Massive Gravity: An Introduction

"暗能量本质及其基本理论"高级研讨班

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提纲

引言

- 描述引力现象的标准模型：广义相对论
  - A. Einstein, "*Die Grundlage der allgemeinen Relativitaetstheorie*", Annalen der Physik **49** (1916), 769-822

Age of the Universe: Infinite

COSMIC TIMES

1919

Size of the Universe: 300,000 Light Years

SUN'S GRAVITY BENDS STARLIGHT

Einstein's Theory Triumphs

"One of the greatest—perhaps the greatest—of achievements in the history of human thought" was what Sir Joseph Thomson, President of the Royal Society of London, called Dr. Albert Einstein's prediction, which was apparently verified during the total eclipse of the Sun May 29 last.

Sir Joseph made his pronouncement during a discussion of the results from observations of the solar eclipse at a joint meeting of the Royal Society and the Royal Astronomical Society in London on Thursday evening, November 6, before a large attendance of astronomers and physicists. The excitement in the air was almost

actly as arranged, but out of the 16 plates taken, only two showed as many as five stars each. Prof. Eddington was also unable to stay several more months to take check-photographs of the star field.

Sir Frank explained in detail the apparatus both expeditions had employed, the way the photographic plates were measured back at the Greenwich Observatory, the corrections that had to be made for various disturbing factors, and the methods by which comparison between the theoretical and observed positions had been made. He convinced the meeting that the results were definite and conclusive, and that deflection did

MT. WILSON ASTRONOMER ESTIMATES MILKY WAY TEN TIMES BIGGER THAN THOUGHT

But Disputes Suggestions that Spiral Nebulae are Other "Island Universes"

The Milky Way is a "discoidal" (disc-shaped) galaxy of stars 10 times bigger than astronomers had previously conceived, according to Mt. Wilson astronomer, Dr. Harlow Shapley. Moreover, he claims, the Sun exists nearer to its edge than to its center. But he disputes the hypotheses of other astronomers that scores of spiral nebulae seen in the starry heavens are other galaxies, or "island universes", that resemble the Milky Way.

In his tour-de-force series of papers throughout 1918 and 1919, the prolific Dr. Shapley examines other recent astronomical work in astonishing detail, as well as presenting the results of

Dr. Shapley has concluded that "our galactic universe appears as a single, enormous, all-comprehending unit, the extent and form of which seem to be indicated through the dimensions of the widely extended assemblage of globular clusters." The center of our discoidal sidereal system "is distant from the Earth some twenty thousand parsecs"—more than 60,000 light-years—"in the direction of the constellation Sagittarius," Dr. Shapley continued.

His conclusions fly in the face of generally accepted astronomical wisdom. "Until the last year or so, most students of stellar problems believed rather vaguely that the Sun was not far from the

- 这个理论仍有很多令人不解之处，例如
  - nonrenormalizable, thus not UV complete; $E \gtrsim M_{\rm pl}$?
  - how to modify it consistently in IR? $\begin{cases} \text{dark energy} \\ \text{UV-IR connection} \end{cases}$

- 从场论观点看，引力子是自旋 $s=2$ 的粒子
  - 只有 $s=0,\ 2$ 的粒子能够在同荷物质之间传递吸引力
  - 以 $s=0$ 作为引力子的理论 (Einstein-Fokker, 1914) 不能解释光线在太阳的引力场中的偏转 ($T_{\text{em}}=0$)，对水星近日点进动的预言也不符合观测 (结果为广义相对论预言的 $-1/6$)
- 以 μ 记引力子的质量，静态弱场情形的引力势能为

$$V(r)\sim\begin{cases}-\dfrac{\kappa^2(0)Mm}{8\pi r}, & \mu=0\\[2ex] -\dfrac{4}{3}\dfrac{\kappa^2(\mu)Mm}{8\pi r}e^{-\mu r}, & \mu\neq 0\end{cases}$$

vDVZ discontinuity

- 牛顿极限的存在及牛顿引力的长程性，要求
  - 引力耦合常数与牛顿常数的关系

$$\frac{\kappa^2(0)}{8\pi}=\frac{\kappa^2(\mu)}{6\pi}=G$$

不同的标度才能给出相同的牛顿极限

  - 即使 $\mu\neq 0$，也应很小 $\Rightarrow$ 在牛顿极限成立的范围 $e^{-\mu r}\approx 1$
  - 在星系团尺度上，天文观测得到保守的质量上限

$$\mu\lesssim 2\times 10^{-65}\,\mathrm{kg}\sim 10^{-29}\,\mathrm{eV}$$

- 光子和引力子质量上限比较
  - A. S. Goldhaber and M. M. Nieto, "*Photon and Graviton Mass Limits*" , Rev. Mod. Phys. **82** (2010) 939

Description of method	$\lambda\!\!\!^{-}_C \gtrsim$ (m)	$\mu \lesssim$ (eV)	$\mu \lesssim$ (kg)	Comments
1 Secure photon mass limits:				
Dispersion in the ionosphere (99)	8×10^5	3×10^{-13}	4×10^{-49}	
Coulomb's law (83)	2×10^7	10^{-14}	2×10^{-50}	
Jupiter's magnetic field (103)	5×10^8	4×10^{-16}	7×10^{-52}	
Solar wind magnetic field (106)	2×10^{11} (1.3 AU)	10^{-18}	2×10^{-54}	
2 Speculative photon mass limits:				
Extended Lakes method (116–118)	3×10^9	7×10^{-17}	10^{-52}	$\lambda\!\!\!^{-}_C \sim 4\, R_\odot$ to 20 AU, depending
	$\Leftrightarrow 3 \times 10^{12}$	$\Leftrightarrow 7 \times 10^{-20}$	$\Leftrightarrow 10^{-55}$	on **B** speculations
Higgs mass for photon (82)	No limit feasible			Strong constraints on 3D Higgs parameter space
Cosmic mag. fields (108; 112), (82)	3×10^{19} (10^3 pc)	6×10^{-27}	10^{-62}	Needs const. **B** in galaxy regions
3 Graviton mass limits:				
Grav. wave dispersion (205)	3×10^{12}	8×10^{-20}	10^{-55}	Question mark for scalar graviton
Pulsar timing (206)	2×10^{16}	9×10^{-24}	2×10^{-59}	Fluctuations due to graviton phase velocity
Gravity over cluster sizes (177)	2×10^{22}	10^{-29}	2×10^{-65}	
Near field constraints (169)	3×10^{24} (10^8 pc)	6×10^{-32}	10^{-67}	For DGP model
Far field constraints (273)	3×10^{26} (10^{10} pc)	6×10^{-34}	10^{-69}	For DGP model

- 作为相对论性场论本身，Massive Gravity 具有诸多令人惊奇的性质，即使从纯理论的角度看也是有趣的
- 去年理论方面的 review 文章

K. Hinterbichler, “*Theoretical Aspects of Massive Gravity*”, `arXiv:1105.3735 [hep-th]`

  - Fierz-Pauli 理论
  - vDVZ 不连续性
  - 自由场的非线性扩充
  - Boulware-Deser 鬼场
  - Vainshtein 机制
  - 无鬼的有质量引力
  - DGP 模型
  - $\cdots$

- Bergshoeff-Fernández-Melgarejo-Rosseel-Townsend: “New Massive Gravity”
- N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Effective Field theory for massive gravitons and gravity in theory space,” Ann. Phys. **305** (2003) 96

引力场的线性近似

- 约定 $\eta_{\mu\nu}=(-+\cdots+)$；在平坦时空中，整数自旋的粒子可用带 Lorentz 张量指标的场 $\Phi^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_p}(x)$ 描述，经典自由场的每个分量均满足 K-G 方程

$$(\Box-\mu^2)\Phi^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_p}(x)=0,\qquad \Box\equiv\eta^{\mu\nu}\partial_\mu\partial_\nu$$

- 实际的场方程应蕴涵 K-G 方程 —— 以上、下指标分别全反称的 (p,q)-型张量场为例，Fierz-Pauli 形式的运动方程写做

$$\begin{aligned}&\left\{\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_{p+q+1}]}_{\nu_{p+q+1}}\partial^{\nu_{p+q+1}}\partial_{\mu_{p+q+1}}-\mu^2\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_{p+q}]}_{\nu_{p+q}}\right\}\\ \times\quad&\Phi^{\nu_{p+1}\cdots\nu_{p+q}}_{\mu_{q+1}\cdots\nu_{p+q}}(x)=0\end{aligned}$$

- 这里反称化并不引入 $\frac{1}{k!}$，如 $\delta^{[\mu_1}_{\nu_1}\delta^{\mu_2]}_{\nu_2}\equiv\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}-\delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2}$

M. Fierz and W. Pauli, “” *On relativistic wave-equations for particles of arbitrary spin in an electromagnetic field*”, Proc. Loy. Soc. London **A173** (1939) 211

- 例①：(0,0)-型标量场运动方程即普通的 Klein-Gordon 方程

$$\left\{\delta^\mu_\nu \partial^\nu \partial_\mu - \mu^2\right\}\Phi(x) = 0 \;\Rightarrow\; (\Box - \mu^2)\Phi(x) = 0$$

- 例②：(1,0)-型矢量场 $\Phi_\mu(x)$ 的运动方程退化为

$$\left\{\delta^{[\mu_1}_{\nu_1}\delta^{\mu_2]}_{\nu_2}\partial^{\nu_2}\partial_{\mu_2} - \mu^2\delta^{\mu_1}_{\nu_1}\right\}\Phi_{\mu_1}(x) = 0$$

$$\Rightarrow\quad (\Box - \mu^2)\Phi_\mu(x) - \partial_\mu\partial^\nu\Phi_\nu(x) = 0$$

  - 与有质量矢量场比较

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu$$

$$\Rightarrow\quad (\Box - \mu^2)A_\mu - \partial_\mu\partial^\nu A_\nu = 0$$

  - 用 ∂^μ 作用到运动方程得

$$\Box\partial^\mu A_\mu - \mu^2\partial^\mu A_\mu - \partial^\mu\partial_\mu\partial^\nu A_\nu = 0 \;\Rightarrow\; \mu^2\partial^\mu A_\mu = 0$$

  - 故当 $\mu \neq 0$ 时

$$(\Box - \mu^2)A_\mu = 0, \quad \partial^\mu A_\mu = 0$$

- 例③： $(p,0)$-型全反称张量场 $\Phi_{\mu_1\cdots\mu_p}(x)$ 的运动方程退化为

$$
\begin{aligned}
&\left\{\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_p}_{\nu_p}\delta^{\mu_{p+1}]}_{\nu_{p+1}}\partial^{\nu_{p+1}}\partial_{\mu_{p+1}}-\mu^2\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_p]}_{\nu_p}\right\}\\
&\times\Phi_{\mu_1\cdots\mu_p}(x)=0\\
\Rightarrow\quad &\Box\Phi_{\mu_1\cdots\mu_p}+(-1)^p\,\partial^\nu\partial_{[\mu_1}\Phi_{\mu_2\cdots\mu_p]\nu}-\mu^2\Phi_{\mu_1\cdots\mu_p}=0
\end{aligned}
$$

- 若 $\mu\neq 0$，方程分裂为

$$
\begin{cases}
(\Box-\mu^2)\Phi_{\mu_1\cdots\mu_p}(x)=0 & \text{K-G 方程}\\
\partial^{\mu_1}\Phi_{\mu_1\cdots\mu_p}(x)=0 & \text{约束条件}
\end{cases}
$$

- 动量表象约束 $k^{\mu_1}\widetilde{\Phi}_{\mu_1\cdots\mu_p}(k)=0$，静止系 $k^\mu=(\mu,0,\cdots,0)$ 中共 C^{p-1}_{D-1} 个约束 $\widetilde{\Phi}_{0i_1\cdots i_{p-1}}=0\Rightarrow$ 物理自由度由小群计数

$$
C^p_D-C^{p-1}_{D-1}=C^p_{D-1}\quad\leftrightarrow\quad\widetilde{\Phi}_{i_1\cdots i_p}\in SO(D-1)\text{-反称张量}
$$

- 当 $\mu=0$ 时有规范对称性 $\Phi_{\mu_1\cdots\mu_p}\to\Phi_{\mu_1\cdots\mu_p}+\partial_{[\mu_1}\chi_{\mu_2\cdots\mu_p]}$

- 规范自由度 $\chi_{\mu_1\cdots\mu_{p-1}}(x)$ 携带的指标全反称；可取 Lorentz 规范 $\partial^{\mu_1}\Phi_{\mu_1\mu_2\cdots\mu_p}=0$，运动方程简化为 $\Box\Phi_{\mu_1\cdots\mu_p}=0$
- Lorentz 规范的剩余对称性

$$\Phi_{\mu_1\cdots\mu_p}\to\Phi_{\mu_1\cdots\mu_p}+\partial_{[\mu_1}\chi_{\mu_2\cdots\mu_p]},\quad \Box\chi_{\mu_2\cdots\mu_p}=0$$

- 动量表象的规范条件 $k^{\mu_1}\widetilde{\Phi}_{\mu_1\cdots\mu_p}=0$，剩余规范对称性

$$\widetilde{\Phi}_{\mu_1\cdots\mu_p}\to\widetilde{\Phi}_{\mu_1\cdots\mu_p}+k_{[\mu_1}\tilde{\chi}_{\mu_2\cdots\mu_p]},\quad k^2\tilde{\chi}_{\mu_2\cdots\mu_p}=0$$

- 在光锥坐标 $k^\mu=(E,0,\cdots,0,E)$ 中

$$\widetilde{\Phi}_{0\mu_2\cdots\mu_p}+\widetilde{\Phi}_{D-1\mu_2\cdots\mu_p}=0\ \begin{cases}\widetilde{\Phi}_{0i_2\cdots i_p}+\widetilde{\Phi}_{D-1i_2\cdots i_p}=0\\ \widetilde{\Phi}_{0D-1i_3\cdots i_p}=0\end{cases}$$

$i_2,\cdots,i_p\in\{1,2,\cdots,D-2\}$ 横向指标

- 自由度的约化

$$\widetilde{\Phi}_{\mu_1\cdots\mu_p}\to\left(\widetilde{\Phi}_{i_1i_2\cdots i_p},\ \widetilde{\Phi}_{0i_2\cdots i_p}\right)$$

- 剩余规范变换

$$
\begin{aligned}
\widetilde{\Phi}_{i_1 i_2\cdots i_p} &\to \widetilde{\Phi}_{i_1 i_2\cdots i_p} + k_{[i_1}\tilde{\chi}_{i_2\cdots i_p]} \\
\widetilde{\Phi}_{0 i_2\cdots i_p} &\to \widetilde{\Phi}_{0 i_2\cdots i_p} - E\tilde{\chi}_{i_2\cdots i_p} \\
\widetilde{\Phi}_{D-1 i_2\cdots i_p} &\to \widetilde{\Phi}_{D-1 i_2\cdots i_p} + E\tilde{\chi}_{i_2\cdots i_p}
\end{aligned}
$$

- 故令 $\tilde{\chi}_{i_2\cdots i_p} = \widetilde{\Phi}_{0 i_2\cdots i_p}/E$，可同时将 $\widetilde{\Phi}_{0 i_2\cdots i_p}$ 和 $\widetilde{\Phi}_{D-1 i_2\cdots i_p}$ 规范固定成零，仅留下横向分量 $\widetilde{\Phi}_{i_1 i_2\cdots i_p}$ 为动力学自由度

自旋为 2 的场

- $(1,1)$-型张量场 $\Phi^{\nu}_{\mu}(x)$ 的迹记做 $\Phi(x) = \Phi^{\lambda}_{\lambda}(x)$；运动方程退化为

$$
\left\{ \delta^{[\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3]}_{\nu_3}\partial^{\nu_3}\partial_{\mu_3} - \mu^2 \delta^{[\mu_1}_{\nu_1}\delta^{\mu_2]}_{\nu_2} \right\} \Phi^{\nu_2}_{\mu_2}(x) = 0
$$

$$
\begin{aligned}
\Rightarrow \quad (\Box - \mu^2)\left(\Phi^{\mu}_{\nu} - \Phi\delta^{\mu}_{\nu}\right) + \partial^{\mu}\partial_{\nu}\Phi - \partial^{\mu}\partial_{\lambda}\Phi^{\lambda}_{\nu} \\
- \partial_{\nu}\partial^{\lambda}\Phi^{\mu}_{\lambda} + \partial_{\lambda}\partial^{\rho}\Phi^{\lambda}_{\rho}\delta^{\mu}_{\nu} = 0
\end{aligned}
$$

- 在有质量 $\mu \neq 0$ 情形，分别用 ∂_{μ} 和 ∂^{ν} 作用于方程，得

$$
\partial_{\lambda}\Phi^{\lambda}_{\mu} = \partial_{\mu}\Phi, \quad \partial^{\lambda}\Phi^{\nu}_{\lambda} = \partial^{\nu}\Phi
$$

- 将这两个约束方程代入运动方程

$$(\Box-\mu^2)\Phi^\mu_\nu=\partial^\mu\partial_\nu\Phi-\mu^2\Phi\delta^\mu_\nu$$

$$\Rightarrow\quad (D-1)\mu^2\Phi(x)=0\ \Rightarrow\ \Phi(x)=0\quad (\text{设 } D>1)$$

$$\Rightarrow\quad \begin{cases}(\Box-\mu^2)\Phi^\mu_\nu=0 & \text{K-G 方程}\\ \partial_\mu\Phi^\mu_\nu=\partial^\mu\Phi^\nu_\mu=\Phi=0 & \text{约束条件}\end{cases}$$

- 零质量场 Φ^μ_ν 遵从的运动方程

$$\Box(\Phi^\mu_\nu-\Phi\delta^\mu_\nu)+\partial^\mu\partial_\nu\Phi-\partial^\mu\partial_\lambda\Phi^\lambda_\nu-\partial_\nu\partial^\lambda\Phi^\mu_\lambda+\partial_\lambda\partial^\rho\Phi^\lambda_\rho\delta^\mu_\nu=0$$

- 规范对称性

$$\Phi^\mu_\nu\to\Phi^\mu_\nu+\partial^\mu\chi_\nu$$

- 将 $\Phi_{\mu\nu}(x)\equiv\eta_{\mu\lambda}\Phi^\lambda_\nu(x)$ 分解成对称部分和反称部分之和

$$\Phi_{\mu\nu}(x)=\frac{1}{2}\left[h_{\mu\nu}(x)+B_{\mu\nu}(x)\right]\qquad h_{\mu\nu}\in(1,1)\oplus(0,0)$$

$$\Rightarrow\quad (\Box-\mu^2)(h_{\mu\nu}-h\eta_{\mu\nu})+\partial_\mu\partial_\nu h-\partial^\lambda\partial_\mu h_{\nu\lambda}$$
$$-\partial^\lambda\partial_\nu h_{\mu\lambda}+\partial^\lambda\partial^\rho h_{\lambda\rho}\eta_{\mu\nu}=0$$

- 有质量的运动方程分裂为

$$\begin{cases} (\Box-\mu^2)h_{\mu\nu}=0 & \text{K-G 方程} \\ \partial^\mu h_{\mu\nu}=h=0 & \text{约束条件} \end{cases}$$

- 零质量运动方程

$$\Box(h_{\mu\nu}-h\eta_{\mu\nu})+\partial_\mu\partial_\nu h-\partial^\lambda\partial_\mu h_{\nu\lambda}-\partial^\lambda\partial_\nu h_{\mu\lambda}+\partial^\lambda\partial^\rho h_{\lambda\rho}\eta_{\mu\nu}=0$$

- 规范对称性

$$h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\chi_\nu+\partial_\nu\chi_\mu$$

- 在规范变换下，散度及迹的梯度变化为

$$\left.\begin{aligned} \partial^\mu h'_{\mu\nu}&=\partial^\mu h_{\mu\nu}+\Box\chi_\nu+\partial_\nu\partial^\mu\chi_\mu \\ \partial_\nu h'&=\partial_\nu h+2\partial_\nu\partial^\mu\chi_\mu \end{aligned}\right\}\Rightarrow$$

$$\partial^\mu h'_{\mu\nu}-\frac{1}{2}\partial_\nu h'=\partial^\mu h_{\mu\nu}-\frac{1}{2}\partial_\nu h+\Box\chi_\nu$$

- 故可取谐和规范 $\partial^\mu h_{\mu\nu}=\frac{1}{2}\partial_\nu h$ 简化运动方程 $\Box h_{\mu\nu}=0$

- 几何解释：把二秩对称张量 $g_{\mu\nu}(x) \equiv \eta_{\mu\nu} - 2\kappa h_{\mu\nu}(x)$ 解释成时空度量，定义 $h^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\rho}h_{\lambda\rho}$，近似到一阶有

$$g^{\mu\nu} \approx \eta^{\mu\nu} + 2\kappa h^{\mu\nu}$$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}(g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa}) \approx \kappa(h_{\mu\nu,}{}^{\lambda} - h^{\lambda}_{\mu,\nu} - h^{\lambda}_{\nu,\mu})$$

$$\begin{aligned} R_{\lambda\mu\rho\nu} &= \frac{1}{2}\left(g_{\lambda\nu,\,\mu\rho} + g_{\mu\rho,\,\lambda\nu} - g_{\lambda\rho,\,\mu\nu} - g_{\mu\nu,\,\lambda\rho}\right) \\ &\quad + g_{\sigma\tau}\left(\Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\tau}{}_{\mu\rho} - \Gamma^{\sigma}{}_{\lambda\rho}\Gamma^{\tau}{}_{\mu\nu}\right) \\ &\approx \; -\kappa\left(h_{\lambda\nu,\,\mu\rho} + h_{\mu\rho,\,\lambda\nu} - h_{\lambda\rho,\,\mu\nu} - h_{\mu\nu,\,\lambda\rho}\right) \end{aligned}$$

$$R_{\mu\nu} = g^{\lambda\rho}R_{\lambda\mu\rho\nu} \approx \kappa(\partial_\mu\partial_\nu h + \Box h_{\mu\nu} - \partial^\lambda\partial_\mu h_{\nu\lambda} - \partial^\lambda\partial_\nu h_{\mu\lambda})$$

$$R = g^{\mu\nu}R_{\mu\nu} \approx 2\kappa(\Box h - \partial^\lambda\partial^\rho h_{\lambda\rho})$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \approx \kappa\Big[&\Box(h_{\mu\nu} - h\eta_{\mu\nu}) + \partial_\mu\partial_\nu h \\ &- \partial^\lambda\partial_\mu h_{\nu\lambda} - \partial^\lambda\partial_\nu h_{\mu\lambda} + \partial^\lambda\partial_\rho h_{\lambda\rho}\eta_{\mu\nu}\Big] \end{aligned}$$

- 故 $\mu=0$ 的运动方程等价于线性化的真空爱因斯坦场方程

$$\left[R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right]_{\text{线性化}}=0$$

- $h_{\mu\nu}$ 的规范变换来自时空无穷小广义坐标变换

$$x^\mu \to x'^\mu = x^\mu + 2\kappa\epsilon^\mu(x)$$
$$\Rightarrow \quad \delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -2\kappa(\nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu)$$

- 当 $g_{\mu\nu}(x)=\eta_{\mu\nu}-2\kappa h_{\mu\nu}(x)$ 时，有 $\delta h_{\mu\nu}\approx\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu$
- 如 $h_{\mu\nu}$ 具有谐和规范，$h'_{\mu\nu}=h_{\mu\nu}+\partial_\mu\chi_\nu+\partial_\nu\chi_\mu$ 也满足谐和规范的条件是 $\Box\chi_\nu=0$，因此“在壳”的 χ_ν 提供了剩余规范对称变换；转换到动量表象

$$\text{谐和规范} \quad k^\mu\tilde{h}_{\mu\nu}=\frac{1}{2}k_\nu\tilde{h}$$
$$\text{剩余对称} \quad \tilde{h}_{\mu\nu}\to\tilde{h}_{\mu\nu}+k_\mu\tilde{\chi}_\nu+k_\nu\tilde{\chi}_\mu,\quad k^2\tilde{\chi}_\nu=0$$

- 取光锥坐标 $k^\mu = (E, 0, \cdots, 0, E)$，谐和规范条件给出

$$\tilde{h}_{00} + \tilde{h}_{D-1,0} = -\frac{1}{2}\tilde{h} = -\frac{1}{2}\left[-\tilde{h}_{00} + \sum_{i=1}^{D-2}\tilde{h}_{ii} + \tilde{h}_{D-1,D-1}\right]$$

$$\tilde{h}_{0,D-1} + \tilde{h}_{D-1,D-1} = \frac{1}{2}\left[-\tilde{h}_{00} + \sum_{i=1}^{D-2}\tilde{h}_{ii} + \tilde{h}_{D-1,D-1}\right]$$

$$\tilde{h}_{0i} + \tilde{h}_{D-1,i} = 0$$

- 前两个方程给出

$$\frac{1}{2}\tilde{h}_{00} + \tilde{h}_{0,D-1} + \frac{1}{2}\tilde{h}_{D-1,D-1} = -\frac{1}{2}\sum_i \tilde{h}_{ii} = \frac{1}{2}\sum_i \tilde{h}_{ii} = 0$$

- 因此谐和规范将自由度 $\tilde{h}_{\mu\nu}$ 约化为无迹对称的横向分量 $\tilde{h}_{ij}$ 及若干其他分量

$$\begin{array}{ccccc} \tilde{h}_{\mu\nu} & \rightarrow & \tilde{h}_{ij}, & \tilde{h}_{0i}, & \tilde{h}_{00}, \quad \tilde{h}_{0,D-1} \\ \frac{D(D+1)}{2} & & \frac{(D-2)(D-1)}{2} - 1 & D-2 & 1 \qquad 1 \end{array}$$

- 剩余规范变换

$$\tilde{h}_{ij} \to \tilde{h}_{ij} + \cancel{k_i \tilde{\chi}_j} + \cancel{k_j \tilde{\chi}_i} \qquad \text{保持不变}$$

$$\tilde{h}_{0i} \to \tilde{h}_{0i} - E\tilde{\chi}_i + \cancel{k_i \tilde{\chi}_0} \qquad \text{可规范掉 } \tilde{h}_{0i}$$

$$\tilde{h}_{00} \to \tilde{h}_{00} - 2E\tilde{\chi}_0 \qquad \text{可规范掉 } \tilde{h}_{00}$$

$$\tilde{h}_{0,D-1} \to \tilde{h}_{0,D-1} - E\tilde{\chi}_{D-1} + E\tilde{\chi}_0 \qquad \text{可规范掉 } \tilde{h}_{0,D-1}$$

- 零质量物理自由度由横向无迹 $\tilde{h}_{ij}$ 给出，$\frac{(D-2)(D-1)}{2} - 1$
- 有质量的自由度数 $\frac{(D-1)D}{2} - 1$：取 $k^\mu = (\mu, 0 \cdots, 0)$

$$k^\mu \tilde{h}_{\mu\nu} = 0 \ \Rightarrow \ \tilde{h}_{0\nu} = 0 \qquad \text{物理自由度 } h_{ij}$$

$$\tilde{h} = 0 \ \Rightarrow \ -\tilde{h}_{00} + \sum_{i=1}^{D-1} \tilde{h}_{ii} \ \Rightarrow \ \sum_{i=1}^{D-1} \tilde{h}_{ii} = 0 \quad \text{无迹}$$

vDVZ 不连续性

- 令 $J^{\mu\nu} = \kappa T^{\mu\nu}$ 为物质源，与 $h_{\mu\nu}$ 耦合的拉氏量

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \hat{K}^{\mu\nu,\lambda\rho} h_{\lambda\rho} - \kappa h_{\mu\nu} T^{\mu\nu} \ \Rightarrow \ \hat{K}^{\mu\nu,\lambda\rho} h_{\lambda\rho} = \kappa T^{\mu\nu}$$

- 在有质量情形，动能算子 $\hat{K}^{\mu\nu,\lambda\rho}$ 在质壳外可逆

$$\hat{K}^{-1}_{\mu\nu,\lambda\rho} = \frac{1}{2(\Box - \mu^2)}\left[\left(\eta_{\mu\lambda} - \frac{\partial_\mu\partial_\lambda}{\mu^2}\right)\left(\eta_{\nu\rho} - \frac{\partial_\nu\partial_\rho}{\mu^2}\right) + \left(\eta_{\mu\rho} - \frac{\partial_\mu\partial_\rho}{\mu^2}\right)\left(\eta_{\nu\lambda} - \frac{\partial_\nu\partial_\lambda}{\mu^2}\right) - \frac{2}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\mu^2}\right)\left(\eta_{\lambda\rho} - \frac{\partial_\lambda\partial_\rho}{\mu^2}\right)\right]$$

- 相应的 Green's 函数

$$G_{\mu\nu,\lambda\rho}(x-y) = -\frac{1}{2}\int \frac{d^D k}{(2\pi)^D}\frac{e^{ik\cdot(x-y)}}{k^2+\mu^2-i\epsilon} \times \left[\underbrace{\Pi_{\mu\lambda}(k)\Pi_{\nu\rho}(k) + \Pi_{\mu\rho}(k)\Pi_{\nu\lambda}(k) - \frac{2}{D-1}\Pi_{\mu\nu}(k)\Pi_{\lambda\rho}(k)}_{2\Pi_{\mu\nu,\lambda\rho}(k)}\right]$$

$$\Pi_{\mu\nu}(k) \equiv \eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}, \quad \Pi_{\mu\nu,\lambda\rho}(k) = \sum_a \bar{e}^{(a)}_{\mu\nu}(k)e^{(a)}_{\lambda\rho}(k)$$

- 在 $D=4$ 维中带物质源的非齐次方程解

$$h_{\mu\nu}(x)=\kappa\int d^4y\,D_{\mu\nu,\lambda\rho}(x-y)T^{\lambda\rho}(y)$$

- 作用在守恒流 $k_\mu T^{\mu\nu}(k)=0$ 上的张量 $\Pi_{\mu\nu}$ 如同 $\eta_{\mu\nu}$，因此静止点源 $T^{00}\sim M\delta^{(3)}(\mathbf{x})$ 产生的场及与粒子 m 的作用能

$$\begin{aligned}
h_{00} &= -\frac{1}{2}\left[(\eta_{00})^2+(\eta_{00})^2-\frac{2}{3}(\eta_{00})^2\right]\frac{\kappa M}{4\pi}\frac{e^{-\mu r}}{r} \\
&= -\frac{4}{3}\cdot\frac{\kappa M}{8\pi}\frac{e^{-\mu r}}{r} \\
h_{ij} &= -\frac{1}{2}\left[\eta_{i0}\eta_{j0}+\eta_{j0}\eta_{i0}-\frac{2}{3}\eta_{ij}\eta_{00}\right]\frac{\kappa M}{4\pi}\frac{e^{-\mu r}}{r} \\
&= -\frac{2}{3}\cdot\frac{\kappa M}{8\pi}\frac{e^{-\mu r}}{r}\delta_{ij} \\
h_{0i} &= 0 \\
V &= \kappa\int d^3\mathbf{x}h_{00}T'^{00}=-\frac{4}{3}\cdot\frac{\kappa^2 Mm}{8\pi r}e^{-\mu r}
\end{aligned}$$

- 零质量场方程有规范对称性，动能算子作用在纯规范上为零

$$\hat{K}^{\mu\nu,\lambda\rho}\left(\partial_\lambda\chi_\nu+\partial_\rho\chi_\lambda\right)=0\ \Rightarrow\ \hat{K}\ \text{不可逆}$$

- 取谐和规范，场方程 $\Box h_{\mu\nu}=0$，作用在满足规范条件的子空间上的动能算子 $\Box$ 可逆，故在拉氏量中引进规范固定项

$$-\frac{1}{2\xi}\left(\partial^\mu h_{\mu\nu}-\frac{1}{2}\partial_\nu h\right)^2 \qquad R_\xi\ \text{规范}$$

- Feynman 规范 $\xi=1$ 下的传播子

$$D_{\mu\nu,\lambda\rho}(k)=-\frac{\Pi^\perp_{\mu\nu,\lambda\rho}(k)}{k^2+\mu^2-i\epsilon}$$

$$\Pi^\perp_{\mu\nu,\lambda\rho}(k)=\sum_{s=\pm2}\bar{e}^{(s)}_{\mu\nu}(k)e^{(s)}_{\lambda\rho}(k)$$

$$=\frac{1}{2}\left[\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}-\eta_{\mu\nu}\eta_{\lambda\rho}\right]\ +\ k^\mu\ \text{依赖项}$$

- 静止点源 $T^{00} \sim M\delta^{(3)}(\mathbf{x})$ 产生的零质量场及与粒子作用势

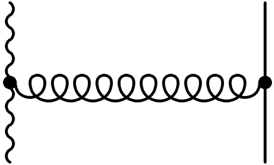
$$
\begin{aligned}
h_{00} &= -\frac{1}{2}\left[(\eta_{00})^2 + (\eta_{00})^2 - (\eta_{00})^2\right]\frac{\kappa M}{4\pi r} = -\frac{\kappa M}{8\pi r} \\
h_{ij} &= -\frac{1}{2}\left[\eta_{i0}\eta_{j0} + \eta_{j0}\eta_{i0} - \eta_{ij}\eta_{00}\right]\frac{\kappa M}{4\pi r} = -\frac{\kappa M}{8\pi r}\delta_{ij} \\
h_{0i} &= 0 \\
V &= -\frac{\kappa^2 M m}{8\pi r}
\end{aligned}
$$

H. van Dam and M. Veltman, "*Massive and massless Yang-Mills and gravitational fields*", Nucl. Phys. **B22** (1974) 397

- 当 $\mu r \ll 1$，可忽略指有质量理论中的数因子，对两种理论的耦合常数 κ^2 采用不同的标度可使牛顿近似均成立；但太阳引力场引起光线偏转的结果不同
- 光线偏转的信息蕴涵在流-流相互作用的振幅之中

$$
\mathcal{A}(\mu) \sim \frac{\kappa^2(\mu)}{2}\int d^4x d^4y T'^{\mu\nu}(x) D_{\mu\nu,\lambda\rho}(x-y) T^{\lambda\rho}(y)
$$

- 引力传播子



$$D_{\mu\nu,\lambda\rho}(x-y)$$

$$\sim D(x-y)\times\begin{cases}\frac{1}{2}\Big[\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}-\eta_{\mu\nu}\eta_{\lambda\rho}\Big], & \mu=0\\ \frac{1}{2}\Big[\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}-\frac{2}{3}\eta_{\mu\nu}\eta_{\lambda\rho}\Big], & \mu\neq 0\end{cases}$$

- 电磁场的能量-动量张量 $T'^{\mu\nu}$ 无迹，传播子中的 vDVZ 跳变项对振幅没有贡献，故当 $\mu\approx 0$ 时振幅 $\mathcal{A}(\mu)$ 与 $\mathcal{A}(0)$ 唯一差别是 $\kappa^2(\mu)=3\kappa^2(0)/4$

$$\Delta\varphi=\begin{cases}F\left[\kappa^2(\mu)\right], & \mu\neq 0\\ F\left[\kappa^2(0)\right], & \mu=0\end{cases}\qquad \text{函数 } F \text{ 相同}$$

$$\Rightarrow\quad \Delta\varphi\Big|_{\mu\neq 0}=\frac{\kappa^2(\mu)M_\odot}{2\pi b}=\frac{3}{4}\Delta\varphi\Big|_{\mu=0}$$

H. van Dam and M. Veltman, "*On the Mass of the Graviton*", Gen. Rel. Grav. **3** (1972) 215

ON THE MASS OF THE GRAVITON†

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In that case (5) and (8), differing only by a term of the form $\delta_{\mu\nu}\delta_{\alpha\beta}$, give equal results. However, the coupling constants are different, and we find that the deflection of a light ray near the sun in the massive theory is 3/4 of the deflection in the mass-less theory (for recent experimental results see [3]).

- 因 $1-3/4=25\%$ 超出了实际观测已达的精度 $\leq 10\%$，可以推断引力子无质量；随之出现的问题:
  - 假定有一个独立的方法探测引力子的质量 μ，作为理论的输入参数，μ 偏离零的一个微小误差将导致输出误差大到 25%
  - 类似于 Wilson 自然性判据的违反

- 关于自然性判据，Susskind 在一九七八年给出的一段表述是

> (自然性) 要求一个理论的可观测性质在其基本参数的微小变动下是稳定的
>
> ... a concept of naturalness which requires the observable properties of a theory to be stable against minute variations of the fundamental parameters

L. Susskind, "*Dynamics of spontaneous symmetry breaking in the Weinberg-Salam theory*", SLAC-PUB-2142 (1978), Phys. Rev. **D20** (1979) 2619

- 解决引力质量的问题可以借助 Vainshtein 机制

A. I. Vainshtein, "*To the problem of nonvanishing gravitation mass*", Phys. Lett. **B39** (1972) 393

C. Deffayet, G. R. Dvali, G. Gabadadze, and A. I. Vainshtein, "*Nonperturbative continuity in graviton mass versus perturbative discontinuity*", Phys. Rev. **D65** (2002) 044026; `arXiv:hep-th/0106001`

- 矢量理论为何无类似跳变？考虑约束 $k^\mu \tilde{A}_\mu(k) = 0$

$$\text{基展开}\quad \tilde{A}_\mu(k) = \sum_{a=1}^{D-1} c_a(k) e_\mu^{(a)}, \quad k^\mu e_\mu^{(a)} = 0$$

$$\text{幺正性}\quad \eta^{\mu\nu} \bar{e}_\mu^{(a)} e_\nu^{(b)} = \eta^{ab} = \delta^{ab}$$

$$\eta^{\mu\nu} \bar{e}_\mu^{(a)} \frac{k_\nu}{m} = 0, \quad \eta^{\mu\nu} \frac{k_\mu}{m} \frac{k_\nu}{m} = \eta_{00}$$

$$\text{完备性}\quad \eta_{00} \frac{k_\mu}{m} \frac{k_\nu}{m} + \delta_{ab} e_\mu^{(a)} \bar{e}_\nu^{(b)} = \eta_{\mu\nu}$$

$$\Rightarrow \sum_{a=1}^{D-1} e_\mu^{(a)} \bar{e}_\nu^{(a)} = \eta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} = \Pi_{\mu\nu}$$

- 有质量情形的极化 $e_\mu^{(a)}$ 通常在静止系 $k^\mu = (m, 0, \cdots, 0)$ 中构造，$e_0^{(a)} = 0$，$e_i^{(a)}$ 是 $D-1$ 维空间中的单位向量，可取

$$e_\mu^{(a)} = (0, \underbrace{0, \cdots, 0, 1, 0, \cdots, 0}_{\text{第 } a \text{ 个位置为 } 1})$$

- 静止系禁止取 $m \to 0$ 极限，而且不是描述物质之间静态力合适的坐标系：$k^\mu \tilde{J}_\mu = 0 \;\Rightarrow\; \tilde{J}_0 = 0$，故在传递相互作用的粒子静止的坐标系中，流守恒的物质源不可能静止
- 在运动坐标系中，$e_\mu^{(a)}$ 的表达式可通过对静止系做 Lorentz boost 得到；设 $\mathbf{v}$ 为飞行速度，$\langle \mathbf{v}|$, $|\mathbf{v}\rangle$ 分别是行和列向量

$$\Lambda^\mu{}_\nu(\mathbf{v}) = \begin{pmatrix} \gamma & \gamma\langle \mathbf{v}| \\ \gamma|\mathbf{v}\rangle & \mathbf{1} + \frac{\gamma-1}{|\mathbf{v}|^2}|\mathbf{v}\rangle\langle\mathbf{v}| \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-|\mathbf{v}|^2}}$$

- $\Lambda(\mathbf{v})$ 对 $m\delta_0^\mu$ 作用得 $k^\mu = (\gamma m, \gamma m \mathbf{v})$；且 $\Lambda(\mathbf{v})^T \eta \Lambda(\mathbf{v}) = \eta$

$$\underbrace{\begin{pmatrix} -\gamma & \gamma\langle \mathbf{v}| \\ -\gamma|\mathbf{v}\rangle & \mathbf{1} + \frac{\gamma-1}{|\mathbf{v}|^2}|\mathbf{v}\rangle\langle\mathbf{v}| \end{pmatrix}}_{\Lambda(\mathbf{v})^T\eta} \underbrace{\begin{pmatrix} \gamma & \gamma\langle \mathbf{v}| \\ \gamma|\mathbf{v}\rangle & \mathbf{1} + \frac{\gamma-1}{|\mathbf{v}|^2}|\mathbf{v}\rangle\langle\mathbf{v}| \end{pmatrix}}_{\Lambda(\mathbf{v})}$$

$$= \begin{pmatrix} -\gamma^2(1-|\mathbf{v}|^2) & [-\gamma^2+\gamma+\gamma(\gamma-1)]\langle\mathbf{v}| \\ [-\gamma^2+\gamma+\gamma(\gamma-1)]|\mathbf{v}\rangle & \mathbf{1} + \left[\frac{2(\gamma-1)+(\gamma-1)^2}{|\mathbf{v}|^2} - \gamma^2\right]|\mathbf{v}\rangle\langle\mathbf{v}| \end{pmatrix}$$

$$= \mathrm{diag}(-1, \mathbf{1})$$

- 用 $\mathbf{n}_{(a)}$ 表示 $D-1$ 维空间中第 a 个正交归一的坐标向量，则经 Lorentz boost 的极化为 $(E=k^0=m\gamma)$

$$
\begin{aligned}
e^{\mu}_{(a)} &= \Lambda(\mathbf{v})\cdot\begin{pmatrix}0\\ \mathbf{n}_{(a)}\end{pmatrix}=\begin{pmatrix}\gamma\mathbf{n}_{(a)}\cdot\mathbf{v}\\ \mathbf{n}_{(a)}+\dfrac{\gamma-1}{|\mathbf{v}|^2}(\mathbf{n}_{(a)}\cdot\mathbf{v})\mathbf{v}\end{pmatrix}\\
&= \begin{pmatrix}\dfrac{1}{m}\mathbf{n}_{(a)}\cdot\mathbf{k}\\ \mathbf{n}_{(a)}-\dfrac{(\mathbf{n}_{(a)}\cdot\mathbf{k})\mathbf{k}}{|\mathbf{k}|^2}+\dfrac{E}{m}\dfrac{(\mathbf{n}_{(a)}\cdot\mathbf{k})\mathbf{k}}{|\mathbf{k}|^2}\end{pmatrix}
\end{aligned}
$$

- 由 $\mathbf{k}\cdot[\mathbf{n}_{(a)}-(\mathbf{n}_{(a)}\cdot\mathbf{k})\mathbf{k}/|\mathbf{k}|^2]=0$，容易验证约束 $k_\mu e^{\mu}_{(a)}=0$
- 注意到 $\mathbf{n}_{(a)}\perp\mathbf{k}$ 的横向极化和 $\mathbf{n}_{(a)}\|\mathbf{k}$ 的纵向极化行为不同

$$
横向\ e^{\mu}_{(a)}=\begin{pmatrix}0\\ \mathbf{n}_{(a)}\end{pmatrix},\quad 纵向\ e^{\mu}_{(a)}=\frac{1}{m}\begin{pmatrix}|\mathbf{k}|\\ E\mathbf{k}/|\mathbf{k}|\end{pmatrix}
$$

- 横向极化不随动量增加而增长，且与 m 无关

- 由简单的代数等式可将纵向极化拆成两部分

$$\frac{E}{m}-\frac{m}{E+|\mathbf{k}|}=\frac{E^2+E|\mathbf{k}|-m^2}{m(E+|\mathbf{k}|)}=\frac{|\mathbf{k}|^2+E|\mathbf{k}|}{m(E+|\mathbf{k}|)}=\frac{|\mathbf{k}|}{m}$$

$$e^{\mu}_{(\parallel)}=\begin{pmatrix}\frac{E}{m}-\frac{m}{E+|\mathbf{k}|}\\ \left[\frac{|\mathbf{k}|}{m}+\frac{m}{E+|\mathbf{k}|}\right]\frac{\mathbf{k}}{|\mathbf{k}|}\end{pmatrix}=\underbrace{\frac{1}{m}\begin{pmatrix}E\\ \mathbf{k}\end{pmatrix}}_{k^{\mu}/m}+\frac{m}{E+|\mathbf{k}|}\begin{pmatrix}-1\\ \frac{\mathbf{k}}{|\mathbf{k}|}\end{pmatrix}$$

- 因 $\Pi^{\mu\nu}\sim e^{\mu}_{(\perp)}\cdot e^{\nu}_{(\perp)}+e^{\mu}_{(\parallel)}e^{\nu}_{(\parallel)}$，纵向极化的第一部分与守恒流不耦合：$k^{\mu}\tilde{J}_{\mu}=0$，第二部分当 $m\to 0$ 贡献为零 $\Rightarrow$ 退耦
- 注：对静止系做 Lorentz boost 得到的坐标系，能够很好地描述纵向自由度与物质源的相互作用，但不适合描述横向自由度与静态源的耦合，因为该系中的 $e^{0}_{(\perp)}=0$，无法用来计算 $\tilde{J}_0 e^{0}_{(\perp)}\cdot e^{0}_{(\perp)}\tilde{J}_0$ 产生的物理效应
- 为了克服上述困难，寻找 Lorentz 协变的方式构造 $e^{\mu}_{(\perp)}\cdot e^{\nu}_{(\perp)}$

- 这相当于建立在任何 Lorentz 系中都成立的协变的完备关系

$$e^{\mu}_{(\perp)} \cdot e^{\nu}_{(\perp)} = \Pi^{\mu\nu}_{\perp}$$

- 注意到 $e^{\mu}_{(\perp)}$ 共有 $D-2$ 个向量，要构造 D 维时空中的完备基，还需要两个与之正交的线性独立的向量，其中之一已选为 (经归一后的) $k^{\mu} = (k^0, \mathbf{k})$，约束 $k_{\mu} e^{\mu}_{(\perp)} = 0$ 保证了正交性；另一个可选成 $l^{\mu} = (k^0, -\mathbf{k})$，静止系 l^{μ} 退化为 k^{μ}，但当 $\mathbf{k} \neq 0$ 时两者线性独立；在前面 boost 得到的坐标系中

$$l_{\mu} e^{\mu}_{(\perp)} = k_0 \cdot 0 + (-\mathbf{k}) \cdot \mathbf{n}_{(\perp)} = 0$$

$$\Rightarrow \quad \text{任意坐标系中 } l_{\mu} e^{\mu}_{(\perp)} = 0$$

- 对纵向极化，一般 $k_{\mu} e^{\mu}_{(\parallel)} = 0 \;\not\Rightarrow\; l_{\mu} e^{\mu}_{(\parallel)} = 0$；如 $\mathbf{k} \neq 0$ 时

$$k_{\mu} e^{\mu}_{(\parallel)} = (-E) \frac{1}{m} \mathbf{n}_{(\parallel)} \cdot \mathbf{k} + \mathbf{k} \cdot \frac{E}{m} \frac{(\mathbf{n}_{(\parallel)} \cdot \mathbf{k})\mathbf{k}}{|\mathbf{k}|^2} = 0$$

$$l_{\mu} e^{\mu}_{(\parallel)} = (-E) \frac{1}{m} \mathbf{n}_{(\parallel)} \cdot \mathbf{k} + (-\mathbf{k}) \cdot \frac{E}{m} \frac{(\mathbf{n}_{(\parallel)} \cdot \mathbf{k})\mathbf{k}}{|\mathbf{k}|^2} \neq 0$$

- 尽管 k^μ, l^μ 在 $\mathbf{k}\neq 0$ 时线性独立，但并不正交；相互正交的是 $k^\mu \pm l^\mu$，因此完备关系可写成

$$e^\mu_{(\perp)}\cdot e^\nu_{(\perp)} = \eta^{\mu\nu} - \frac{k^\mu l^\nu + l^\mu k^\nu}{2k^\lambda l_\lambda} \equiv \Pi^{\mu\nu}_\perp$$

- 验证：在静止系中有

$$e^0_{(\perp)}\cdot e^0_{(\perp)} = \eta^{00} - \frac{k^0k^0 + k^0k^0}{-2(k^0)^2} = 0 \quad \text{一般坐标系} \neq 0$$

$$e^0_{(\perp)}\cdot e^i_{(\perp)} = \eta^{0i} - \frac{-k^0k^i + k^0k^i}{-2(k^0)^2} = 0$$

$$e^i_{(\perp)}\cdot e^j_{(\perp)} = \delta^{ij} - \frac{-k^ik^j - k^jk^i}{-2(k^0)^2} = \delta^{ij} = n^i_{(\perp)}\cdot n^j_{(\perp)}$$

- 取 $k^0 = 0, \mathbf{k}\neq 0$ 的离壳坐标系，$e^i_{(\perp)}\cdot e^j_{(\perp)} = \delta^{ij} - k^ik^j/|\mathbf{k}|^2$ 是 $D-1$ 空间到其横向子空间的投影算子，这正是预期的
- 一旦写出协变的 $\Pi^{\mu\nu}_\perp$，可将其与静态源 $\tilde{J}_\mu = (\tilde{J}_0, \mathbf{0})$ 耦合，作用在守恒流上 $\Pi^{\mu\nu}_\perp \sim \eta^{\mu\nu}$，与规范理论的计算吻合

- 类似可讨论自旋为 2 的有质量场 $h_{\mu\nu}$ 在 $m \to 0$ 时纵向自由度的贡献；$m \neq 0$ 的约束有 $D+1$ 个：$\partial^\mu h_{\mu\nu} = 0$，$h = 0$

$$\text{动力学自由度} \quad \frac{D(D+1)}{2} - (D+1) = \frac{(D-2)(D+1)}{2} \xrightarrow{D=4} 5$$

$$\text{纵向自由度} \quad \frac{(D-2)(D+1)}{2} - \left[\frac{(D-2)(D-1)}{2} - 1\right] = D - 1 \xrightarrow{D=4} 3$$

- 设 $D=4$；将满足约束的 $\tilde{h}_{\mu\nu}(k)$ 按对称张量基 $e^{(a)}_{\mu\nu} = e^{(a)}_{\nu\mu}$ 做展开

$$\tilde{h}_{\mu\nu} = \sum_{a=1}^{5} c_a(k) e^{(a)}_{\mu\nu}, \quad k^\mu e^{(a)}_{\mu\nu} = 0, \quad \eta^{\mu\nu} e^{(a)}_{\mu\nu} = 0$$

$$\bar{e}^{(a)}_{\mu\nu} e^{(b)\mu\nu} = \delta^{ab}, \quad \sum_{a=1}^{5} e^{(a)}_{\mu\nu} \bar{e}^{(a)}_{\lambda\rho} = \Pi_{\mu\nu,\lambda\rho}$$

- 在静止系 $k^\mu = (m, 0, 0, 0)$ 中

$$k^\mu e^{(a)}_{\mu\nu} = m e^{(a)}_{0\nu} = 0 \ \Rightarrow \ e^{(a)}_{0\nu} = 0$$

- 符合约束条件及正交归一性的一组静止系解为

$$e^{\mu\nu}_{(1)} = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e^{\mu\nu}_{(2)} = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e^{\mu\nu}_{(3)} = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad e^{\mu\nu}_{(4)} = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$e^{\mu\nu}_{(5)} = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$e^{0\mu} = 0 \ \Rightarrow$ 收缩时不出现 η_{00}

$e_{(a)} \cdot e_{(b)} = \mathrm{Tr}\big(e_{(a)} e_{(b)}\big) = \delta_{ab}$

- 求解约束条件后得到的这些场分量填充小群 $SO(3)$ 的自旋为 $s=2$ 的不可约表示，z-轴方向投影值为 $s_3=\pm 2, \pm 1, 0$
- 为了组合出 s_3 的本征态，考虑绕 z-轴的空间转动

$$\Lambda^{\mu}{}_{\nu}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in SO(2) \subset SO(3)$$

$$e^{\mu\nu}_{(a)} \to e'^{\mu\nu}_{(a)} = \Lambda^{\mu}{}_{\lambda}(\theta)\Lambda^{\nu}{}_{\rho}(\theta)e^{\lambda\rho}_{(a)}$$

矩阵记号

$$e'_{(a)} = \Lambda(\theta)e_{(a)}\Lambda(\theta)^T$$

- 例如，矩阵乘法和简单的三角函数公式给出

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}$$

- 因此经空间转动后的极化 $e^{\mu\nu}_{(1)}$，$e^{\mu\nu}_{(2)}$ 为

$$e'^{\mu\nu}_{(1)} = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta & 0 \\ 0 & -\sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e'^{\mu\nu}_{(2)} = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin 2\theta & \cos 2\theta & 0 \\ 0 & \cos 2\theta & -\sin 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- 引进线性组合 $\epsilon^{\mu\nu}_{\pm 2} \equiv (e^{\mu\nu}_{(1)} \pm i e^{\mu\nu}_{(2)})/\sqrt{2}$，有

$$\left. \begin{aligned} \epsilon^{\mu\nu}_{\pm 2} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \epsilon'^{\mu\nu}_{\pm 2} &= e^{\pm 2i\theta} \epsilon^{\mu\nu}_{\pm 2} \end{aligned} \right\} \Rightarrow \epsilon^{\mu\nu}_{\pm 2} \text{ 是 } s_3 = \pm 2 \text{ 本征态}$$

- 类似引进线性组合 $\epsilon^{\mu\nu}_{\pm 1} \equiv (e^{\mu\nu}_{(3)} \pm i e^{\mu\nu}_{(4)})/\sqrt{2}$，直接验证在环绕 z-轴的转动下

$$\epsilon'^{\mu\nu}_{\pm 1} = e^{\pm i\theta} \epsilon^{\mu\nu}_{\pm 1} \quad \Rightarrow \quad \epsilon^{\mu\nu}_{\pm 1} \text{ 是 } s_3 = \pm 1 \text{ 本征态}$$

- 最后，$\epsilon^{\mu\nu}_0 \equiv e^{\mu\nu}_{(5)}$ 在绕 z-轴的转动下不发生改变

$$\epsilon'^{\mu\nu}_0 = \epsilon^{\mu\nu}_0 \quad \Rightarrow \quad \epsilon^{\mu\nu}_0 \text{ 是 } s_3 = 0 \text{ 本征态}$$

- 自旋分量 $s_3 = \pm 2$ 的极化与指标对称的守恒流 $\tilde{J}_{\mu\nu}(k)$ 耦合

$$\begin{aligned} \tilde{J}_{(\pm 2)} \equiv \epsilon^{\mu\nu}_{\pm 2} \tilde{J}_{\mu\nu} &= \frac{1}{2} \mathrm{Tr} \left[\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \begin{pmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{12} & \tilde{J}_{22} \end{pmatrix} \right] \\ &= \frac{1}{2} (\tilde{J}_{11} - \tilde{J}_{22} \pm 2i \tilde{J}_{12}) \end{aligned}$$

- 交换 $s_3 = \pm 2$ 粒子态对两个流 $\tilde{J}_{\mu\nu}$, $\tilde{J}'_{\mu\nu}$ 相互作用的贡献

$$\begin{aligned} \sum_{s_3 = \pm 2} \tilde{J}'^{*}_{(s_3)} \tilde{J}_{(s_3)} &= \frac{1}{4} (\tilde{J}'^{*}_{11} - \tilde{J}'^{*}_{22} - 2i \tilde{J}'^{*}_{12})(\tilde{J}_{11} - \tilde{J}_{22} + 2i \tilde{J}_{12}) \\ + (\tilde{J}'^{*}_{\mu\nu} \leftrightarrow \tilde{J}_{\mu\nu}) &= \frac{1}{2} (\tilde{J}'^{*}_{11} - \tilde{J}'^{*}_{22})(\tilde{J}_{11} - \tilde{J}_{22}) + 2 \tilde{J}'^{*}_{12} \tilde{J}_{12} \end{aligned}$$

- 重新归并最后一式中的各项，有

$$\sum_{s_3=\pm 2} \tilde{J}'^*_{(s_3)} \tilde{J}_{(s_3)} = (\tilde{J}'^*_{11}\tilde{J}_{11} + \tilde{J}'^*_{22}\tilde{J}_{22} + 2\tilde{J}'^*_{12}\tilde{J}_{12}) - \frac{1}{2}(\tilde{J}'^*_{11} + \tilde{J}'^*_{22})(\tilde{J}_{11} + \tilde{J}_{22}) \equiv \tilde{J}'^*_{\mu\nu}\, \Pi^{\mu\nu,\lambda\rho}_{\pm 2}\, \tilde{J}_{\lambda\rho}$$

$$\Pi^{\mu\nu,\lambda\rho}_{\pm 2} = \begin{cases} \frac{1}{2}(\eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\lambda\rho}), & \text{指标横向} \\ 0, & \text{其他} \end{cases}$$

- 注：一般非静止坐标系中 $\Pi^{\mu\nu,\lambda\rho}_{\pm 2}$ 由 $\sum_{s_3=\pm 2} \epsilon^{\mu\nu}_{s_3} \epsilon^{\lambda\rho}_{s_3}$ 的完备关系给出，其独立于 k^μ, l^μ 的项恰与上式第一行相同，并且无须限制 μ, ν, λ, ρ 为横向指标；依赖动量的附加项保证了协变的 $\Pi^{\mu\nu,\lambda\rho}_{\pm 2}$ 在质壳上为横向空间投影算子，这些附加项与守恒流没有耦合 $\Rightarrow s_3 = \pm 2$ 的 sector 在 $m \to 0$ 的极限下光滑地过渡到零质量的结果
- 与矢量场不同，剩余自由度当质量趋近于零时不完全退耦

- 自旋分量 $s_3=\pm1$ 的极化与指标对称的守恒流 $\tilde{J}_{\mu\nu}(k)$ 耦合

$$
\begin{aligned}
\tilde{J}_{(\pm1)} \equiv \epsilon^{\mu\nu}_{\pm1}\tilde{J}_{\mu\nu} &= \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & \pm i\\ 1 & \pm i & 0\end{pmatrix}\begin{pmatrix}\tilde{J}_{11} & \tilde{J}_{12} & \tilde{J}_{13}\\ \tilde{J}_{12} & \tilde{J}_{22} & \tilde{J}_{23}\\ \tilde{J}_{13} & \tilde{J}_{23} & \tilde{J}_{33}\end{pmatrix}\right]\\
&= \tilde{J}_{13} \pm i\tilde{J}_{23}
\end{aligned}
$$

- 交换 $s_3=\pm1$ 粒子态对两个流 $\tilde{J}_{\mu\nu}$, $\tilde{J}'_{\mu\nu}$ 相互作用的贡献

$$
\begin{aligned}
\sum_{s_3=\pm1}\tilde{J}'^{*}_{(s_3)}\tilde{J}_{(s_3)} &= (\tilde{J}'^{*}_{13} - i\tilde{J}'^{*}_{23})(\tilde{J}_{13} + i\tilde{J}_{23}) + (\tilde{J}'^{*}_{\mu\nu} \leftrightarrow \tilde{J}_{\mu\nu})\\
&= 2(\tilde{J}'^{*}_{13}\tilde{J}_{13} + \tilde{J}'^{*}_{23}\tilde{J}_{23})
\end{aligned}
$$

- 分量 $s_3=0$ 的情形可类似处理：定义 $\tilde{J}_{(0)} \equiv \epsilon^{\mu\nu}_{0}\tilde{J}_{\mu\nu}$，有

$$
\begin{aligned}
\tilde{J}_{(0)} &= \sqrt{\frac{2}{3}}\mathrm{Tr}\left[\begin{pmatrix}1/2 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\tilde{J}_{11} & \tilde{J}_{12} & \tilde{J}_{13}\\ \tilde{J}_{12} & \tilde{J}_{22} & \tilde{J}_{23}\\ \tilde{J}_{13} & \tilde{J}_{23} & \tilde{J}_{33}\end{pmatrix}\right]\\
&= \sqrt{\frac{2}{3}}\left[\frac{1}{2}(\tilde{J}_{11} + \tilde{J}_{22}) - \tilde{J}_{33}\right] = \frac{1}{\sqrt{6}}(\tilde{J}_{11} + \tilde{J}_{22} - 2\tilde{J}_{33})
\end{aligned}
$$

- 因此 $s_3=0$ 粒子态对流-流相互作用的贡献为

$$\tilde{J}'^*_{(0)}\tilde{J}_{(0)}=\frac{1}{6}(\tilde{J}'^*_{11}+\tilde{J}'^*_{22}-2\tilde{J}'^*_{33})(\tilde{J}_{11}+\tilde{J}_{22}-2\tilde{J}_{33})$$

- 纯含横向指标项 $(\tilde{J}'^*_{11}+\tilde{J}'^*_{22})(\tilde{J}_{11}+\tilde{J}_{22})/6$ 并至 $s_3=\pm 2$ 相应的项中，那里的系数 $-1/2$ 改变为

$$-\frac{1}{2}\to-\frac{1}{2}+\frac{1}{6}=-\frac{1}{3}\ \Rightarrow\ \text{vDVZ 跳变}$$

- 为论证 $s_3=0$ 的量子态在 $m\to 0$ 时不退耦，沿着 z-轴方向做 Lorentz boost：取 $\mathbf{v}=(0,0,v)$, $\gamma=1/\sqrt{1-v^2}$

$$\Lambda^\mu{}_\nu(\mathbf{v})=\begin{pmatrix}\gamma & \gamma\langle\mathbf{v}| \\ \gamma|\mathbf{v}\rangle & \mathbf{1}+\dfrac{\gamma-1}{|\mathbf{v}|^2}|\mathbf{v}\rangle\langle\mathbf{v}|\end{pmatrix}=\begin{pmatrix}\gamma & 0 & 0 & \gamma v\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \gamma v & 0 & 0 & \gamma\end{pmatrix}$$

- 在此运动坐标系中 $k^\mu=(\gamma m,0,0,\gamma m v)=(k^0,0,0,k^3)$

- 极化 ϵ_0 在 Lorentz boost 之下变换成 $\epsilon_0' = \Lambda(v)\epsilon_0\Lambda(v)^T$

$$
\epsilon_0'^{\mu\nu} = \sqrt{\frac{2}{3}} \begin{pmatrix} \frac{k^0}{m} & 0 & 0 & \frac{k^3}{m} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{k^3}{m} & 0 & 0 & \frac{k^0}{m} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
$$

$$
\times \begin{pmatrix} \frac{k^0}{m} & 0 & 0 & \frac{k^3}{m} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{k^3}{m} & 0 & 0 & \frac{k^0}{m} \end{pmatrix} = \sqrt{\frac{2}{3}} \begin{pmatrix} -\frac{(k^3)^2}{m^2} & 0 & 0 & -\frac{k^0k^3}{m^2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{k^3k^0}{m^2} & 0 & 0 & -\frac{(k^0)^2}{m^2} \end{pmatrix}
$$

$$
= \sqrt{\frac{2}{3}} \begin{pmatrix} 1 - \frac{(k^0)^2}{m^2} & -\frac{k^0k^1}{m^2} & -\frac{k^0k^2}{m^2} & -\frac{k^0k^3}{m^2} \\ -\frac{k^1k^0}{m^2} & \frac{1}{2} - \frac{(k^1)^2}{m^2} & -\frac{k^1k^2}{m^2} & -\frac{k^1k^3}{m^2} \\ -\frac{k^2k^0}{m^2} & -\frac{k^2k^1}{m^2} & \frac{1}{2} - \frac{(k^2)^2}{m^2} & -\frac{k^2k^3}{m^2} \\ -\frac{k^3k^0}{m^2} & -\frac{k^3k^1}{m^2} & -\frac{k^3k^2}{m^2} & -1 - \frac{(k^3)^2}{m^2} \end{pmatrix}
$$

- 由此将 $\epsilon_0'^{\mu\nu}$ 分解成两部分：第一部分与守恒流 $\tilde{J}_{\mu\nu}$ 不发生耦合，第二部分在 $m \to 0$ 的极限下保持有限

$$\epsilon_0'^{\mu\nu} = \sqrt{\frac{2}{3}}\left[-\frac{k^\mu k^\nu}{m^2} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right]$$

- 注意到系数 1/6 的贡献在 $m \to 0$ 时并不退耦：按照前面的步骤，$\epsilon_0'^{\mu\nu}\tilde{J}_{\mu\nu}$ 可写成

$$\begin{aligned} & \sqrt{\frac{2}{3}}\mathrm{Tr}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \tilde{J}_{00} & \tilde{J}_{01} & \tilde{J}_{02} & \tilde{J}_{03} \\ \tilde{J}_{01} & \tilde{J}_{11} & \tilde{J}_{12} & \tilde{J}_{13} \\ \tilde{J}_{02} & \tilde{J}_{12} & \tilde{J}_{22} & \tilde{J}_{23} \\ \tilde{J}_{03} & \tilde{J}_{13} & \tilde{J}_{23} & \tilde{J}_{33} \end{pmatrix}\right] \\ = & \sqrt{\frac{2}{3}}\left[\tilde{J}_{00} + \frac{1}{2}(\tilde{J}_{11} + \tilde{J}_{22}) - \tilde{J}_{33}\right] \\ \Rightarrow & (\tilde{J}'^{*}_{\mu\nu}\epsilon_0'^{\mu\nu})(\epsilon_0'^{\lambda\rho}\tilde{J}_{\lambda\rho}) = \frac{1}{6}(\tilde{J}'^{*}_{11} + \tilde{J}'^{*}_{22})(\tilde{J}_{11} + \tilde{J}_{22}) + \cdots \end{aligned}$$

引力子的自作用

- 尽管零质量自旋为 2 的线性理论有很好的牛顿极限，并且光线偏转的计算也与广义相对论吻合，这个理论不能正确描述水星进动的观测值
- 检验粒子作用量与张量场的耦合

$$S_{\mathrm{M}} + S_{\mathrm{I}} = -m \int_{\ell} d\tau \left[\sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} + \kappa \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} h_{\mu\nu} \right]$$

$$\equiv \quad -m \int_{\ell} d\tau \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} + \mathrm{O}(\kappa^2), \quad g_{\mu\nu} = \eta_{\mu\nu} - 2\kappa h_{\mu\nu}$$

- 引力场中粒子的经典轨道是度量 $g_{\mu\nu}$ 的时空中的测地线

$$g_{\mu\nu} = \begin{pmatrix} -1 + \dfrac{r_g}{r} & 0 \\ 0 & \left(1 + \dfrac{r_g}{r}\right) \delta_{ij} \end{pmatrix}, \quad r_g \equiv 2GM$$

- 球极坐标系中的表达式

$$\begin{pmatrix} -1+\frac{r_g}{r} & 0 & 0 & 0 \\ 0 & 1+\frac{r_g}{r} & 0 & 0 \\ 0 & 0 & \left(1+\frac{r_g}{r}\right)r^2 & 0 \\ 0 & 0 & 0 & \left(1+\frac{r_g}{r}\right)r^2\sin^2\vartheta \end{pmatrix}$$

- 测地线的 Hamilton-Jacobi 方程

$$\frac{\partial S}{\partial x^\mu} = p_\mu \qquad \begin{cases} \dfrac{\partial S}{\partial x^i} = p_i \\ \dfrac{\partial S}{\partial t} = p_0 = -p^0 = -H \end{cases}$$

$$\Rightarrow \quad g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = 0$$

- 考虑 $\vartheta =$ 常值的平面轨道：$\partial_\vartheta S = 0$，$\sin^2\vartheta = 1$

- 度量与 t 和 φ 无关，测地线方程在 $t \to t+a$, $\varphi \to \varphi + b$ 变换下不变，存在守恒能量 E 和角动量 L；Hamilton-Jacobi 方程求解的标准方法是采用如下的 Ansatz

$$\left.\begin{array}{l} \dfrac{\partial S}{\partial t} = -E \\ \dfrac{\partial S}{\partial \varphi} = L \end{array}\right\} \Rightarrow S(t,r,\varphi) = S_0(r;E,L) - Et + L\varphi$$

- 可将原方程约化为常微分方程

$$\left(1-\frac{r_g}{r}\right)^{-1} E^2 - \left(1+\frac{r_g}{r}\right)^{-1}\left[S_0'(r)^2 + \frac{L^2}{r^2}\right] = m^2$$

$$\Rightarrow \quad S_0'(r)^2 = \frac{1+r_g/r}{1-r_g/r}E^2 - (1+r_g/r)m^2 - \frac{L^2}{r^2}$$

$$\Rightarrow \quad S_0(r) = \int dr \sqrt{\frac{1+r_g/r}{1-r_g/r}E^2 - \frac{L^2}{r^2} - (1+r_g/r)m^2}$$

- 与行星轨道尺度 r 比较，引力半径 r_g 是个很小的量，例如太阳的 $r_g \sim 3\text{km}$，地球的 $r_g \sim 0.9\text{cm}$；故可对 r_g/r 做展开

$$\frac{1+r_g/r}{1-r_g/r} \approx 1+2\left[\frac{r_g}{r}+\left(\frac{r_g}{r}\right)^2+\cdots\right]$$

$$\frac{1+r_g/r}{1-r_g/r}E^2-\frac{L^2}{r^2}-\left(1+\frac{r_g}{r}\right)m^2$$

$$\approx \quad E^2-m^2+(2E^2-m^2)\frac{r_g}{r}+\left(2E^2-\frac{L^2}{r_g^2}\right)\left(\frac{r_g}{r}\right)^2+\cdots$$

- 代入 $S_0(r)$ 的表达式

$$S_0(r) \approx \int dr\sqrt{E^2-m^2+(2E^2-m^2)\frac{r_g}{r}-\frac{L^2-2E^2r_g^2}{r^2}}$$

- 牛顿理论中的 Hamilton-Jacobi 方程：令 $\tilde{S}=\tilde{S}_0-\tilde{E}t+L\varphi$

$$\frac{\partial \tilde{S}}{\partial t}+\tilde{H}\left(x^i;\frac{\partial \tilde{S}}{\partial x^j}\right)=0 \ \Rightarrow\ \tilde{H}\left(x^i;\frac{\partial \tilde{S}_0}{\partial x^j}\right)=\tilde{E}$$

- 在球极坐标下，平面轨道 $\vartheta = \pi/2$ 相应的方程为

$$\tilde{H}(x^i; p_j) = \frac{1}{2m}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \vartheta}\right) + U(r)$$

$$\Rightarrow \quad \frac{1}{2m}\left[\left(\frac{\partial \tilde{S}_0}{\partial r}\right)^2 + \frac{L^2}{r^2}\right] - \frac{GMm}{r} = \tilde{E}$$

$$\Rightarrow \quad \tilde{S}_0(r) = \int dr \sqrt{2m\tilde{E} + m^2 \frac{r_g}{r} - \frac{L^2}{r^2}}$$

- 对于低速运动的行星，相对论能量 E 与非相对论能量 $\tilde{E}$ 之间的近似关系是 $E \simeq m + \tilde{E}$, $m \gg \tilde{E}$；因此

$$E^2 - m^2 \approx (m + \tilde{E})^2 - m^2 \approx 2m\tilde{E}, \quad 2E^2 - m^2 \approx m^2$$

$$\Rightarrow \quad S_0(r) \approx \int dr \sqrt{2m\tilde{E} + m^2 \frac{r_g}{r} - \frac{L^2 - 2m^2 r_g^2}{r^2}}$$

- 行星轨道半径对时间的依赖 $r = r(t)$ 可从 $t = \partial S_0/\partial E$ 得出；类似地，行星在赤道上的角位置 φ 由下式计算

$$\varphi - \varphi_0 = \begin{cases} -\dfrac{\partial S_0(r)}{\partial L} & \text{张量理论} \\ -\dfrac{\partial \tilde{S}_0(r)}{\partial L} & \text{牛顿理论} \end{cases}$$

$$= \begin{cases} L\displaystyle\int_{r_0}^{r} \frac{dr}{r^2\sqrt{2m\tilde{E} + m^2 r_g/r - (L^2 - 2m^2 r_g^2)/r^2}} \\ L\displaystyle\int_{r_0}^{r} \frac{dr}{r^2\sqrt{2m\tilde{E} + m^2 r_g/r - L^2/r^2}} \end{cases}$$

- 将张量理论看作牛顿理论的修正，展开 $S_0(r) = \tilde{S}_0(r) + \cdots$

$$S_0(r) \approx \int dr \sqrt{\underbrace{\left(2m\tilde{E} + m^2\frac{r_g}{r} - \frac{L^2}{r^2}\right)}_{A(r)}\left(1 + \frac{2m^2 r_g^2}{r^2 A(r)}\right)}$$

- 第二个括弧可以展成

$$\sqrt{1+\frac{2m^2r_g^2}{r^2A(r)}}\approx 1+\frac{m^2r_g^2}{r^2A(r)},\ A(r)\equiv 2m\tilde{E}+\frac{m^2r_g}{r}-\frac{L^2}{r^2}$$

$$\sqrt{A(r)}\sqrt{1+\frac{2m^2r_g^2}{r^2A(r)}}\approx\sqrt{A(r)}+\frac{m^2r_g^2}{r^2\sqrt{A(r)}}$$

$$S_0(r)\approx\tilde{S}_0(r)+m^2r_g^2\int\frac{dr}{r^2\sqrt{A(r)}}$$

$$=\tilde{S}_0(r)+\frac{m^2r_g^2}{L}\cdot(\varphi-\varphi_0)_{牛顿}$$

- 两边以 $-\partial/\partial L$ 作用，得

$$(\varphi-\varphi_0)_{张量}-(\varphi-\varphi_0)_{牛顿}=\frac{m^2r_g^2}{L^2}\cdot(\varphi-\varphi_0)_{牛顿}$$

- 当行星每绕太阳一周，$(\varphi-\varphi_0)_{牛顿}=2\pi$ 时，张量理论预言

$$\Delta\varphi=\frac{2\pi m^2r_g^2}{L^2}=\frac{8\pi G^2M^2m^2}{L^2}$$

- 在广义相对论中取空间各向同性形式的 Schwarzschild 度量

$$g_{\mu\nu} = \begin{pmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & B(r)r^2 & 0 \\ 0 & 0 & 0 & B(r)r^2\sin^2\vartheta \end{pmatrix}$$

$$A(r) = \left(\frac{1 - r_g/4r}{1 + r_g/4r}\right)^2, \quad B(r) = \left(1 + \frac{r_g}{4r}\right)^4$$

- 测地线上的 Hamilton-Jacobi 方程

$$-m^2 = g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = -\frac{1}{A(r)}\left(\frac{\partial S}{\partial t}\right)^2$$
$$+ \quad \frac{1}{B(r)}\left\{\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial S}{\partial \vartheta}\right)^2 + \frac{1}{\sin^2\vartheta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right]\right\}$$

- 如前在 $\vartheta = \pi/2$ 轨道平面寻求 $S = S_0 - Et + L\varphi$ 形式的解

$$\frac{E^2}{A(r)} - \frac{1}{B(r)}\left[S_0'(r)^2 + \frac{L^2}{r^2}\right] = m^2$$

- 展开到 r_g/r 的二阶

$$B(r) \approx 1 + 4\cdot\frac{r_g}{4r} + 6\cdot\left(\frac{r_g}{4r}\right)^2 = 1 + \frac{r_g}{r} + \frac{3}{8}\left(\frac{r_g}{r}\right)^2$$

$$\frac{B(r)}{A(r)} = \frac{(1+r_g/4r)^6}{(1-r_g/4r)^2} \approx 1 + \frac{2r_g}{r} + \frac{15}{8}\left(\frac{r_g}{r}\right)^2$$

$$\Rightarrow \quad S_0(r) = \int dr\sqrt{E^2\frac{B(r)}{A(r)} - m^2B(r) - \frac{L^2}{r^2}}$$

$$\approx \int dr\sqrt{2m\tilde{E} + \frac{m^2r_g}{r} - \frac{L^2 - 3m^2r_g^2/2}{r^2}}$$

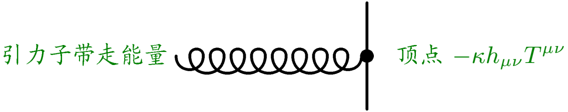
- 因此广义相对论的结果是张量理论的 3/4 倍

$$\Delta\varphi = \frac{6\pi G^2M^2m^2}{L^2}$$

观测进动就能判断引力子有自相互作用

非线性项的确定

- 无质量的引力子与物质的自洽耦合 $\hat{K}_{\mu\nu}{}^{\lambda\rho}h_{\lambda\rho} = \kappa T_{\mu\nu}$ 要求流守恒 $\partial^\mu T_{\mu\nu} = 0$
- 但耦合系统的守恒量是 $T_{\mu\nu} + t_{\mu\nu}$，必须计及引力场本身的能量-动量张量 $t_{\mu\nu}$，物质能量-动量张量 $T_{\mu\nu}$ 并不单独守恒



- 以沿测地线的点源能量-动量张量为例，$g_{\mu\nu} \sim \eta_{\mu\nu} - 2\kappa h_{\mu\nu}$

$$T^{\mu\nu}(x) = M \int d\tau \frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\delta^{(4)}(x - z(\tau))$$

$$\Rightarrow \quad \partial_\mu T^{\mu\nu}(x) = -M \int d\tau \frac{dz^\nu}{d\tau}\frac{d}{d\tau}\delta^{(4)}(x - z(\tau))$$

$$= \quad M \int d\tau \frac{d^2 z^\nu}{d\tau^2}\delta^{(4)}(x - z(\tau))$$

$$\Rightarrow \quad (\eta_{\nu\sigma} - 2\kappa h_{\nu\sigma})\partial_\mu T^{\mu\nu} = \kappa(h_{\sigma\lambda,\rho} + h_{\sigma\rho,\lambda} - h_{\lambda\rho,\sigma})T^{\lambda\rho}$$

- 自洽的耦合系统须含引力自作用：$S_K \sim \frac{1}{2}\int h_{\mu\nu}\hat{K}^{\mu\nu,\lambda\rho}h_{\lambda\rho}$ 应做非线性扩充 $S_K \rightarrow S_G = S_K + \Xi$
- 注意到 $\partial^\mu T_{\mu\nu} \sim \mathrm{O}(\kappa)$，若令 $\overset{(0)}{t}_{\mu\nu} = -\delta\Xi^{(1)}/\delta h_{\mu\nu}$ 满足

$$\eta_{\nu\sigma}\partial_\mu \frac{\delta\Xi^{(1)}}{\delta h_{\mu\nu}} = \kappa(h_{\sigma\lambda,\rho} + h_{\sigma\rho,\lambda} - h_{\lambda\rho,\sigma})T^{\lambda\rho}$$

$$\Rightarrow \quad \partial^\mu(T_{\mu\nu} + \overset{(0)}{t}_{\mu\nu}) \sim \mathrm{O}(\kappa^2)$$

- Feynman 用零级近似 $\kappa T^{\mu\nu} = \hat{K}^{\mu\nu,\lambda\rho}h_{\lambda\rho}$ 代入方程右边，通过迭代解出自作用项的一阶修正

"*Feynman Lectures on Gravitation*", Addison-Wesley, Reading (1995)

$$\begin{aligned}\Xi^{(1)} = -\int \Big[& h^{\mu\nu}\bar{h}^{\lambda\rho}h_{\mu\nu,\lambda\rho} + h_\lambda{}^\nu h^{\lambda\mu}\Box\bar{h}_{\mu\nu} \\ & - 2h^{\mu\nu}h_{\nu\rho}\bar{h}_{\mu\lambda,}{}^{\lambda\rho} + 2\bar{h}_{\mu\nu}\bar{h}^{\lambda\mu}{}_{,\lambda}\bar{h}^{\rho\nu}{}_{,\rho} \\ & + \left(\frac{1}{2}h_{\mu\nu}h^{\mu\nu} + \frac{1}{4}h^2\right)\bar{h}^{\lambda\rho}{}_{,\lambda\rho}\Big]\end{aligned}$$

$S_K + \kappa\Xi^{(1)}$：爱因斯坦理论线性近似的下一阶

- 在自洽的理论中有

$$\frac{\delta S_G}{\delta h_{\mu\nu}} = \kappa T^{\mu\nu}, \quad \partial_\mu \frac{\delta S_G}{\delta h_{\mu\nu}} = \kappa \partial_\mu T^{\mu\nu}$$

- 要求上列方程与能量-动量张量的散度条件相容，即得

$$g_{\nu\sigma}\partial_\mu \frac{\delta S_G}{\delta h_{\mu\nu}} = \kappa(h_{\sigma\mu,\nu} + h_{\sigma\nu,\mu} - h_{\mu\nu,\sigma})\frac{\delta S_G}{\delta h_{\mu\nu}} \quad \Rightarrow$$

$$\int d^4x \frac{\delta S_G}{\delta h_{\mu\nu}}\Big[(\xi^\sigma g_{\sigma\nu})_{,\mu} + \kappa\xi^\sigma(h_{\sigma\mu,\nu} + h_{\sigma\nu,\mu} - h_{\mu\nu,\sigma})\Big] = 0$$

- 这个条件等价于作用量具有某种不变性

$$\tilde{\delta}_\xi S_G[h_{\mu\nu}] = \int d^4x \frac{\delta S_G}{\delta h_{\mu\nu}} \tilde{\delta}_\xi h_{\mu\nu} = 0$$

$$\tilde{\delta}_\xi h_{\mu\nu} \equiv \xi_{\mu,\nu} + \xi_{\nu,\mu} - 2\kappa(\xi^\sigma{}_{,\nu} h_{\sigma\mu} + \xi^\sigma{}_{,\mu} h_{\sigma\nu} + \xi^\sigma h_{\mu\nu,\sigma})$$

- 与 $g_{\mu\nu}$ 在变换 $x^\mu \to x'^\mu = x^\mu + 2\kappa\xi^\mu(x)$ 下的张量性质比较

$$\delta_\xi g_{\mu\nu} = -2\kappa(g_{\mu\sigma}\xi^\sigma{}_{,\nu} + g_{\nu\sigma}\xi^\sigma{}_{,\mu} + g_{\mu\nu,\sigma}\xi^\sigma)$$

$$\Leftrightarrow \quad \delta_\xi h_{\mu\nu} = g_{\mu\sigma}\xi^\sigma{}_{,\nu} + g_{\nu\sigma}\xi^\sigma{}_{,\mu} + g_{\mu\nu,\sigma}\xi^\sigma = \tilde{\delta}_\xi h_{\mu\nu}$$

- 结论：自洽的 S_G 必具有广义协变性，且展开到 $h^2 + h^3$ 阶与 Einstein-Hilbert 作用量一致 $\Rightarrow$ 零质量自旋为 2 的引力理论在红外端唯一地由爱因斯坦理论描述 (Rigidity)
- 注：非线性项可以更方便地在一阶框架内迭代构造

S. Deser, "*Self-interaction and gauge invariance*", Gen. Rel. Grav. **1** (1970) 9

- 以爱因斯坦理论为例，一阶框架的度量 $g_{\mu\nu}$ 和仿射联络 $\Gamma^{\lambda}_{\mu\nu}$ 是独立的场变量 (Palatini formalism)

$$I_G[g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}] = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma)$$

$$R_{\mu\nu}(\Gamma) \equiv \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\lambda}$$

  - 作用量对 $\Gamma^{\lambda}_{\mu\nu}$ 变分为零给出仿射联络与度量的关系
  - 对 $g^{\mu\nu}$ 变分为零给出真空爱因斯坦场方程

$$\delta(\sqrt{-g}g^{\mu\nu}) = \sqrt{-g}\left[\delta g^{\mu\nu} - \frac{1}{2}g_{\lambda\rho}g^{\mu\nu}\delta g^{\lambda\rho}\right]$$

$$\Rightarrow \quad R_{\mu\nu}(\Gamma) - \frac{1}{2}R(\Gamma)g_{\mu\nu} = 0$$

- 零质量张量理论来自爱因斯坦理论的线性化，其一阶作用量可取为 ($g_{\mu\nu} = \eta_{\mu\nu} - 2\kappa h_{\mu\nu}$, $g^{\mu\nu} = \eta^{\mu\nu} + 2\kappa h^{\mu\nu} + \mathrm{O}(h^2)$)

$$S_K = \frac{1}{2\kappa^2} \int d^D x \left[\eta^{\mu\nu} R^{\mathrm{Q}}_{\mu\nu}(\Gamma) + 2\kappa \bar{h}^{\mu\nu} R^{\mathrm{L}}_{\mu\nu}(\Gamma)\right]$$

$$\begin{cases} R^{\mathrm{L}}_{\mu\nu}(\Gamma) \equiv \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma_\mu \\ R^{\mathrm{Q}}_{\mu\nu} \equiv \Gamma_\lambda \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\rho} \Gamma^\rho_{\mu\lambda} \end{cases} \qquad \Gamma_\mu \equiv \Gamma^\lambda_{\mu\lambda}$$

- 对 $\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h$ 变分

$$\frac{\delta S_K}{\delta \bar{h}^{\mu\nu}} = \frac{1}{2\kappa}\left[R^{\mathrm{L}}_{\mu\nu}(\Gamma) + R^{\mathrm{L}}_{\nu\mu}(\Gamma)\right]$$

- 对 $\Gamma^\lambda_{\mu\nu}$ 变分有

$$\Gamma^\lambda_{\mu\nu} = \kappa \eta^{\lambda\rho}\left(h_{\mu\nu,\rho} - h_{\rho\mu,\nu} - h_{\rho\nu,\mu}\right)$$

- 代入 $R^{\mathrm{L}}_{\mu\nu}(\Gamma)$，变分原理 $\delta S_K / \delta \bar{h}^{\mu\nu} = 0$ 即给出线性化的真空爱因斯坦场方程

- 纯引力系统场方程的一阶修正

$$\frac{1}{2\kappa}\left[R^{\mathrm{L}}_{\mu\nu}(\Gamma)+R^{\mathrm{L}}_{\nu\mu}(\Gamma)\right]=\kappa\bar{t}_{\mu\nu},\quad \bar{t}_{\mu\nu}\equiv t_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}t$$

- 这里 $t_{\mu\nu}$ 为引力场的能量-动量张量，可以从 Noether 流导出，也可等价地引进辅助度量 $\eta_{\mu\nu}\to\gamma_{\mu\nu}$ 并由变分得到

$$t_{\mu\nu}=-\frac{2}{\sqrt{-\gamma}}\frac{\delta S_K[\gamma]}{\delta\gamma^{\mu\nu}}\bigg|_{\gamma=\eta}$$

$$S_K[\gamma]\equiv\frac{1}{2\kappa^2}\int d^Dx\sqrt{-\gamma}\left[\gamma^{\mu\nu}\overset{\gamma}{R}{}^{\mathrm{Q}}_{\mu\nu}(\Gamma)+2\kappa\bar{h}^{\mu\nu}\overset{\gamma}{R}{}^{\mathrm{L}}_{\mu\nu}(\Gamma)\right]$$

- 上式的 $\overset{\gamma}{R}_{\mu\nu}$ 通过在 $R_{\mu\nu}(\Gamma)$ 中做置换 $\partial_\mu\to\overset{\gamma}{\nabla}_\mu$ 来定义，$\overset{\gamma}{\nabla}_\mu$ 是由 $\gamma_{\mu\nu}$ 构造的协变导数
- 注意到一阶近似并不能区分 $\bar{h}^{\mu\nu}$ 是张量抑或张量密度

$$\sqrt{-g}g^{\mu\nu}\approx\sqrt{-\eta}\eta^{\mu\nu}+\delta(\sqrt{-g}g^{\mu\nu})\big|_{g=\eta}$$

$$=\quad\sqrt{-\eta}\eta^{\mu\nu}+2\kappa\sqrt{-\eta}\left[h^{\mu\nu}-\frac{1}{2}\eta_{\lambda\rho}\eta^{\mu\nu}h^{\lambda\rho}\right]=\eta^{\mu\nu}+2\kappa\bar{h}^{\mu\nu}$$

- 如果将 $\bar{h}^{\mu\nu}$ 理解成张量密度，协变的作用量应写成

$$S'_K[\gamma] = \frac{1}{2\kappa^2}\int d^Dx \left[\sqrt{-\gamma}\gamma^{\mu\nu}\overset{\gamma}{R}{}^{\mathrm{Q}}_{\mu\nu}(\Gamma) + 2\kappa\bar{h}^{\mu\nu}\overset{\gamma}{R}{}^{\mathrm{L}}_{\mu\nu}(\Gamma)\right]$$

- 这两种理解引起能量-动量张量上的差异“在壳”为零

$$t_{\mu\nu} = \overset{\mathrm{Q}}{t}_{\mu\nu} + \overset{\mathrm{L}}{t}_{\mu\nu}, \quad t'_{\mu\nu} = -\frac{2}{\sqrt{-\gamma}}\frac{\delta S'_K[\gamma]}{\delta\gamma^{\mu\nu}}\bigg|_{\gamma=\eta} = \overset{\mathrm{Q}}{t}_{\mu\nu} + \overset{\mathrm{L}}{t'}_{\mu\nu}$$

$$\frac{\delta S_K[\gamma]}{\delta\bar{h}^{\mu\nu}} = \frac{1}{\kappa}\sqrt{-\gamma}\overset{\gamma}{R}{}^{\mathrm{L}}_{\mu\nu}(\Gamma) \ \Rightarrow\ \bar{h}^{\mu\nu}\overset{\gamma}{R}{}^{\mathrm{L}}_{\mu\nu}(\Gamma) = \frac{\kappa\bar{h}^{\mu\nu}}{\sqrt{-\gamma}}\frac{\delta S_K[\gamma]}{\delta\bar{h}^{\mu\nu}}$$

$$\overset{\mathrm{L}}{t}_{\mu\nu} = \left[-\frac{2}{\sqrt{-\gamma}}\int d^Dx\,\bar{h}^{\lambda\rho}\frac{\delta^2}{\delta\bar{h}^{\lambda\rho}\delta\gamma^{\mu\nu}}S_K[\gamma]\right]_{\gamma=\eta}$$

$$\overset{\mathrm{L}}{t'}_{\mu\nu} = \left[-\frac{2}{\sqrt{-\gamma}}\frac{\delta}{\delta\gamma^{\mu\nu}}\int d^Dx\frac{\bar{h}^{\lambda\rho}}{\sqrt{-\gamma}}\frac{\delta S_K[\gamma]}{\delta\bar{h}^{\lambda\rho}}\right]_{\gamma=\eta} = \overset{\mathrm{L}}{t}_{\mu\nu}$$
$$+\left[-\frac{2}{\sqrt{-\gamma}}\int d^Dx\left(\frac{\delta}{\delta\gamma^{\mu\nu}}\frac{1}{\sqrt{-\gamma}}\right)\bar{h}^{\lambda\rho}\frac{\delta S_K[\gamma]}{\delta\bar{h}^{\lambda\rho}}\right]_{\gamma=\eta}$$

- 不妨假定系统的引力能量-动量张量为 $t'_{\mu\nu}$；因 $\overset{\gamma}{R}{}^{\mathrm{Q}}_{\mu\nu}(\Gamma)$ 中不含由 $\gamma_{\mu\nu}$ 定义的协变导数，变分时保持不变，故有

$$\overset{\mathrm{Q}}{t}_{\mu\nu} = \left[-\frac{2}{\sqrt{-\gamma}}\cdot\frac{\sqrt{-\gamma}}{2\kappa^2}\left(\overset{\gamma}{R}{}^{\mathrm{Q}}_{\mu\nu}(\Gamma) - \frac{1}{2}\gamma_{\mu\nu}\gamma^{\lambda\rho}\overset{\gamma}{R}{}^{\mathrm{Q}}(\Gamma)_{\lambda\rho}\right)\right]_{\gamma=\eta}$$

$$= -\frac{1}{\kappa^2}\left[R^{\mathrm{Q}}_{\mu\nu}(\Gamma) - \frac{1}{2}\eta_{\mu\nu}R^{\mathrm{Q}}(\Gamma)\right]$$

$$\overset{\mathrm{Q}}{t}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overset{\mathrm{Q}}{t} = -\frac{1}{\kappa^2}\left[R^{\mathrm{Q}}_{\mu\nu} - \frac{4-D}{4}\eta_{\mu\nu}R^{\mathrm{Q}}\right] = -\frac{1}{\kappa^2}R^{\mathrm{Q}}_{\mu\nu}$$

- 另一方面，应用公式 (所有指标均用 $\eta_{\mu\nu}$ 升降)

$$\delta\left[\gamma^{\lambda\rho}\left(\gamma_{\rho\mu,\nu} + \gamma_{\rho\nu,\mu} - \gamma_{\mu\nu,\rho}\right)\right] = \delta\gamma^{\lambda\rho}\left(\gamma_{\rho\mu,\nu} + \gamma_{\rho\nu,\mu} - \gamma_{\mu\nu,\rho}\right)$$

$$+\,\gamma^{\lambda\rho}\left[(\delta\gamma_{\rho\mu})_{,\nu} + (\delta\gamma_{\rho\nu})_{,\mu} - (\delta\gamma_{\mu\nu})_{,\rho}\right] \;\Rightarrow\; \delta(\overset{\gamma}{\nabla}_\mu\Gamma) \;\Rightarrow$$

$$\overset{\mathrm{L}}{t'}_{\mu\nu} = \left[-\frac{2}{\sqrt{-\gamma}}\frac{\delta}{\delta\gamma^{\mu\nu}}\frac{1}{\kappa}\int d^Dx\,\bar{h}^{\lambda\rho}\overset{\gamma}{R}{}^{\mathrm{L}}_{\lambda\rho}(\Gamma)\right]_{\gamma=\eta} = -\partial^\rho\times$$

$$\left[2\bar{h}_\rho{}^\lambda\Gamma_{(\mu\nu)\lambda} + 2\bar{h}_{(\mu}{}^\lambda\Gamma_{\rho\lambda\nu)} - 2\bar{h}_{(\mu}{}^\lambda\Gamma_{\lambda)\rho\lambda} - 2\bar{h}_{\rho(\mu}\Gamma_{\nu)} + \bar{h}_{\mu\nu}\Gamma_\rho\right]$$

- 由于 $\overset{\mathrm{L}}{t'}_{\mu\nu}$ 是个全导数，Deser 设想可以将其略去而不影响物理，$t'_{\mu\nu} \sim \overset{\mathrm{Q}}{t}_{\mu\nu} \;\Rightarrow\; \bar{t}'_{\mu\nu} = -R^{\mathrm{Q}}_{\mu\nu}(\Gamma)/\kappa^2$；这个能量-动量张量与 $\bar{h}^{\mu\nu}$ 耦合后添加到拉氏量中作为自洽性的补偿项

$$\mathcal{L}_K^{(1)} = \mathcal{L}_K - \kappa\bar{h}^{\mu\nu}\bar{t}'_{\mu\nu} = \frac{1}{2\kappa^2}\left[\eta^{\mu\nu}R^{\mathrm{Q}}_{\mu\nu}(\Gamma) + 2\kappa\bar{h}^{\mu\nu}R^{\mathrm{L}}_{\mu\nu}(\Gamma)\right]$$
$$+ \frac{1}{\kappa}\bar{h}^{\mu\nu}R^{\mathrm{Q}}_{\mu\nu}(\Gamma) = \frac{1}{2\kappa^2}\left[\eta^{\mu\nu}R^{\mathrm{Q}}_{\mu\nu}(\Gamma) + 2\kappa\bar{h}^{\mu\nu}R_{\mu\nu}(\Gamma)\right]$$

- 最后注意到 $\eta^{\mu\nu}R^{\mathrm{L}}_{\mu\nu}(\Gamma)$ 是全导数，可添加在拉氏量中，使第一项的 $\eta^{\mu\nu}R^{\mathrm{Q}}_{\mu\nu}(\Gamma) \to \eta^{\mu\nu}[R^{\mathrm{Q}}_{\mu\nu}(\Gamma) + R^{\mathrm{L}}_{\mu\nu}(\Gamma)] = \eta^{\mu\nu}R_{\mu\nu}(\Gamma)$

$$\mathcal{L}_K^{(1)} = \frac{1}{2\kappa^2}(\eta^{\mu\nu} + 2\kappa\bar{h}^{\mu\nu})R_{\mu\nu}(\Gamma)$$

- 在一阶框架中，$g_{\mu\nu}$ (以及 $h_{\mu\nu}$) 与 $\Gamma^{\lambda}_{\mu\nu}$ 独立，故可定义 $g_{\mu\nu}$ 所对应的张量密度为 $\eta^{\mu\nu} + 2\kappa\bar{h}^{\mu\nu}$，一步迭代即得爱因斯坦理论

$$\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + 2\kappa\bar{h}^{\mu\nu} \;\Rightarrow\; \mathcal{L}_K^{(1)} = \frac{1}{2\kappa^2}\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma)$$

场的自由度计数

- 既然无质量引力子只有两个物理自由度；$s_3 = \pm 2$，其非线性扩充几乎唯一地给出广义相对论，引力的红外端修改 $\Leftrightarrow$ 添加新的物理自由度
  - 例：$f(R)$-gravity；设 $f'' \neq 0$

$$\frac{1}{16\pi G}\int_M d^4x\sqrt{-g}R \rightarrow \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}f(R)$$

$$\cong \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[\underbrace{f'(\phi)R - (\phi f'(\phi) - f(\phi))}_{\text{对 }\phi\text{ 变分: }\phi = R}\Big]$$

Weyl rescaling $\tilde{g}_{\mu\nu} = f'(\phi)g_{\mu\nu} \sim$ GR + 标量场

- 添加质量使线性理论的自由度增加到 5
- Boulware-Deser 发现，如果在有质量的引力理论中引进非线性项，自由度将进一步增至 6，新增的是鬼场

D. G. Boulware and S. Deser, "*Can gravitation have a finite range?*", Phys. Rev. **D6** (1972) 3368

- 自由度的计数可以从相空间的分析得出

- 首先考虑 FP 理论的 Hamilton 形式

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - 2\partial_{(i}h_{j)0} + 2\partial_k h_{k0}\delta_{ij}$$

$$\Rightarrow \quad \dot{h}_{ij} = \pi_{ij} - \frac{1}{D-2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0}$$

- 用正则场变量将 FP 理论写成

$$S = \int d^D x \ \pi_{ij}\dot{h}_{ij} - \mathcal{H} + 2h_{0i}\left(\partial_j \pi_{ij}\right) + m^2 h_{0i}^2$$

$$+h_{00}\left(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\right)$$

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 \\ +\frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} \\ -\frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2(h_{ij}h_{ij} - h_{ii}^2)\end{aligned}$$

- $m=0$ 时，h_{0i}，h_{00} 相当于拉氏乘子，故下列约束条件成立

$$\partial_i \pi_{ij} = 0, \quad \vec{\nabla}^2 h_{ii} - \partial_i \partial_j h_{ij} = 0$$

- 在 $D=4$ 维时空中

$$\# h_{ij} = \# \pi_{ij} = 6, \quad \#\text{相空间约束} = 4, \quad \#\text{规范变换} = 4$$

- 故相空间自由度为 $6+6-4-4=4$，场自由度为 2，对应着两个横向的极化自由度
- 如 $m \neq 0$，运动方程给出

$$h_{0i} = -\frac{1}{m^2} \partial_j \pi_{ij}$$

$$\Rightarrow \quad S = \int d^D x \; \pi_{ij} \dot{h}_{ij} - \mathcal{H} + h_{00} \left(\vec{\nabla}^2 h_{ii} - \partial_i \partial_j h_{ij} - m^2 h_{ii} \right)$$

$$\mathcal{H} \to \mathcal{H} + \frac{1}{m^2} (\partial_j \pi_{ij})^2$$

- 拉氏乘子 h_{00} 给出二类初级约束 $\mathcal{C}$，自由度 $(12-2)/2 = 5$

$$\mathcal{C} = -\vec{\nabla}^2 h_{ii} + \partial_i \partial_j h_{ij} + m^2 h_{ii} = 0 \;\Rightarrow\; \{H, \mathcal{C}\}_{\mathrm{PB}} = \frac{1}{D-2} m^2 \pi_{ii} + \partial_i \partial_j \pi_{ij}$$

- 将有质量引力非线性地扩充为

$$S = \frac{1}{2\kappa^2}\int d^D x\left[(\sqrt{-g}R) - \frac{1}{4}m^2\eta^{\mu\alpha}\eta^{\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta}\right)\right]$$

- ADM formalism：选择类空曲面 Σ_t

$$g_{00} = -N^2 + g^{ij}N_i N_j, \quad g_{0i} = N_i$$

- 爱因斯坦作用量部分

$$\frac{1}{2\kappa^2}\int d^D x\sqrt{g}N\left[{}^{(d)}R - K^2 + K^{ij}K_{ij}\right]$$

$${}^{(d)}R \quad \text{空间曲率}$$

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right) \quad \text{外曲率}$$

协变导数用空间分量 g_{ij} 定义

- g_{ij} 的 Legendre 变换

$$p^{ij} = \frac{\delta L}{\delta \dot{g}_{ij}} = \frac{1}{2\kappa^2}\sqrt{g}\left(K^{ij} - Kg^{ij}\right)$$

- 作用量以一阶形式写出

$$2\kappa^2 L = \left(\int_{\Sigma_t} d^d x \ p^{ij} \dot{g}_{ij}\right) - H$$

$$H = \left(\int_{\Sigma_t} d^d x \ p^{ab} \dot{g}_{ab}\right) - L = \int_{\Sigma_t} d^d x \ N\mathcal{C} + N_i \mathcal{C}^i$$

$$\mathcal{C} = \sqrt{g}\left[{}^{(d)}R + K^2 - K^{ij}K_{ij}\right]$$

$$\mathcal{C}^i = 2\sqrt{g}\nabla_j \left(K^{ij} - Kh^{ij}\right)$$

- 在一阶形式中，将外曲率看成 p^{ij}, g_{ij} 的函数

$$K_{ij} = \frac{2\kappa^2}{\sqrt{g}}\left(p_{ij} - \frac{1}{D-2} p h_{ij}\right)$$

- 所有指标均用 g_{ij}, g^{ij} 升降收缩
- 当质量为零时，N 和 N_i 都是乘子，$\mathcal{C}$, $\mathcal{C}^i$ 为一类约束
- $\# = 12 - 4 - 4 = 4$ 相空间自由度 $\Rightarrow$ 非线性项不改变自由度数

- 质量项的贡献

$$
\begin{aligned}
& \eta^{\mu\alpha}\eta^{\mu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\mu\beta}\right) \\
= \quad & \delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} \\
- \quad & 2N^2\delta^{ij}h_{ij} + 2N_i\left(g^{ij} - \delta^{ij}\right)N_i, \quad h_{ij} \equiv g_{ij} - \delta_{ij}
\end{aligned}
$$

- 作用量

$$
\begin{aligned}
& S = \frac{1}{2\kappa^2}\int d^Dx\Big\{p^{ab}\dot{g}_{ab} - N\mathcal{C} - N_i\mathcal{C}^i \\
& -\frac{m^2}{4}\Big[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} \\
& +2N_i\left(g^{ij} - \delta^{ij}\right)N_i\Big]\Big\}
\end{aligned}
$$

- 自量项含 N, N_i 的二次，由拉氏乘子变为辅助场；代数求解

$$
N = \frac{\mathcal{C}}{m^2\delta^{ij}h_{ij}}, \qquad N_i = \frac{1}{m^2}\left(g^{ij} - \delta^{ij}\right)^{-1}\mathcal{C}^j
$$

- 代回 Hamiltonian

$$H = \frac{1}{2\kappa^2}\int d^dx\ \frac{1}{2m^2}\frac{\mathcal{C}^2}{\delta^{ij}h_{ij}} + \frac{1}{2m^2}\mathcal{C}^i\left(g^{ij}-\delta^{ij}\right)^{-1}\mathcal{C}^j$$
$$+\frac{m^2}{4}\left[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl}-h_{ik}h_{jl}\right)+2\delta^{ij}h_{ij}\right]\neq 0$$

- 相空间中的 12 个自由度等价于 6 个场自由度，比线性理论中 5 个自由度多 1

Vainshtein 半径

- Vainshtein 发现，存在一个新尺度 $R_V \sim (GM/\mu^4)^{1/5}$，只有当 $r \gg R_V$ 时，有质量的张量理论才能略去自作用的非线性
- 另一方面，当 $r \lesssim R_V$ 时，强耦合的非线性效应 $\Rightarrow$ 实验观测无法区分有质量的张量理论和爱因斯坦理论
- 随着 μ 逐渐趋近于零，基于线性理论的低阶计算变得越来越不可靠，非微扰修正将消除 vDVZ 跳变
- 太阳 $GM \sim r_g \sim 1\text{km}$, $1/\mu \gtrsim 10^{19}\text{km}$, $R_V \sim 1.6\times10^{15}\text{km}$; $r_{es} \sim 1.5\times10^{8}\text{km}$

- 回顾爱因斯坦理论中非线性效应的微扰展开：球对称度量

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

- 保持球对称性最一般的坐标变换是 $r \to r'$，这种变换允许固定 (空间各向同性的) 规范 $C(r) = B(r)$
- 真空爱因斯坦场方程 ($r \neq 0$) 的 tt 分量和 rr 分量给出非线性方程组

$$3r(B')^2 - 4B(2B' + rB'') = 0$$

$$4A'B^2 + 2(2A + rA')B'B + rA(B')^2 = 0$$

- 引进微扰参数 ϵ，在平坦背景 $A_0(r) = B_0(r) = 1$ 上做展开

$$A(r) = A_0(r) + \epsilon A_1(r) + \epsilon^2 A_2(r) + \cdots$$

$$B(r) = B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots$$

- 平坦背景自动满足场方程；在 $O(\epsilon)$ 阶，场方程线性化为

$$A_1' + B_1' = 0, \quad B_1'' + \frac{2B_1'}{r} = 0$$

- 考虑渐近平坦的解 $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$，这个渐近条件固定了三个常数中的两个；以 M 记另一积分常数

$$A_1 = -\frac{2GM}{r}, \quad B_1 = \frac{2GM}{r}$$

- 利用已求得的 $A_1(r)$ 和 $B_1(r)$，可将 $\mathrm{O}(\epsilon^2)$ 阶的场方程写成

$$\frac{3G^2M^2}{r^4} - \frac{2B_2'}{r} - B_2'' = 0, \quad \frac{7G^2M^2}{r^3} + A_2' + B_2' = 0$$

- 三个任意常数仍由渐近平坦性质固定两个，剩下的一个常数被渐近条件 $A \sim B \sim \mathrm{O}(1/r^2)$ 所固定

$$A_2 = 2\left(\frac{GM}{r}\right)^2, \quad B_2 = \frac{3}{2}\left(\frac{GM}{r}\right)^2$$

- 由此求出

$$A(r) = 1 - \frac{2GM}{r} + \frac{2G^2M^2}{r^2} + \cdots = \left(\frac{1 - r_g/4r}{1 + r_g/4r}\right)^2$$

$$B(r) = 1 + \frac{2GM}{r} + \frac{3G^2M^2}{2r^2} + \cdots = \left(1 + \frac{r_g}{4r}\right)^4$$

- 因缺乏规范对称性，有质量张量场理论的非线性扩充并不唯一，最简单的选择是在爱因斯坦理论中添加质量项

$$S=\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}R-\frac{\mu^2}{2}\int d^4x\,\eta^{\mu\lambda}\eta^{\nu\rho}(h_{\mu\nu}h_{\lambda\rho}-h_{\mu\lambda}h_{\nu\rho})$$

- 令 $g_{\mu\nu}=\eta_{\mu\nu}-2\kappa h_{\mu\nu}$ 并将上述理论线性化即得有质量的自由张量场理论
- 仍考虑球对称度量，由于不能使用广义协变性质，$C(r)$ 在这里无法规范固定成 $B(r)$

$$ds^2=-A(r)dt^2+B(r)dr^2+C(r)r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2)$$

- 独立的场方程来自 tt、rr 和 $\vartheta\vartheta$ 分量，如 tt 分量为

$$\begin{aligned}&4\mu^2r^2AB^2C^3+\left[2\mu^2r^2A(B-3)B^2-4\sqrt{ABC^2}(B-rB')\right]C^2\\&+2\sqrt{ABC^2}\left[2B^2-2rB(3C'+rC'')+r^2B'C'\right]C\\&+r^2B\sqrt{ABC^2}(C')^2=0\end{aligned}$$

- 当 $\mu\to 0$ 时 $\vartheta\vartheta$ 分量可以由其他两个方程导出，并不独立

- 仍寻求渐近平坦解 $A_0(r) = B_0(r) = C_0(r) = 1$，并在此背景上做微扰展开

$$A(r) = A_0(r) + \epsilon A_1(r) + \epsilon^2 A_2(r) + \cdots$$

$$B(r) = B_0(r) + \epsilon B_1(r) + \epsilon^2 B_2(r) + \cdots$$

$$C(r) = C_0(r) + \epsilon C_1(r) + \epsilon^2 C_2(r) + \cdots$$

- 在 $\mathrm{O}(\epsilon)$ 阶，场方程线性化为

$$2(\mu^2 r^2 - 1)C_1 + (\mu^2 r^2 + 2)B_1 + 2r(-3C_1' + B_1' - rC_1'') = 0$$

$$-\frac{1}{2}\mu^2 A_1' + \left(\frac{1}{r^2} - \mu^2\right) C_1 + \frac{r(A_1' + C_1') - B_1}{r^2} = 0$$

$$\mu^2 r(A_1 + B_1 + C_1) - A_1' + B_1' - 2C_1' - r(A_1'' + C_1'') = 0$$

- 方程的一阶微扰解

$$A_1(r) = -\frac{8GM}{3}\frac{e^{-\mu r}}{r}, \quad B_1(r) = -\frac{8GM}{3}\frac{e^{-\mu r}}{r}\frac{1 + \mu r}{\mu^2 r^2}$$

$$C_1(r) = \frac{4GM}{3}\frac{e^{-\mu r}}{r}\frac{1 + \mu r + \mu^2 r^2}{\mu^2 r^2}$$

- 进一步求出 $O(\epsilon^2)$ 阶之解，在指数因子 $e^{-\mu r}$ 可忽略时，有

$$A(r) = 1 - \frac{8}{3}\frac{GM}{r}\left(1 - \frac{1}{6}\frac{GM}{\mu^4 r^5} + \cdots\right)$$
$$B(r) = 1 - \frac{8}{3}\frac{GM}{\mu^2 r^3}\left(1 - 14\frac{GM}{\mu^4 r^5} + \cdots\right)$$
$$C(r) = 1 + \frac{4}{3}\frac{GM}{\mu^2 r^3}\left(1 - 4\frac{GM}{\mu^4 r^5} + \cdots\right)$$

- 当 $r^5 \lesssim GM/\mu^4 \equiv R_V^5$ 时微扰展开不可信任，此时基于线性理论的计算有可能是“人造”的
- 线性近似仅当 $r \gg R_V$ 是好近似，$\mu \to 0$ 蕴涵 $R_V \to \infty$
- 有效场论分析：当 $r \lesssim R_V$ 时，Boulware-Deser 鬼场的质量变得很轻，开始提供长距作用；由于动能项是负的，产生的斥力与 $s_3 = 0$ 的纵向模提供的引力抵消，有效的自由度仅为横向自由度