On some issues of non-extremal vs extremal branes

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Introduction

For a charged black hole (brane), the BPS bound

$$M \geq Q. \tag{1.1}$$

Non-extremal black hole (brane) $\xrightarrow{\text{extremal limit}}$ extremal correspondence

Question:

While a well-defined mathematical limit,
can the two be related to each other by a
well-defined smooth physical process?

Introduction

Can Hawking radiation play such a role?

The answer appears to be negative for the following reason:

- if the pair of particles or branes created are neutral, the process can only reduce the mass of non-extremal object and at most can give rise to an near extremal object,
- if the pair created are charged (usually $q \gg m$, for example, for electron $e/m \sim 10^{21}$), it can also give rise to an near extremal object
- either of the above can only give rise to an near-extremal but not extremal black hole (brane).

Introduction

Further more,

- Non-extremal and extremal black objects have different topologies and so one doesn't expect that the two can be related to each other by a smooth physical process,
- In terms of their respective thermodynamics, the non-extremal black object has well-defined temperature, entropy etc and its zero temperature limit which is usually taken wrongly as that of extremal black object (even in the so called extremal limit, the non-extremal one has a different topology from the corresponding extremal one). In fact, the extremal black object doesn't have its own intrinsical temperature and can reach thermal equilibrium with its environment at any temperature, just like the so-called "hot empty space".

Introduction

- In many aspects, the extremal limit of non-extremal black object is just like 1) accelerating massive particle such that its speed approaches that of light but can never reach that, 2) $T \to 0$ for a thermal system but can never reach $T = 0$ (the third law of thermal laws), 3) the property of $m \to 0$ for a massive particle is usually not the same as that of $m = 0$ for a massless particle.
- In a susy theory, non-extremal black object breaks all susy while extremal one preserves certain fraction of susy, so one doesn't expect that the susy can be restored by a continuous limit (the fermions in the two cases obey different boundary conditions)

Introduction

More evidence can be listed showing the difference between extremal and non-extremal black objects, indicating that the two cannot be related to each other smoothly.

While one knows that the non-extremal black objects can be formed by gravitational collapse, if the extremal ones cannot be formed from the non-extremal ones, how can they be generated from the first place?

The extremal objects are basic building blocks in any susy theories and their possible source of creation is from pair creation (?).

Basic setup

For a non-extremal black hole (brane),

$$S_{\rm BH} \sim A, \quad T_{\rm BH} \sim \kappa, \tag{2.1}$$

Precisely,

$$T_{\rm BH} = \frac{\kappa}{2\pi} \quad \left(= \frac{\hbar c \kappa}{2\pi k}\right), \qquad S = \frac{A}{4} \quad \left(= \frac{k c^3 A}{4 G \hbar}\right). \tag{2.2}$$

⇒ Quantum Thermodynamics?

⇒ Part of Quantum Gravity?

Basic setup

However, for an extremal black hole (brane),

$$S_{\rm extre} = 0(?), \tag{2.3}$$

$$T_{\rm extre} = 0(?). \tag{2.4}$$

Basic setup

Well-known, asymptotically flat black hole (brane) is unstable thermodynamically.

For example, a Schwarzschild black,

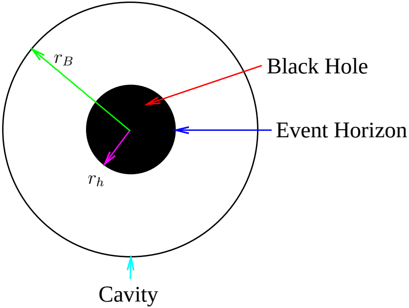
$$S_{\rm BH} = 4\pi M^2, \qquad T_{\rm BH} = \frac{1}{8\pi M} \tag{2.5}$$

with M the ADM energy carried by the black hole. This system is actually thermodynamically unstable (the specific heat $C < 0$)!!!

Basic setup

- In order to give a proper consideration of asymptotically flat black hole (brane) thermodynamics, we need first to suitably stabilize the black hole thermally.
- In other words, we need to consider ensembles that include not only the black hole under consideration but also its environment.
- Further, as self-gravitating systems are spatially inhomogeneous, any specification of such ensembles requires not just thermodynamic quantities of interest but also the place at which they take specific values.

Basic setup



Black hole (r_h) placed in a cavity (r_B) with fixed T and V.

Basic setup

The stability can be analyzed using the Helmholz free energy which can be calculated following Gibbons and Hawking (PRD15(1977)2752) that the partition function Z contains the first-order classical Euclidean Einstein action of the hole as its leading term.

In other words,

$$Z = e^{-\beta F} \approx e^{-I_E} \tag{2.6}$$

$\Rightarrow$

$$I_E(r_B, T, e; r_+) = \beta F = \beta E(r_B, T, e; r_+) - S(r_+) \tag{2.7}$$

with $\beta = 1/T$ and E the internal energy of the cavity.

Basic setup

Define,

$$\bar{I}_E \equiv \frac{I_E}{4\pi r_B^2}, \qquad x \equiv \frac{r_+}{r_B}, \quad q \equiv \frac{e}{r_B}, \quad \bar{b} \equiv \frac{\beta}{4\pi r_B},$$
$$q < x < 1, \quad (r_+ > e,\ r_B > r_+). \tag{2.8}$$

$$\frac{\partial \bar{I}_E}{\partial x} \sim (\bar{b} - b_q(x)), \tag{2.9}$$

whose extremal points (equation of state)

$$\Rightarrow \ \bar{b} = b_q(\bar{x}), \tag{2.10}$$

gives the thermal equilibrium!

Basic setup

$$\left.\frac{d^2 \bar{I}_E}{dx^2}\right|_{x=\bar{x}} \sim -\frac{db_q(\bar{x})}{d\bar{x}}, \tag{2.11}$$

$$\begin{aligned} \frac{db_q(x)}{dx} > 0, \qquad & \frac{d^2 \bar{I}_E}{dx^2} < 0 \qquad (\text{unstable}) \\ \frac{db_q(x)}{dx} < 0, \qquad & \frac{d^2 \bar{I}_E}{dx^2} > 0 \qquad (\text{stable}) \end{aligned} \tag{2.12}$$

Chargeless black hole

Consider the simplest spherical symmetric Schwarzschild black hole in Euclidean signature

$$ds_E^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2, \qquad (2.13)$$

with the horizon radius $r_h = 2M$. If we place this black hole in a large spherical hot cavity at a given $r = r_B$ with the temperature at the wall fixed at T_B, this will define a canonical ensemble *a la* York (PRD33 (1986) 2092) for this hole.

Chargeless case

$$\bar{I}_E = \bar{b}\left(1 - (1-x)^{1/2}\right) - \frac{1}{4}x^2. \tag{2.14}$$

$$\left(\bar{I}_E = 0\,(x=0) \quad \Leftrightarrow \quad \text{hot flat space}\right). \tag{2.15}$$

$$\frac{d\bar{I}_E}{dx} = \frac{1}{2(1-x)^{1/2}}\left(\bar{b} - b(x)\right), \tag{2.16}$$

$$b(x) = x(1-x)^{1/2} > 0. \tag{2.17}$$

Note

$$b(x \to 0) \to 0, \qquad b(x \to 1) \to 0. \tag{2.18}$$

Chargeless case

$$\frac{d\bar{I}_E}{dx} = 0 \Rightarrow \bar{b} = b(\bar{x}) = \bar{x}(1-\bar{x})^{1/2} \tag{2.19}$$

$$\Rightarrow T_B = T(r_B) = (8\pi M)^{-1}\left(1 - \frac{2M}{r_B}\right)^{-1/2}. \tag{2.20}$$

Chargeless case (Schwarzschild black hole)

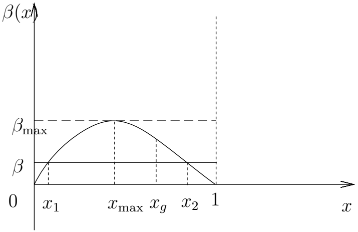


Figure 1: The typical behavior of $\beta(x)$ vs x ($x \equiv r_h/r$).

Charged case

The charged (Reissner-Nordström) black hole is

$$ds_E^2 = V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_2^2, \tag{2.21}$$

with

$$V(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}, \qquad \Phi = \frac{e}{r}. \tag{2.22}$$

It has two horizons given at (V (r) = 0)

$$r_\pm = M \pm \sqrt{M^2 - e^2}, \tag{2.23}$$

which implies

$$M \geq e, \quad \text{(BPS Bound)} \tag{2.24}$$

Charged non-extremal case

$$\bar{I}_E(\bar{b}, q; x) = \bar{b}\left(1 - \sqrt{(1-x)\left(1 - \frac{q^2}{x}\right)}\right) - \frac{1}{4}x^2. \qquad (2.25)$$

$$\frac{d\bar{I}_E}{dx} = \frac{1 - \frac{q^2}{x^2}}{2(1-x)^{1/2}\left(1 - \frac{q^2}{x}\right)^{1/2}}\left(\bar{b} - b_q(x)\right), \qquad (2.26)$$

where

$$b_q(x) = \frac{x(1-x)^{1/2}\left(1 - \frac{q^2}{x}\right)^{1/2}}{1 - \frac{q^2}{x^2}}. \qquad (2.27)$$

Note

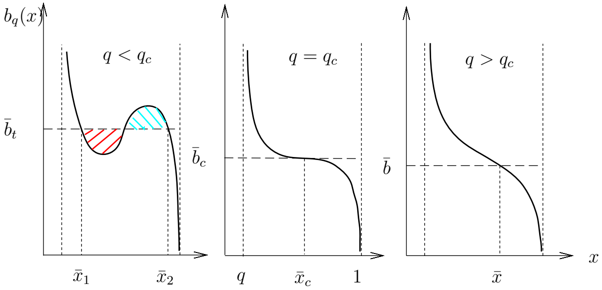
$$b_q(x \to q) \to \infty, \quad b_q(x \to 1) \to 0. \qquad (2.28)$$

Charged non-extremal case

Note that

$$b_q(x \to q) \to \infty, \quad b_q(x \to 1) \to 0, \tag{2.29}$$

there exists a critical charge $q_c = \sqrt{5} - 2$ ($\bar{x}_c = 5 - 2\sqrt{5}$, $\bar{b}_c = 0.429$) and



The typical behaviors of $b(x)$ vs x for $q < q_c, q = q_c, q > q_c$.

Charged extremal case

Now $M = e$,

$$ds_E^2 = \left(1 - \frac{e}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{e}{r}\right)^2} + r^2 d\Omega_2^2, \tag{2.30}$$

where the horizon is at $r = e$ which is infinitely away from any direction.

We can see this by setting $r = e(1 + \epsilon)$ with $\epsilon \ll 1$ and defining $\rho = -e \ln \epsilon$,

$$ds_E^2 \sim e^{-2\rho/e} dt^2 + d\rho^2 + e^2 d\Omega^2, \tag{2.31}$$

there is no conical singularity at the horizon ($\rho \to \infty$) and so the period of t can be anything.

Charged extremal case

So the extremal brane plays the similar role as 'hot empty space' in chargeless case that it can take any temperature as the cavity. The entropy can be calculated to be zero.

In this case, Note that the extremal black hole $(x = q)$ always has the largest free energy for given cavity temperature, therefore can never be a stable phase.

Charged case

Note that for non-extremal black hole, the topology for (t, r) is R^2, so the fermions if any must be anti-periodic along t, while for extremal one it is $S^1 \times R$ for which the fermions are periodic along t.

p-brane

A p-brane is a p-dimensional hyperspace ($p = 0, 1, \cdots, 9$) residing at the bulk spacetime with dimension D ($D \geq p + 1$) and can carry either electric-like $d + 1$-form charge with $d = p + 1$ as

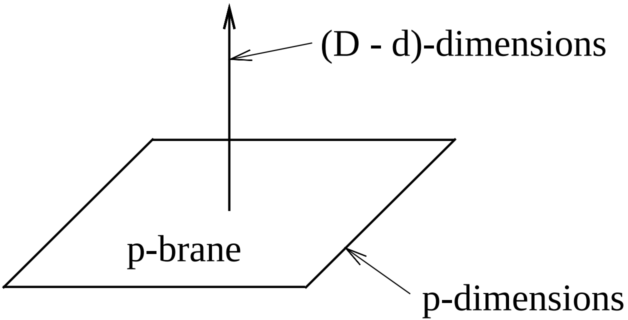
$$e_d \sim \int {}^*F_{d+1} \tag{2.32}$$

or magnetic-like $\tilde{d} + 1$-form charge with $\tilde{d} = D - 2 - d$ as

$$g_{\tilde{d}} \sim \int F_{\tilde{d}+1}. \tag{2.33}$$

p-brane

The spatial dimensions transverse to the p-brane is $D - d = \tilde{d} + 2$ and note $1 \leq \tilde{d} \leq 7$.

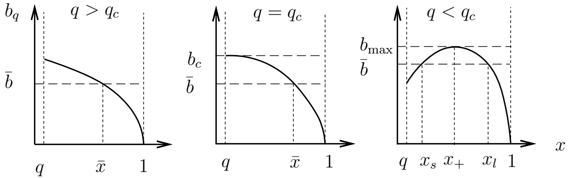


Phase structure and transition

For each given $\tilde{d} > 2$, the phase structure here is the same as the charged black hole though the detail is different. There exists a critical charge q_c, depending on $\tilde{d}$, and we have also the three cases for each given $\tilde{d} > 2$,

Phase structure

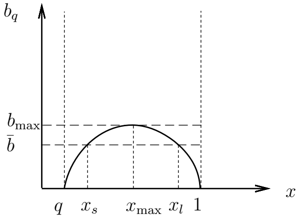
The $\tilde{d} = 2$ case ($q_c = 1/3$):



The typical behaviors of $b_q(x)$ vs x for $\tilde{d} = 2$.

Phase structure

The $\tilde{d} = 1$ case:



The typical behavior of $b_q(x)$ vs x for $\tilde{d} = 1$.

Phase structure

Summary

- Therefore for either $\tilde{d} = 2$ or $\tilde{d} = 1$, the extremal case has to be considered to give a complete phase structure and the non-extremal and the extremal branes can only be related to each other by a first order phase transition.
- If we denote (t, r) as representing the Euclidean time and radial directions, once again, the non-extremal one has a topology along (t, r) as R^2 and the fermions in this theory must obey anti-periodic boundary condition, while for extremal one, the topology is $S^1 \times R$ and the fermions obey periodic boundary condition since the extremal one preserves 1/2 susy in this theory. In other words, the two cases have different topologies and different spin structures.

Phase structure

- So in general non-extremal brane cannot turn into extremal one smoothly and vice-versa, in support of the claims given at the beginning.

THANK YOU!