Scattering Amplitude

Bo Feng

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Motivation

- For LHC (Large Hadron Collide) data analysis: both backgrounds and signals.
- Challenge faced by us:
  - With a lot of QCD processes, one-loop corrections are general needed.
  - With high energy, multiple jets processes are important.
- QCD $+$ Multiple jets $+$ One-loop $=$ Very Difficult Task

Motivation

Table: The LHC “priority" wishlist for which a NLO computation seems now feasible.

process ($V \in \{Z, W, \gamma\}$)	relevant for
1. $pp \to V\,V\,\mathrm{jet}$	$t\bar{t}H$, new physics
2. $pp \to t\bar{t}\,b\bar{b}$	$t\bar{t}H$
3. $pp \to t\bar{t} + 2\,\mathrm{jets}$	$t\bar{t}H$
4. $pp \to V\,V\,b\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
5. $pp \to V\,V + 2\,\mathrm{jets}$	VBF$\to H \to VV$
6. $pp \to V + 3\,\mathrm{jets}$	various new physics signatures
7. $pp \to V\,V\,V$	SUSY trilepton

Motivation

- Last several years, there are HUGE developments in the evaluation of general one-loop amplitudes. They can be accumulated into two categories: OPP method and Unitarity cut method.
  [Bern, Dixon, Dunbar , Kosower, 1994] [Ossola, Papadopoulos, Pittau, 2006] [Anastasiou, Britto, Feng, Kunszt, Mastrolia, 2006]
- This section I will introduce above two methods

Passarino-Veltman reduction

- Passarino-Veltman reduction reduce any one-loop amplitudes into the form $\sum_i c_i I_i$ where I_i is basis and c_i is coefficient.

  [Passarino, Veltman, 1979]

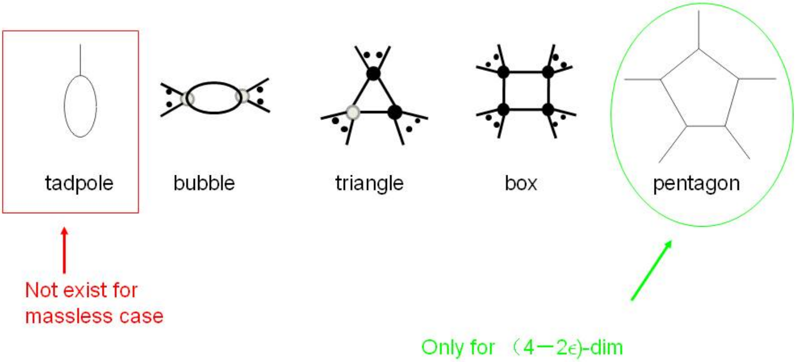
- The basis is scalar one-loop integration with following topology: tadpole, bubble, triangle, box and pentagon. For example

$$I_{bubble} = \int d^{4-2\epsilon} p \frac{1}{(p - M_1^2)((p-K)^2 - M_2^2)}$$

  [Bern, Dixon, Kosower, 1993]

- Two separate tasks: studying of basis and finding coefficients.

Basis of one-loop amplitude:



OPP method

- PV-reduction guarantee integrand to be decomposition:

$$
\begin{aligned}
I \;=\; & \sum_{i}^{m-1}(a(i)+\widetilde{a}(q;i))I^{(i)}+\sum_{i<j}^{m-1}(b(i,j)+\widetilde{b}(q;i,j))I^{(i,j)} \\
& +\sum_{i<j<r}^{m-1}(c(i,j,r)+\widetilde{c}(q;i,j,r))I^{(i,j,r)} \\
& +\sum_{i<j<r<s}^{m-1}(d(i,j,r,s)+\widetilde{d}(q;i,j,r,s))I^{(i,j,r,s)} \\
& +\sum_{i<j<r<s<k} p(i,j,r,s,k)I^{(i,j,r,s,K)}
\end{aligned}
$$

OPP method

- OPP realized that: (1) coefficients can be divided into two types: physical one and spurious one. (2) Physical one depends only external momenta and polarization vectors. (3) Spurious one do depend on loop momentum, but its integration is zero, so do not give physical contribution.
- More importantly, they categorized all ℓ-dependence of spurious coefficients
- A naive method is to take enough points of ℓ to solve all coefficients. However, solving with huge matrix is time consuming.

OPP method

- A better solution is to use the concept of leading singularities
- Step One— Box coefficient: (1) Mutliplying four D_i at both sides; (2) Then taking the value of loop momentum to set four D to zero. (3) Right hand sides has only one term contributes with two unknown coefficients: they are nothing, but exactly the physical and spurious coefficient related to the specific box basis.

OPP method

- Step Two— triangle coefficient: Having solved all boxes, we can subtract all boxes from both side (at the level of integrand). The lefted leading singularity is triangle. Multiplying three D_i at both sides and choose loop momenta to set three D to be zero. Then we solve 7 coefficients for this particular triangles.
- Similarly for bubble and tadpole coefficients.

OPP method

Good points of OPP method

- OPP method reduce the loop integration into pure algebraic calculations.
- The method is standard and easy to understand
- It is easy to program
- The efficiency depends crucially with the input expressions: the simpler the better!!!

OPP method

Unsatisfied points of OPP method

- We need to solve both physical and spurious coefficients although later one is not needed for loop calculations.
- We must solve systematically, i.e., to find bubble coefficient we need to solve boxes first and then triangles.

Unitarity cut method

General idea:

[Bern, Dixon, Dunbar , Kosower, 1994]

- Using $A_n = \sum_i c_i I_i$, we take **imaginary part** at both sides
- c_i is rational, so imaginary part of right handed side is given by ΔI_i which can easily calculated. Moreover, ΔI_i is unique for each basis like **figure print**
- The left handed side is given by

$$\Delta A_n = \int d^4\ell\delta(\ell^2)\delta((\ell - K)^2)A_L^{tree}A_R^{tree}$$

- Comparing both sides, we can read out coefficients c_i.

Unitarity cut method

- **Good point:** The left input A_L^{tree}, A_R^{tree} are on-shell tree-level amplitudes, so their expressions are simpler and gauge invariant
- **Unsatisfied point:** There is no easy way to carry out reduced phase space integration at the left handed side

Unitarity cut method

- A breakthrough putting unitarity cut method into practical use is the holomorphic anomaly: it reduces the integration to read out residue of corresponding pole
  [Cachazo, Svrcek, Witten, 2004; Britto, Buchbinder,Cachazo, Feng, 2005]
- Systematically study identified which pole corresponds to which basis, thus we can read out coefficient of given basis without calculating other's unlike the OPP method.
  [Britto, Feng, 2006]

Unitarity cut method

Further generalization:

- Unitarity cut method to general D-dimension.

  [Anastasiou, Britto, Feng, Kunszt, Mastrolia, 2006]

- Forde's method: A combination of OPP method and unitarity cut method

  [Forde, 2007]

- Generalized OPP method: generalize the OPP method to general D-dimension with features from D-dimensional unitarity cut method:

  [R.K. Ellis, W.T. Giele, Z.Kunszt, 2007]

Leading singularity

- Unitarity cut method takes imaginary part, which is equivalent to cut two inner propagators
- Could we put more inner propagators on-shell? If we could, what is the physical interpretation?
- Yes! We can! The concept is leading singularity!
- For one-loop in 4D, we can cut at most four inner propagators and fix momentum ℓ completely. For this case, we produce product of four on-shell tree-level amplitudes, which is nothing but the coefficient of box basis.

[Britto, Cachazo, Feng, 2004]

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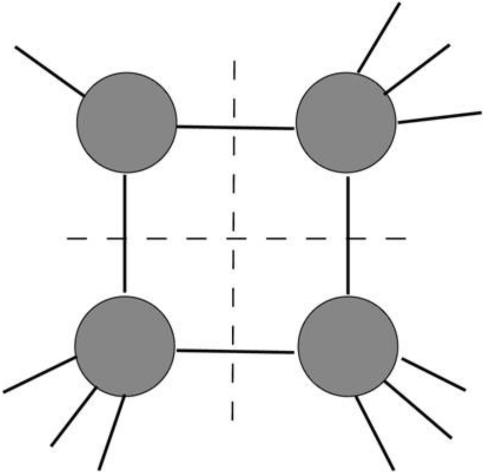
Leading singularity and quadruple cut:



- Leading singularity makes some coefficients easy to calculate at any loop. For example, box coefficient at one-loop.
- For general theory, besides these coefficients calculated by leading singularities, there are other coefficients which are little bit nontrivial to extract.
- However, there are some special theories, for which only these easily calculated coefficients are non zero. One of such example is the $\mathcal{N} = 4$ SYM theory.
- One of most important breakthrough in last couple year is the expression of leading singularities for all loops in $\mathcal{N} = 4$ SYM theory.

$\mathcal{N} = 4$ theory

[Series Work by N. Arkani-Hamed and F. Cachazo]

- Twistor variables $\mathcal{W} = (\widetilde{\mu}, \widetilde{\lambda}; \widetilde{\eta})$:

$$\begin{aligned} \mathcal{W} &= g^{+} + \widetilde{\eta}_A \psi_{+}^{A} + \frac{1}{2}\widetilde{\eta}^A \eta^B S_{AB} \\ &+ \frac{1}{3!}\widetilde{\eta}^A \eta^B \widetilde{\eta}^C \epsilon_{ABCD} \psi_{-}^{D} + \widetilde{\eta}^1 \eta^2 \widetilde{\eta}^3 \widetilde{\eta}^4 g^{-} \end{aligned}$$

- For n-particle with k-negative helicities we construct matrix $C_{\alpha a}$ as the $k \times n$ matrix with indices $a = 1, .., n$ for the particles, $\alpha = 1, ..., k$ to denote the negative helicity. We use $(m_1, ..., m_k)$ as the determinant of $k \times k$ sub-block of matrix (which we will call *minor*)

$\mathcal{N} = 4$ theory

The conjecture: Following expression

$$\mathcal{L}_{n;k}(\mathcal{W}_a) = \frac{1}{GL(k)} \int \frac{d^{kn} C_{\alpha a}}{(12..k)(23...(k+1))...(n1...(k-1))} \prod_{\alpha=1}^{k} \delta^{4|4}(\sum_{a=1}^{n} C_{\alpha a} \mathcal{W}_a) \quad (1)$$

contains leading singularities to all loops!
First few remarks:

- Cyclic symmetry is manifest!
- Parity symmetry can be obtained by transpose of matrix $C_{\alpha a}$

$\mathcal{N} = 4$ theory

Understanding One — Gauge fixing

- The expression has the $GL(k)$ invariance, which should be mod out and dimension of integration is $k(n-k)$.
- Different choice of gauge fixing corresponds to different choice of BCFW-deformed pair. Thus although individual term is different, their sum will be same!
- Different physical property will be manifest at different gauge choice. One choice with manifest dual conformal symmetry while another one, manifest particle interpretation.

$\mathcal{N} = 4$ theory

Geometrical picture:

- Each line $C_{\alpha a}$ with $a = 1, 2..., n$ defines a vector in n-dimensional particle space
- Whole matrix $C_{\alpha a}$ defines a k-plane (or k-dimensional frame) in n-dimensional particle space
- The moduli space of k-lane in n-dimensional space is known as Grassmannian manifold $G(k, n)$. The dimension of manifold is exactly $k(n - k)$

$\mathcal{N} = 4$ theory

Let us do Fourier transformation back to momentum space $\int d\mu_i^2 e^{\mu_i \lambda_i}$ we get

$$\mathcal{L}_{n;k} = \frac{1}{\mathrm{vol}(\mathrm{GL}(k))} \int \frac{d^{k\times n} C_{\alpha a}}{(C_1 C_2 \cdots C_k)\cdots(C_n \cdots C_{k-1})} \prod_{\alpha=1}^{k} \delta^4(C_{\alpha a}\tilde{\eta}_a)\delta^2(C_{\alpha a}\tilde{\lambda}_a)\int d^2\rho_\alpha \delta^2(\lambda_a - C_{\alpha a}\rho_\alpha) \quad (2)$$

- Factor $\delta^2(C_{\alpha a}\widetilde{\lambda}_a)$ tells us that two n-vectors expanded by $\widetilde{\lambda}$ and k-plane expanded $C_{\alpha a}$ are **orthogonal.**
- Factor $\delta^2(\lambda_a - C_{\alpha a}\rho_\alpha)$ tells us two n-vectors contain inside k-plane
- Putting together, we have momentum conservation $\sum p_i = \sum \lambda_i \widetilde{\lambda}_i$

$\mathcal{N}=4$ theory

Gauge Choice One:

- Use $GL(k)$ to fix k k-dimensional vectors to be the orthogonal basis. Thus n-indices a can be split into $I=1,...,k$ and $i=1,...,(n-k)$ and the gauge fixed Grassmanian integration is

$$\mathcal{L}_{n;k}(\mathcal{W}_a)|_{fixed} = \int \frac{d^{k(n-k)}C_{Ii}}{(12..k)(23...(k+1))...(n1...(k-1))} \prod_{I=1}^{k} \delta^{4|4}(\mathcal{W}_I + \sum_{i=1}^{n-k} C_{Ii}\mathcal{W}_i)$$

$\mathcal{N}=4$ theory

In momentum space we have

$$\mathcal{L}_{n;k}(\mathcal{W}_a)|_{fixed} = \int \frac{d^{k(n-k)}C_{li}\prod_{i=1}^{n-k}\delta^2(\lambda_i-\sum_{l=1}^{k}C_{li}\lambda_l)}{(12..k)(23...(k+1))...(n1...(k-1))} \prod_{l=1}^{k}[\delta^2(\widetilde{\lambda}_l+\sum_{i=1}^{n-k}C_{li}\widetilde{\lambda}_i)\delta^4(\widetilde{\eta}_l+\sum_{i=1}^{n-k}C_{li}\widetilde{\eta}_i)]$$

About the integration:

- $k(n-k)-2(n-k)-2k+4=(k-2)(n-k-2)$ true variables!
- Leading singularities are obtained by residue of multiple variables, which is the generalization of familiar theory based on single complex variable z.

$\mathcal{N} = 4$ theory

Multivariable Residues: A simple example

- $f(z_1, z_2) = \frac{h(z_1,z_2)}{(az_1+bz_2+c)(ez_1+fz_2+g)}$ and integral $\int f(z_1, z_2)dz_1 dz_2$.
- Define $u_1 = (az_1 + bz_2 + c)$, $u_2 = (ez_1 + fz_2 + g)$, we have

$$\int \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{h(z_1(u), z_2(u))}{\det\left(\frac{\partial(u_1,u_2)}{\partial(z_1,z_2)}\right)}$$

- Residue at $(u_1, u_2) = (0, 0)$ is given by

$$\mathrm{Res}[f](z_1^*, z_2^*) = \frac{h(z_1^*, z_2^*)}{\mathcal{J}(z_1^*, z_2^*)}, \quad \mathcal{J}(z_1^*, z_2^*) = \det\left(\frac{\partial(u_1, u_2)}{\partial(z_1, z_2)}\right)$$

$\mathcal{N} = 4$ theory

- With N-variables, the generalization if following. With

$$f = \frac{g}{p_1 \ldots p_M}, \quad M \geq N$$

Taking N-factors from p_i and finding solutions z_i^*, the residue is given by

$$\mathrm{Res}[f](z^*) = \frac{g(z^*)}{\prod_{i \neq i_1, \ldots, i_N} p_i(z^*) \det\left(\frac{\partial(p_i)}{\partial(z_i)}\right)}$$

$\mathcal{N} = 4$ theory

Composed Residue

- For $\frac{1}{x(x+yz)}$ we have only two factors but three variables
- The solution is that when $x = 0$, we have $x + yz \to yz$, i.e., effectively three factors. In other word, we have

$$\frac{1}{x(x+yz)} \to \frac{1}{xyz}$$

$\mathcal{N} = 4$ theory

Global Residue theory:

- Consider $\frac{h}{f_1 \ldots f_N}$ where we can combine arbitrary many factors into N-groups.
- In general, the common solutions of $f_1 = \ldots = f_N = 0$ will not be unique and we denote solutions z_a with $a = 1, \ldots, A$.
- For each solution, the residue is given by

$$\mathrm{Res}_a = \frac{h(z)}{\mathcal{J}_a}, \quad \mathcal{J} = \left| \frac{\partial(f_1, \ldots, f_N)}{\partial(z_1, \ldots, z_N)} \right|$$

- If degree of numerator is less than degree of denominator, then

$$\sum_a \mathrm{Res}_a = 0$$

$\mathcal{N} = 4$ theory

- In BCFW recursion, different choice of deformation pair corresponds different sum over some residues.
- Global Residue theory guarantee that these sums are equivalent to each other.
- Global Residue theory guarantee the decoupling of spurious poles: Some spurious poles show up at the left handed side but not the right handed side of equation!

$\mathcal{N} = 4$ theory

- There are other gauge choices. For example, we can choose the first two rows in matrix $C_{\alpha a}$. Using this choice, the proof of dual superconformal symmetry is easier.
- Instead of twistor variables, another variable **momentum twistor** is also very useful.

$$Z^I = (Z^\alpha; \chi^a) = (\lambda_A, \mu^{A'}; \chi_A) = (|\lambda_A\rangle\,, |\mu_A]\,, |\chi_A])$$

$$|\mu_A] = |x_A|\lambda_A\rangle\,, \quad |\chi_A] = |\theta_A|\lambda_A\rangle\,,$$

Under this variable, the momentum conservation is automatically satisfied!

$\mathcal{N}=4$ theory

Other important results for $\mathcal{N}=4$ SYM-theory:

- Explicit expression for tree-level amplitude with all helicity configuration.
- Each term in the tree-level amplitude has been given the interpretation of leading singularity of corresponding higher loop basis
- Each term can be written as manifest dual superconformal invariant form using Hodge's momentum twistor variables.

$\mathcal{N} = 4$ theory

Other important results for $\mathcal{N} = 4$ SYM-theory:

- All loop integrands can be calculated using the recursion relation
- Loop integrands have been given a nice geometrical picture: they are volume of polygons in momentum twistor space.
- Symbol method is developed to do the integration for higher loops

$\mathcal{N} = 8$ theory

Finite conjecture of $\mathcal{N} = 8$ SUGRA-theory:

- It has been checked up to four loops for four gravitons. The behavior is better than previous dimensional counting.
- We still do not know if the conjecture if true or false
- The loop level calculation is fascinated by generalization of KLT relation at the loop level.

Other topics:

- Wilson loop and amplitude duality
- Correlation function in AdS space and Mellin amplitude