(z) = I_6(z=0)$.
- Same contour can be evaluated using the finite poles plus boundary contribution.

- To see boundary part (around infinity)

$$
\begin{aligned}
I_6(z) &= I_1 + I_2 \\
I_1 &= s_{2\widehat{1}}\Big[A(\widehat{1},2,3,4,5,\widehat{6}) + A(\widehat{1},3,2,4,5,\widehat{6}) \\
&\quad + A(\widehat{1},3,4,2,5,\widehat{6}) + A(\widehat{1},3,4,5,2,\widehat{6})\Big] \\
&= -s_{2\widehat{1}}A(\widehat{1},3,4,5,\widehat{6},2) \to \frac{1}{z}
\end{aligned}
$$

where KK-relation has been used, while for

$$
I_2 \to \frac{1}{z}
$$

- Result $\oint_{z=\infty} \frac{dz}{z} I_6(z) = 0$

Finite pole part, expansion by on-shell recursion relation

- $$\begin{aligned} A(\widehat{1},2,3,4,5,\widehat{6}) &\to \quad s_{2\widehat{1}}A_3(\widehat{1},2,P) \quad A(-P,3,4,5,\widehat{6}) \\ A(\widehat{1},3,2,4,5,\widehat{6}) &\to \quad -A_3(\widehat{1},3,P) \quad A(-P,2,4,5,\widehat{6})(s_{24}+s_{25}+s_{2\widehat{6}}) \\ A(\widehat{1},3,4,2,5,\widehat{6}) &\to \quad -A_3(\widehat{1},3,P) \quad A(-P,4,2,5,\widehat{6})(s_{25}+s_{2\widehat{6}}) \\ A(\widehat{1},3,4,5,2,\widehat{6}) &\to \quad -A_3(\widehat{1},3,P) \quad A(-P,4,5,2,\widehat{6})(s_{2\widehat{6}}) \end{aligned}$$

- $$\begin{aligned} A(\widehat{1},2,3,4,5,\widehat{6}) &\to \quad s_{2\widehat{1}}A_3(\widehat{1},2,3,P) \quad A(-P,4,5,\widehat{6}) \\ A(\widehat{1},3,2,4,5,\widehat{6}) &\to \quad (s_{2\widehat{1}}+s_{23})A_3(\widehat{1},3,2,P) \quad A(-P,4,5,\widehat{6}) \\ A(\widehat{1},3,4,2,5,\widehat{6}) &\to \quad -A_3(\widehat{1},3,4,P) \quad A(-P,2,5,\widehat{6})(s_{25}+s_{2\widehat{6}}) \\ A(\widehat{1},3,4,5,2,\widehat{6}) &\to \quad -A_3(\widehat{1},3,4,P) \quad A(-P,5,2,\widehat{6})(s_{2\widehat{6}}) \end{aligned}$$

-

$$
\begin{aligned}
A(\widehat{1},2,3,4,5,\widehat{6}) &\rightarrow & s_{2\widehat{1}}A_3(\widehat{1},2,3,4,P) & \quad A(-P,5,\widehat{6})\\
A(\widehat{1},3,2,4,5,\widehat{6}) &\rightarrow & (s_{2\widehat{1}}+s_{23})A_3(\widehat{1},3,2,4,P) & \quad A(-P,5,\widehat{6})\\
A(\widehat{1},3,4,2,5,\widehat{6}) &\rightarrow & (s_{2\widehat{1}}+s_{23}+s_{24})A_3(\widehat{1},3,4,2,P) & \quad A(-P,5,\widehat{6})\\
A(\widehat{1},3,4,5,2,\widehat{6}) &\rightarrow & -A_3(\widehat{1},3,4,5,P) & \quad A(-P,2,\widehat{6})(s_{2\widehat{6}}
\end{aligned}
$$

- For the general n, the proof will be exactly same

-

$$
\begin{aligned}
A(\widehat{1},2,3,4,5,\widehat{6}) &\to & s_{2\widehat{1}}A_3(\widehat{1},2,3,4,P) & \quad A(-P,5,\widehat{6}) \\
A(\widehat{1},3,2,4,5,\widehat{6}) &\to & (s_{2\widehat{1}}+s_{23})A_3(\widehat{1},3,2,4,P) & \quad A(-P,5,\widehat{6}) \\
A(\widehat{1},3,4,2,5,\widehat{6}) &\to & (s_{2\widehat{1}}+s_{23}+s_{24})A_3(\widehat{1},3,4,2,P) & \quad A(-P,5,\widehat{6}) \\
A(\widehat{1},3,4,5,2,\widehat{6}) &\to & -A_3(\widehat{1},3,4,5,P) & \quad A(-P,2,\widehat{6})(s_{2\widehat{6}}
\end{aligned}
$$

- For the general n, the proof will be exactly same

Example Two: KLT relation

Let us compare gauge theory and gravity theory:

- Gauge symmetry is symmetry for inner quantities while gravity theory is based on the space-time symmetry, the general equivalence principal for the choice of coordinate.
- The spin of gauge bosons is one while the spin of graviton is two.
- More importantly, the Lagrangian of gauge theory is polynomial with finite interaction terms while the Einstein Lagrangian is highly non-linear and infinite interaction terms after perturbative expansion.

However, we must be careful about these differences we have talked:

- The Lagrangian description is a off-shell description. What happens if we constraint to only on-shell quantities?
- We have clues from string theory:
    - Graviton given by closed string; Gluons given by open string.
    - Closed string $===$ left-moving open mode $\times$ right moving open mode
    - In one word, on-shell Graviton $==$ $[\text{Gluon}]^2$

- One accurate description of above claim is the KLT relation for tree-level scattering amplitude, which is obtained from string theory. For example

$$\begin{aligned}\mathcal{M}_3(1,2,3) &= A_3(1,2,3)\widetilde{A}_3(1,2,3), \\ \mathcal{M}_4(1,2,3,4) &= A_4(1,2,3,4)s_{12}\widetilde{A}_4(3,4,2,1)\end{aligned}$$

[Kawai, Lewellen, Tye; 1985] [Bern, Dixon, Perelstein, Rozowsky; 1999]

- Question: Could we understand this relation directly in the framework of quantum field theory?

Idea of field theory proof of KLT

Now we can give the idea of field theory proof of KLT relation:

- First using only the Lorentz invariance and spin symmetry we have $\mathcal{M}_3(1,2,3) = A_3(1,2,3)\widetilde{A}_3(1,2,3)$.
- Using BCFW-relation to expand gluon amplitudes and then recombine them to give the BCFW expansion of graviton amplitude. Thus by the induction method, we have the pure field theory proof.

Example One: four gravitons with relation

$$M_4(1,2,3,4) = (-)s_{12}A(1,3,4,2)A(1,4,3,2)$$

- Step one: Using $(1,2)$-BCFW-shifting to make

$$I = \oint \frac{dz}{z}(-)s_{12}A(\widehat{1},3,4,\widehat{2})A(\widehat{1},4,3,\widehat{2}) = 0$$

- BCFW expansion to get

$$\sum_h s_{12}A_3(\widehat{1},3,-\widehat{P}^h_{13})\frac{1}{s_{13}}A_3(\widehat{P}^{-h}_{13},4,\widehat{2})A_4(\widehat{1}(z_{13}),4,3,\widehat{2}(z_{13}))$$
$$+ \sum_h s_{12}A_4(\widehat{1}(z_{14}),3,4,\widehat{2}(z_{14}))A_3(\widehat{1},4,-\widehat{P}^h_{14})\frac{1}{s_{14}}A_3(\widehat{P}^h_{14},3,\widehat{2})$$

- For the first line we can use the BCJ relation $s_{12}A_4(\widehat{1}(z_{13}),4,3,\widehat{2}(z_{13}))=s_{13}(z_{13})A_4(4,\widehat{2}(z_{13}),3,\widehat{1}(z_{13})$ to write it as

$$A_3(\widehat{1},3,-\widehat{P}^h_{13})\frac{1}{s_{13}}A_3(\widehat{P}^{-h}_{13},4,\widehat{2})s_{13}(z_{13})A_4(4,\widehat{2}(z_{13}),3,\widehat{1}(z_{13})\,.$$

- Naively in the cut z_{13} we will have $s_{13}(z_{13})=0$. However, notice that

$$\begin{aligned} & A_4(4,2,3,1) \\ = & \sum_h \frac{A_3(4,\widehat{2}(z_{13}),\widehat{P}_{13}(z_{13}))A_3(-\widehat{P}_{13}(z_{13}),3,\widehat{1}(z_{13}))}{s_{13}} \\ & +\frac{A_3(\widehat{1},4,\widehat{P}_{23}(z_{13})A_3(-\widehat{P}_{23}(z_{14}),3,\widehat{1}(z_{14}))}{s_{14}} \end{aligned}$$

- Thus we see that

$$s_{13}(z_{13})A(4,\widehat{2}(z_{13}),3,\widehat{1}(z_{13})$$
$$=\sum_{h}A(4,\widehat{2}(z_{13}),P_{23})A(-P_{23}(z_{13}),3,\widehat{1}(z_{13}))$$

- Doing similarly for the second term we obtain

$$\sum_{h,\widetilde{h}}A_3(\widehat{1},3,-\widehat{P}_{13}^{h})\frac{1}{s_{13}}A_3(\widehat{P}_{13}^{-h},4,\widehat{2})A_3(\widehat{1},3,-\widehat{P}_{13}^{\widetilde{h}})A_3(\widehat{P}_{13}^{-\widetilde{h}},4,\widehat{2})$$
$$+\sum_{h,\widetilde{h}}A_3(\widehat{1},4,-\widehat{P}_{14}^{\widetilde{h}})A_3(\widehat{P}_{14}^{-\widetilde{h}},3,\widehat{2})A_3(\widehat{1},4,-\widehat{P}_{14}^{h})\frac{1}{s_{14}}A_3(\widehat{P}_{14}^{-h},3,\widehat{2})$$

- The double sum $\sum_{h,\widetilde{h}}$ can be written as two sums $\sum_{\widetilde{h}=h}$ and $\sum_{\widetilde{h}=-h}$.
- We have also vanishing identity for flipped helicity

$$A_3(\widehat{1},3,-\widehat{P}_{13}^{+})A_3(\widehat{1},3,-\widehat{P}_{13}^{-})=0\ .$$

- Using three point result we can combine to get

$$M_4(1,2,3,4)=\sum_{h=+,-}M_3(\widehat{1},3,-\widehat{P}_{13}^{h})\frac{1}{s_{13}}M_3(\widehat{P}_{13}^{-h},4,\widehat{2})$$
$$+M_3(\widehat{1},4,-\widehat{P}_{14}^{h})\frac{1}{s_{14}}M_3(\widehat{P}_{14}^{-h},3,\widehat{2})$$

Function S:

- To write down the general KLT relation, we need following function

$$\mathcal{S}[i_1,...,i_k|j_1,j_2,...,j_k]_{p_1} = \prod_{t=1}^{k}(s_{i_t 1}+\sum_{q>t}^{k}\theta(i_t,i_q)s_{i_t i_q})$$

where $\theta(i_t,i_q)=0$ is zero when pair (i_t,i_q) has same ordering at both set $\mathcal{I},\mathcal{J}$ and otherwise, it is one.. Set $\mathcal{J}$ is the reference ordering set.

$$\begin{aligned}\mathcal{S}[2,3,4|2,4,3] &= s_{21}(s_{31}+s_{34})s_{41},\\ \mathcal{S}[2,3,4|4,3,2] &= (s_{21}+s_{23}+s_{24})(s_{31}+s_{34})s_{41}\end{aligned}$$

- Property:

$$\mathcal{S}[i_1, ..., i_k | j_1, j_2, ..., j_k] = \mathcal{S}[j_k, ..., j_1 | i_k, .., i_1]$$

- Dual function

$$\widetilde{\mathcal{S}}[i_2, .., i_{n-1} | j_2, ..., j_{n-1}]_{p_n} = \prod_{t=2}^{n-1} (s_{j_t n} + \sum_{q<t} \theta(j_t, j_q) s_{j_t j_q}) .$$

$\widetilde{\mathcal{S}}$ and $\mathcal{S}$ are related as follows:

$$\widetilde{\mathcal{S}}[\mathcal{I} | \mathcal{J}]_{p_n} = \mathcal{S}[\mathcal{J}^T | \mathcal{I}^T]_{p_n}$$

$$\widetilde{\mathcal{S}}[2, 3, 4 | 4, 3, 2] = s_{45}(s_{35} + s_{34})(s_{25} + s_{23} + s_{24})$$

- A crucial property

$$I = \sum_{\alpha \in S_k} \mathcal{S}[\alpha(i_1, ..., i_k)|j_1, j_2, ..., j_k] A(k+2, \alpha(i_1, ..., i_k), 1) = 0$$

by BCJ relation.

General KLT relations:

- The manifest $(n-3)!$ symmetric form

$$
\begin{aligned}
& M_n \\
= \quad & (-)^{n+1} \sum_{\sigma\in S_{n-3}} \sum_{\alpha\in S_j} \sum_{\beta\in S_{n-3-j}} A(1,\{\sigma_2,..,\sigma_j\},\{\sigma_{j+1},..,\sigma_{n-2}\},n-1,n) \\
& \mathcal{S}[\alpha(\sigma_2,..,\sigma_j)|\sigma_2,..,\sigma_j]_{p_1} \widetilde{\mathcal{S}}[\sigma_{j+1},..,\sigma_{n-2}|\beta(\sigma_{j+1},..,\sigma_{n-2})]_{p_{n-1}} \\
& \widetilde{A}(\alpha(\sigma_2,..,\sigma_j),1,n-1,\beta(\sigma_{j+1},..,\sigma_{n-2}),n)
\end{aligned}
$$

[Bern, Dixon, Perelstein, Rozowsky; 1999]

- The set I or set J can be empty, so we have two more symmetric forms:

$$M_n = (-)^{n+1} \sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A(1,\sigma(2,n-2),n-1,n)$$
$$\mathcal{S}[\widetilde{\sigma}(2,n-2))|\sigma(2,n-2))]_{p_1}\widetilde{A}(n-1,n,\widetilde{\sigma}(2,n-2),1)$$

as well as

$$M_n = (-)^{n+1} \sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A(1,\sigma(2,n-2),n-1,n)$$
$$\widetilde{\mathcal{S}}[\sigma(2,n-2))|\widetilde{\sigma}(2,n-2))]_{p_{n-1}}\widetilde{A}(1,n-1,\widetilde{\sigma}(2,n-2),n)$$

The $(n-2)!$ symmetric new KLT formula:

$$M_n = (-)^n \sum_{\gamma,\beta} \widetilde{A}(n, \gamma(2, ..., n-1), 1)$$
$$\mathcal{S}[\gamma(2, ..., n-1)|\beta(2, .., n-1)]_{p_1} A(1, \beta(2, ..., n-1), n)/s_{123..(n-1)}$$

and

$$M_n = (-)^n \sum_{\beta,\gamma} A(1, \beta(2, ..., n-1), n)$$
$$\widetilde{\mathcal{S}}[\beta(2, ..., n-1)|\gamma(2, .., n-1)]_{p_n} \widetilde{A}(n, \gamma(2, ..., n-1), 1)/s_{2...n}$$

New vanishing identities:

If we use the (n_+, n_-) to denote the number of positive (negative) helicities in A having been flipped in $\widetilde{A}$, then when $n_+ \neq n_-$, we obtain zero, i.e.,

$$0 = (-)^n \sum_{\gamma,\beta} \widetilde{A}_{n_+ \neq n_-}(n, \gamma(2, ..., n-1), 1)$$
$$\mathcal{S}[\gamma(2, ..., n-1)|\beta(2, .., n-1)]_{p_1} A(1, \beta(2, ..., n-1), n)/s_{123..(n-1)}$$

The BCFW proof of the new KLT formula: First step, the pole structure analysis of a general one, for example, $s_{12..k}$

- The pole appears in only one of the amplitudes $\widetilde{A}_n$ and A_n.
- The pole appears in both amplitudes $\widetilde{A}_n$ and A_n.

The second step is to show the structure (A) giving zero:

- The BCFW expansion is given by

$$\frac{(-1)^{n+1}}{s_{\widehat{1}2..n-1}}\sum_{\gamma,\sigma,\beta}\frac{\sum_h \widetilde{A}_{n-k+1}(\widehat{n},\gamma,-\widehat{P}^h)\widetilde{A}_{k+1}(\widehat{P}^{-h},\sigma,\widehat{1})}{s_{12..k}}$$
$$\times\mathcal{S}[\gamma\sigma|\beta_{2,..,n-1}]A_n(\widehat{1},\beta_{2,..,n-1},\widehat{n})\,,$$

- Important observation:

$$\mathcal{S}[\gamma\sigma|\beta_{2,..,n-1}]=\mathcal{S}[\sigma|\rho_{2,k}]\times(\text{a factor independent of }\sigma)\,,$$

- By BCJ relation

$$\sum_{\sigma}\widetilde{A}_{k+1}(\widehat{P}^{-h},\sigma,\widehat{1})\mathcal{S}[\sigma|\rho_{2,k}]=0\,,$$

The third step is to show the part (B) giving the desired result:

- The BCFW expansion is now

$$\frac{(-1)^{n+1}}{s_{\widehat{1}2...(n-1)}}\sum_{\gamma,\beta,\sigma,\alpha}\left[\frac{\sum_h \widetilde{A}(\widehat{n},\gamma,\widehat{P}^{-h})\widetilde{A}(-\widehat{P}^{h},\sigma,\widehat{1})}{s_{12..k}}\right]\mathcal{S}[\gamma\sigma|\alpha\beta]$$
$$\left[\frac{\sum_h A(\widehat{1},\alpha,-\widehat{P}^{h})A(\widehat{P}^{-h},\beta,\widehat{n})}{s_{\widehat{1}2..k}}\right],$$

- Using $\mathcal{S}[\gamma\sigma|\alpha\beta] = \mathcal{S}[\sigma|\alpha] \times \mathcal{S}_{\widehat{P}}[\gamma|\beta]$ we obtain

$$\frac{(-1)^{n+1}}{s_{12..k}} \sum_h \left[\left(\sum_{\sigma,\alpha} \frac{\widetilde{A}(-\widehat{P}^h, \sigma, \widehat{1}) \mathcal{S}[\sigma|\alpha] A(\widehat{1}, \alpha, -\widehat{P}^h)}{s_{\widehat{1}2..k}} \right) \left(\sum_{\gamma,\beta} \frac{\widetilde{A}(\widehat{n}, \gamma, \widehat{P}^{-h}) \mathcal{S}_{\widehat{P}}[\gamma|\beta] A(\widehat{P}^{-h}, \beta, \widehat{n})}{s_{\widehat{P}k+1..(n-1)}} \right) \right] + (h, -h),$$

- It is nothing but

$$-\frac{\sum_h M_{k+1}(\widehat{1}, 2, \ldots, k, -\widehat{P}^h) M_{n-k+1}(\widehat{P}^{-h}, k+1, \ldots, \widehat{n})}{s_{12..k}},$$

The proof of $(n-3)!$ form will be almost same:

- Divide the pole structure into (A) and (B) part.
- Using the BCJ to show the (A) part to be zero.
- Using the $(n-2)!$ form to show that the part (B) is nothing, but the BCFW expansion.

Some remarks

- The on-shell structure of $[gravity] = [gluon]^2$ is extremely important. One can apply it to construct the loop amplitude of SUGRV.
- The reason we have the simple proof is because on-shell recursion relation has got rid of complicated off-shell information