$f(R)$ Theories of Supergravities

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Based on the following paper:

Liu, Lü and Wang, arXiv:1201.2417

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I. Motivation

- Einstein gravity is not a quantum complete theory (at least as the folklore says.)
- The principle of General Relativity is too good to throw away.
- Introduce higher derivative term, while keeping the principle and the symmetry. The $f(R)$ formalism is the simplest generalization.
- Enlarge the symmetry to include supersymmetry, yielding supergravities. (Boosted by string theory, as low-energy effective theories of strings)
- Although both are generalizing GR, little overlap between the two.

I. Motivation, continued

A rather fundamental property of supergravity is that it admits Killing spinor equations, whose defining property is that the Γ-matrix projected integrability condition give rise to equations of motion.

It is hard to imagine such a property for $f(R)$ theories, and hence the two subjects rarely overlap.

However, in a recent paper (arXiv:1111.6602), we (Liu, Lü and Wang) demonstrate that there exist large classes of $f(R)$ theories that do admit Killing spinor equations.

I. Killing Spinor Equations in $f(R)$

Condition:

$$f'' - \frac{\Big(4(D-1)(D-2)W^2 + R\Big)W'}{\Big(4D(D-1)W^2 + R\Big)W} f' + \frac{W'}{\Big(4D(D-1)W^2 + R\Big)W} f = 0\,.$$

Killing spinor equations:

$$\mathcal{D}_\mu \epsilon \equiv \Big(D_\mu + W(R)\Gamma_\mu\Big)\epsilon = 0\,,$$

$$\Big(\Gamma^\mu \nabla_\mu F + U(R)\Big)\epsilon = 0\,,$$

where

$$U = -\frac{(4D(D-1)W^2 + R)f''(R)}{4(D-1)W'}$$

The integrability condition projected by a Γ-matrix:

$$\Gamma^\mu \mathcal{R}_{\mu\nu}\epsilon = 0\,.$$

I. Motivation, continued

- The existence of KSE is not unique to, but an important feature of supergravities
- Thus the $f(R)$ formulation of gravity is not totally incompatible with supersymmetry
- Hence the $f(R)$ formalism is not totally incompatible with string theory
- Higher-order curvature terms are little known in supergravities or in string theories
- May shed some light on the supersymmetrizing of higher order curvature terms.

Two reviews

Before going to discuss $f(R)$ formalism of supergravities, we first give some short reviews on the following two subjects:

- Why is Killing spinor equation so important in supergravities?
- What is $f(R)$ theory?

Why is KSE so important in supergravity?

This is a new way of looking at constructing supergravities that you do not seen in supergravity books.

Introduce a fermionic partner for the graviton, a gravitno ψ_μ.

Note that

$$\delta(\sqrt{-g}R) = G_{\mu\nu}\delta g^{\mu\nu}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ Thus, any symmetry transformation has to generate $G_{\mu\nu}$ to cancel the above one. The kinetic term for the fermions has one derivative, so the the transformation rule for the fermion has to have another.

General coordinate transformation:

$$\delta g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}\,.$$

Analogous transformation

$$\delta\psi_\mu = D_\mu\epsilon$$

where the covariant derivative on fermion is

$$D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon$$

Property of covariant derivatives

On vector:

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

On spinor:

$$[D_\mu, D_\nu]\epsilon = \frac{1}{4} R^{ab}{}_{\mu\nu} \Gamma_{ab} \epsilon$$

Describe parallel transports around an infinitesimal closed path.

Projections

$$\Gamma^{\nu}[D_\mu, D_\nu]\epsilon = \tfrac{1}{2}R_{\mu\nu}\Gamma^{\nu}\epsilon$$

$$\Gamma_{\mu}{}^{\nu\rho}[D_\nu, D_\rho]\epsilon = G_{\mu\nu}\Gamma^{\nu}\epsilon$$

We see that the Einstein tensor $G_{\mu\nu}$ appears!

Gravitino kinetic term

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho$$

Consider a transformation $\delta\psi_\mu = D_\mu \epsilon$, then we have

$$\delta\mathcal{L} = \sqrt{-g}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \delta\psi_\rho = \sqrt{-g}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu D_\rho \epsilon$$

$$= \frac{1}{2}\sqrt{-g}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}[D_\nu, D_\rho]\epsilon = \frac{1}{2}\sqrt{-g}G_{\mu\nu}\bar{\psi}^\mu \Gamma^\nu \epsilon$$

Graviton/Gravitino system: pseudo-supergravity

$$\mathcal{L} = \sqrt{-g}\,(R + \tfrac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho)$$

transformation rules:

$$\delta\psi_\mu = D_\mu \epsilon$$

$$\delta e_\mu^a = \tfrac{1}{4}\bar{\psi}_\mu \Gamma^a \epsilon\,, \quad \text{so} \quad \delta g_{\mu\nu} = \tfrac{1}{2}\bar{\psi}_{(\mu}\Gamma_{\nu)}\,\epsilon$$

The Lagrangian is invariant under the above transformation rules if we do not look beyond quadratic order in fermions.

Any bosonic gravity theories that admit consistent Killing spinor equations can pseudo-supersymmetrized up to and including the quadratic order of fermion fields in the action.

$f(R)$ theories of gravity

Replace the Ricci scalar R in the Einstein-Hilbert action with a generic function $f(R)$.

$f(R)$ can be an arbitrarily custom-made function to suit whatever the purpose.

If one is not a cosmologist, one might feel embarrassed to be associated with it.

Properties of $f(R)$ theories

It is equivalent to a special class of the Brans-Dicke theory.

Introducing an auxiliary field χ, then the Lagrangian

$$e^{-1}\mathcal{L} = f(\chi) + f_{,\chi}(\chi)(R-\chi)\,.$$

Variation of χ gives to $f_{,\chi\chi}(R-\chi) = 0$. Provided that $f_{,\chi\chi} \neq 0$, we have $\chi = R$, and substitute this to the Lagrangian, we obtain $f(R)$ gravity.

The Lagrangian is nothing but a scalar/gravity system with the scalar χ that has no kinetic term. To see it more clearly, we can define $\varphi = f_{,\chi}(\chi)$, and hence the Lagrangian becomes

$$e^{-1}\mathcal{L} = \varphi R + f(\chi(\varphi)) - \varphi\,\chi(\varphi)\,.$$

The conversion requires to find the inverse function for $f'(R)$ and the close-form conversion may not always exist.

Converting scalar/gravity to $f(R)$

In supergravities, we typically have scalar/gravity system in the following form

$$e^{-1}\mathcal{L}_D = R - \tfrac{1}{2}(\partial\phi)^2 - V(\phi)\,,$$

The equations of motion for dilaton ϕ cannot be solved algebraically like the φ in the previous case.

However, let's make the following conformal transformation

$$g_{\mu\nu} \to e^{-2\alpha\phi} g_{\mu\nu}\,, \qquad \varphi = e^{\beta\phi}\,,$$

where

$$\alpha = -\frac{1}{\sqrt{2(D-1)(D-2)}}\,, \qquad \beta = \sqrt{\frac{D-2}{2(D-1)}}\,.$$

The Lagrangian becomes

$$e^{-1}\mathcal{L}_D = \varphi R - \widetilde{V}(\varphi)\,,$$

where

$$\widetilde{V}(\varphi) = \varphi^{\frac{D}{D-2}} V(\phi(\varphi))\,.$$

The Lagrangian has no derivative on φ, and hence it is an auxiliary field. The variation of φ gives rise to an algebraic equation on φ, which can solved in principle. Substitute the solution of φ into the Lagrangian gives rise to an $f(R)$ theory.

Thus we define the frame where the dilation has no kinetic term is the "$f(R)$" frame.

Equivalence and Inequivalence

Three Lagrangians

1. $e^{-1}\mathcal{L}_D = R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\,,$
2. $e^{-1}\mathcal{L}_D = \varphi R - \widetilde{V}(\varphi)\,,$
3. $e^{-1}\mathcal{L}_D = f(R)$

Lagrangian 2 and Lagrangian 3 are classically equivalent. It is analogous to the relationship between the Polyakov action and the Nambu-Goto action in string theory.

Lagrangian 1 is not equivalent to 2 or 3, because the conformal scaling can be singular in the solution space.

In the $f(R)$ theory, the trace mode is turned on. The purpose of the auxiliary φ in Lagrangian 2 is that it excites the trace mode. Thus Lagrangian 2 should be viewed as a gravity theory, rather than gravity plus matter.

The $f(R)$ frame and Kaluza-Klein theory

Recall that the conformal transformation and field redefinition to cast the theory into the $f(R)$ frame:

$$g_{\mu\nu} \to e^{-2\alpha\phi} g_{\mu\nu}\,, \qquad \varphi = e^{\beta\phi}\,,$$

where

$$\alpha = -\frac{1}{\sqrt{2(D-1)(D-2)}}\,, \qquad \beta = \sqrt{\frac{D-2}{2(D-1)}}\,.$$

For those who familiar with Kaluza-Klein reduction, the constants (α, β) are familiar: Consider $D+1$ Lagrangian

$$\hat{e}^{-1}\mathcal{L}_{D+1} = \hat{R}$$

and the reduction ansatz

$$ds^2 = e^{2\alpha\phi} ds_D^2 + e^{2\beta\phi}(dz + \mathcal{A}_{(1)})^2$$

The D-dimensional Lagrangian is

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}_{(2)}^2\,.$$

Thus, if we consider reduction ansatz with the D-dimensional metric unscaled, then we have

$$ds^2 = ds_D^2 + \varphi^2(dz + \mathcal{A}_{(1)})^2$$

The Kaluza-Klein theory becomes

$$e^{-1}\mathcal{L}_D = \varphi R - \tfrac{1}{4}\varphi^3\mathcal{F}_{(2)}^2\,.$$

In other words, the $f(R)$ frame is the same as the $(D+1)$-dimensional frame. And the breathing mode is auxiliary. If there is a cosmological constant in $D+1$ dimensions, we have

$$e^{-1}\mathcal{L}_D = \varphi(R - 2\Lambda) - \tfrac{1}{4}\varphi^3\mathcal{F}_{(2)}^2\,.$$

Solving for φ, we have

$$e^{-1}\mathcal{L}_D = \tfrac{4}{3}(R - 2\Lambda)\sqrt{\frac{R-2\Lambda}{3\mathcal{F}_{(2)}^2}}\,.$$

This is the $f(R)$ description of the Kaluza-Klein theory. It is classically equivalent to the usual Kaluza-Klein theory up to a conformal scaling that can be singular.

The breathing mode is auxiliary

String theory can be obtained from M-theory *via* KK reduction on S^1 or S^1/Z_2. The expectation value of the breathing mode is related to the string coupling constant. The possibility that the breathing mode is auxiliary suggests a non-perturbative formulation of the string theories.

Is this an artefact of low-energy effective action? When the Riemann square term is included in $D+1$ dimensions, the breathing mode ceases to be auxiliary. However, in the reduction of Gauss-Bonnet term, the breathing mode retains its auxiliary feature.

$$
\begin{aligned}
& e^{-1}\mathcal{L}_D \\
&= \varphi(R - \Lambda_0 + \alpha E_{\rm GB}) - \tfrac{1}{4}\varphi^3\mathcal{F}^2_{(2)} + \alpha\varphi^3\Big(- R^{abcd}(\mathcal{F}_{ab}\mathcal{F}_{cd} - \mathcal{F}_{ac}\mathcal{F}_{db}) \\
&+2\nabla_a\mathcal{F}_{bc}\nabla^a\mathcal{F}^{bc} + \tfrac{1}{3}\Box\mathcal{F}^2_{(2)} - 2\nabla_a\nabla_b(\mathcal{F}^2)^{ab} + 2\nabla_b\mathcal{F}^{ba}\nabla^c\mathcal{F}_{ca} \\
&-\tfrac{10}{3}\nabla_a\mathcal{F}_{bc}\nabla^b\mathcal{F}^{ac} + 4F^{ab}\nabla_a\nabla^c\mathcal{F}_{cb} - \tfrac{4}{3}\mathcal{F}^{ab}\nabla^c\nabla_b\mathcal{F}_{ac}\Big) \\
&+\alpha\varphi^5\Big(\tfrac{1}{4}(\mathcal{F}^2_{(2)})^2 + \tfrac{1}{2}(\mathcal{F}^2_{(2)})^{ab}(\mathcal{F}^2_{(2)})_{ab}\Big)\,,
\end{aligned}
$$

String dilaton an auxiliary field?

The breathing mode from $D = 11$ to $D = 10$ is the string loop coupling field.

At least for heterotic string theory, it can be demonstrated that the dilaton is an auxiliary up to and including the α' correction.

For type IIA or type IIB, it is too complicated to determine at this moment.

The $f(R)$-frame vs. string frame

If we view type IIA string theory from $D = 11$ compactified on S^1, the $f(R)$ frame is the most natural frame, especially when higher-order curvature terms are included.

On the other hand, from string theory point of view, type IIA supergravity is natural in the string frame.

A paradox?

Depending on the perturbative or non-perturbative approach to strings.

The $f(R)$ theories of $D = 10$ supergravities

In order to convert the usual supergravities to the $f(R)$ description, it is necessary first to go to the $f(R)$ frame.

All ten-dimensional supergravities are related to $D = 11$ one way or the other. The $f(R)$ frame is nothing but the M-theory frame. This implies that all ten-dimensional supergravities have a natural $f(R)$ description.

Example $\mathcal{N}=1$ $D=10$ supergravity

Bosonic sector in the Einstein frame

$$e^{-1}\mathcal{L}_{10} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{12}e^{-\phi}F_{(3)}^2\,.$$

In the $f(R)$ frame:

$$e^{-1}\mathcal{L}_{10} = \varphi\,R - \tfrac{1}{12}\varphi^{-1}F_{(3)}^2\,.$$

The $f(R)$ theory:

$$e^{-1}\mathcal{L}_{10} = f(R) = \sqrt{-\tfrac{1}{3}RF_{(3)}^2}\,.$$

Including the fermions

Full Lagrangian in the $f(R)$ frame:

$$e^{-1}\mathcal{L} = \varphi(R + K) - \frac{1}{12}\varphi^{-1}F_{(3)}^2 + X_3, \quad (1)$$

where K and the Yukawa term X_3 associated with $F_{(3)}$ are given by

$$\begin{aligned} K &= \tfrac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho - \tfrac{2\sqrt{2}\,\mathrm{i}}{3}\bar{\lambda}\Gamma^{\mu\nu}D_\mu\psi_\nu + \tfrac{1}{2}D^\mu(\psi_\nu\Gamma^\nu\psi_\mu)\,, \\ X_3 &= \Big(-\tfrac{1}{48}\bar{\psi}_\mu\Gamma^{\mu\nu\rho\sigma\lambda}\psi_\lambda - \tfrac{1}{8}\psi^\nu\Gamma^\rho\psi^\sigma + \tfrac{\sqrt{2}\,\mathrm{i}}{12}\bar{\lambda}\Gamma^{\nu\rho}\psi^\sigma\Big)F_{\nu\rho\sigma}\,. \end{aligned}$$

The supersymmetric transformation rules in the $f(R)$ frame are given by

$$\begin{aligned} \delta\psi_\mu &= D_\mu\epsilon + \tfrac{1}{54}F_{\nu\rho\sigma}\Gamma_\mu{}^{\nu\rho\sigma}\epsilon - \tfrac{1}{12}F_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon\,, \\ \delta\lambda &= \tfrac{3\mathrm{i}}{4\sqrt{2}}\varphi^{-1}(\Gamma^\mu\partial_\mu\varphi - \tfrac{1}{18}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho})\epsilon\,, \\ \delta e_\mu^a &= \tfrac{1}{4}\bar{\psi}_\mu\Gamma^a\epsilon\,, \qquad \delta g_{\mu\nu} = \tfrac{1}{2}\bar{\psi}_{(\mu}\gamma_{\nu)}\epsilon\,, \qquad \delta\varphi = -\tfrac{\mathrm{i}}{3\sqrt{2}}\varphi\bar{\lambda}\epsilon\,, \\ \delta A_{\mu\nu} &= \varphi(-\tfrac{1}{2}\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} + \tfrac{\mathrm{i}}{3\sqrt{2}}\bar{\epsilon}\Gamma_{\mu\nu}\lambda)\,. \end{aligned}$$

The $f(R)$ description of ten-D supergravity

Integrating out φ, we find that the $f(R)$ theory of the $\mathcal{N}=1$, $D=10$ $f(R)$ supergravity is given by

$$
\begin{aligned}
e^{-1}\mathcal{L} &= \sqrt{-\tfrac{1}{3}(R+K)F_{(3)}^2} + X_3\,,\\
\delta\psi_\mu &= D_\mu\epsilon + \tfrac{1}{54}F_{\nu\rho\sigma}\Gamma_\mu{}^{\nu\rho\sigma}\epsilon - \tfrac{1}{12}F_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon\,,\\
\delta\lambda &= \frac{3\mathrm{i}}{4\sqrt{2}}F^{-1}(\Gamma^\mu\partial_\mu F - \tfrac{1}{18}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho})\epsilon\,,\\
\delta e_\mu^a &= \tfrac{1}{4}\psi_\mu\Gamma^a\epsilon\,, \qquad \delta g_{\mu\nu} = \tfrac{1}{2}\bar\psi_{(\mu}\gamma_{\nu)}\epsilon_0\,,\\
\delta A_{\mu\nu} &= F\,(-\tfrac{1}{2}\bar\epsilon\Gamma_{[\mu}\psi_{\nu]} + \frac{\mathrm{i}}{3\sqrt{2}}\bar\epsilon\Gamma_{\mu\nu}\lambda)\,.
\end{aligned}
$$

One can add further the matter Yang-Mills multiplet (A_μ,χ). In the $f(R)$ frame, the extra parts of the Lagrangian and supersymmetric transformation rules are given by

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\mathrm{YM}} &= -\tfrac{1}{4}F_{(2)}^2 + \tfrac{1}{2}\varphi\bar\chi\Gamma^\mu D_\mu\chi - \tfrac{1}{48}F_{\mu\nu\rho}\bar\chi\Gamma^{\mu\nu\rho}\chi - \frac{1}{4\sqrt{2}}\varphi^{\frac{1}{2}}F_{\nu\rho}\bar\chi\Gamma^\mu\Gamma^{\nu\rho}\psi_\mu\\
\delta\chi &= \frac{1}{4\sqrt{2}}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon\,, \qquad \delta A_\mu = -\frac{1}{2\sqrt{2}}\varphi^{\frac{1}{2}}\bar\chi\Gamma_\mu\epsilon\,,\\
\delta_{\mathrm{extra}}A_{\mu\nu} &= \frac{1}{2\sqrt{2}}\varphi^{\frac{1}{2}}\bar\chi A_{[\mu}\Gamma_{\nu]}\epsilon\,.
\end{aligned}
$$

It is again straightforward to integrate out the auxiliary φ and obtain the $f(R)$ theory of heterotic supergravity.

The $f(R)$ description of ten-D supergravity

We find that all the $D = 10$ supergravities can be cast into the $f(R)$ descriptions.

However, this is not universal. Most of lower dimensional examples do not land themselves naturally in $f(R)$ theories.

For example, $D = 7$ gauged supergravity in the $f(R)$ frame is given by

$$\begin{aligned} e^{-1}\mathcal{L}_7 &= \varphi R - g^2\varphi^{\frac{7}{5}}\Big(\tfrac{1}{4}\varphi^{-\frac{8\sqrt{6}}{5}} - 2\varphi^{-\frac{3\sqrt{6}}{5}} - 2\varphi^{\frac{2\sqrt{6}}{5}}\Big) \\ &\quad -\tfrac{1}{48}\varphi^{-\frac{1}{5}+\frac{4\sqrt{6}}{5}}F_{(4)}^2 - \tfrac{1}{4}\varphi^{-\frac{4}{5}+\frac{2\sqrt{6}}{5}}(F_{(2)}^i)^2 + e^{-1}\mathcal{L}_{FFA}\,, \end{aligned}$$

The irrational power suggests that this frame is not natural to describe the system.

The majority of supergravities in fact will have such irrational powers in $f(R)$ frames.

An example in lower dimensions

$\mathcal{N}=2$, $D=5$ gauged supergravity with a vector multiplet:

$$e^{-1}\mathcal{L}_5 = R - \tfrac{1}{2}(\partial\phi)^2 + 4g^2\Big(2e^{-\frac{1}{\sqrt{6}}\phi} + e^{\frac{2}{\sqrt{6}}\phi}\Big) - \tfrac{1}{4}e^{-\frac{2}{\sqrt{6}}\phi}F_{(2)}^2 - \tfrac{1}{4}e^{\frac{4}{\sqrt{6}}\phi}\mathcal{F}_{(2)}^2 + \tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}\mathcal{A}_\lambda\,,$$

In the $f(R)$ frame:

$$e^{-1}\mathcal{L} = \varphi(R+8g^2) + \varphi^3(4g^2 - \tfrac{1}{4}\mathcal{F}_{(2)}^2) - \tfrac{1}{4}\varphi^{-1}F_{(2)}^2 - \tfrac{1}{4}e^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}\mathcal{A}_\lambda\,.$$

Supersymmetric transformation rules

$$\begin{aligned}\delta\psi_\mu &= [D_\mu - \tfrac{\mathrm{i}}{2}g(\sqrt{2}A_\mu + \mathcal{A}_\mu)]\epsilon + \tfrac{1}{3}g\varphi\Gamma_\mu\epsilon \\ &\quad + \tfrac{3\sqrt{2}\,\mathrm{i}}{16}\varphi^{-1}F_{\nu\rho}\Gamma_\mu\Gamma^{\nu\rho}\epsilon - \tfrac{3\mathrm{i}}{4}(\sqrt{2}\,\varphi^{-1}F_{\mu\nu} + \varphi\mathcal{F}_{\mu\nu})\Gamma^\nu\epsilon\,, \\ \delta\lambda &= -\frac{\mathrm{i}}{\sqrt{6}}\Big(\varphi^{-1}\Gamma^\mu\nabla_\mu\varphi - g(\varphi - \varphi^{-1}) - \tfrac{3\mathrm{i}}{8}(\sqrt{2}\varphi^{-1}F_{\mu\nu} - 2\varphi\mathcal{F}_{\mu\nu})\Gamma^{\mu\nu}\Big)\epsilon\,.\end{aligned}$$

So what?

Is there any new physics emerging?

Even at the classical level, the solution space of the usual supergravity is different from that of the corresponding $f(R)$ theory.

Example 1: $\mathcal{N}=2$, $D=10$

The $f(R)$ theory admits the following AdS$_7 \times S^3$ solution

$$ds^2 = \ell^2\Big(\frac{dr^2}{r^2} + r\, dx^\mu dx_\mu + d\Omega_{(3)}^2\Big)\,, \qquad d\Omega_3^2 = d\theta^2 + \sin^2\theta d\Omega_2^2\,,$$
$$F_{(3)} = 3\ell^3 \cos\theta \sin^2\theta d\theta\wedge\Omega_{(2)}\,, \qquad \varphi = \ell\cos\theta\,.$$

Note that $\varphi=0$ at the S^3 equator $\theta=\pi/2$. Thus in the original description, the solution has a power-law curvature singularity at the equator, and hence $\theta\in[0,\pi/2)$. The charge is $Q_5=\int F_{(3)} = 4\pi q$.

In this $f(R)$ frame, the geodesic is not complete at the equator, and hence $\theta\in[0,\pi]$. However if we allow that, we have $Q_5 = 4\pi q - 4\pi q = 0$ and also φ is negative at the southern hemisphere.

To avoid ghosts, we impose at a delta function source at the equator

$$F_{(3)} = 3\ell^3 |\cos\theta| \sin^2\theta\, d\theta\wedge\Omega_{(2)}\,, \qquad \varphi = \ell|\cos\theta|\,.$$

This solution with such global property clearly does not exist in the original theory.

An analogy: AdS$_6$ in the D4/D8 system

Does it make sense to talk about solutions with a source sitting at the equator? In fact that is how AdS$_6$ is embedded in string theory. D1/D8 system in massive type IIA:

$$
\begin{aligned}
ds_{10}^2 &= (\cos\theta)^{\frac{2}{9}}(ds_{AdS_6}^2 + 2d\theta^2 + 2\sin^2\theta d\Omega_3^2)\,, \\
F_{(4)} &= \tfrac{5\sqrt{2}}{6}(\cos\theta)^{\frac{1}{3}}\sin^3\theta\, d\theta\wedge\Omega_{(3)}\,, \qquad \varphi = (\cos\theta)^{-\frac{5}{9}}\,.
\end{aligned}
$$

Thus the solution has a power-law curvature singularity at the equator $\theta = \pi/2$ of the S^4. The solution becomes regular in the D4-brane frame, in which the Lagangian takes the form

$$
e^{-1}\mathcal{L} = \tilde{\varphi}(R - 2(\partial\log\tilde{\varphi})^2 - \tfrac{1}{1440}F_{(6)}^2) + \text{more}\,.
$$

The metric of the D4/D8 solution is then simply the AdS$_6\times S^4$ without the pre-factor. Furthermore, $\tilde{\varphi} = (\sin\theta)^{1/3}$. Thus it is necessary that θ runs from 0 to π with the $(\cos\theta)^{1/3}$ factor in both $F_{(4)}$ and $\tilde{\varphi}$ added an absolute-value sign, namely $|(\cos\theta)^{1/3}|$. This requires a delta-function source on the equator.

Another example

It is well-known that $\mathcal{N} = 2$, $D = 5$ gauge supergravity has no BPS static black holes. The static BPS solution has a naked singularity. The solutions are called "super stars." The resolution either turns it to be a bubbling solution or rotating solution with less supersymmetry.

In the corresponding $f(R)$ theory, the solution becomes

$$ds_5^2 = -H^{-1} h dt^2 + H\Big(\frac{dr^2}{h} + r^2 d\Omega_3^2\Big), \qquad F_{(2)} = \sqrt{2}\, dt \wedge dH^{-1},$$
$$h = 1 + g^2 r^2 H^2, \qquad \varphi = \frac{|r|}{\sqrt{r^2+q^2}}, \qquad H = 1 + \frac{q}{r^2}.$$

Since we have

$$H^{-1} h = \frac{r^2 + g^2(r^2+q^2)^2}{r^2+q^2},$$

it follows that the solution describes a wormhole with r runs from $-\infty$ to $+\infty$. The positivity of φ requires that a delta-function matter source at $r = 0$ is needed for supporting this wormhole.

The corresponding $D = 10$ type IIB solution is given by

$$
\begin{aligned}
ds_{10}^2 &= \sqrt{\Delta}\Big(-\frac{r^2 + g^2(r^2+q^2)^2}{r^2+q^2} dt^2 + \frac{(r^2+q^2)dr^2}{r^2+g^2(r^2+q^2)^2} \\
&\qquad + (r^2+q^2)d\Omega_3^2 + g^{-2}d\theta^2\Big) \\
&\quad + \frac{1}{g^2\sqrt{\Delta}}\Big(\tfrac{1}{4}\sin^2\theta\,(\sigma_1^2+\sigma_2^2+(\sigma_3+2A_{(1)})^2) \\
&\qquad + \frac{r^2}{r^2+q^2}\cos^2\theta\, d\phi^2\Big)\,, \\
\Delta &= \cos^2\theta + \frac{r^2}{r^2+q^2}\sin^2\theta\,.
\end{aligned}
$$

It is thus clear that the coordinate r runs from 0 to ∞ in $D = 10$.

Thus

Usual supergravity and its corresponding $f(R)$ theory do not have the same solution space.

The local solutions can be related by some conformal factors that can become singular. However, the global structures do not survive.

In a gravity theory, a solution is not only specified by a local metric, but also by its global structure.

Wormholes emerge naturally in $f(R)$ theory whilst they are more or less impossible in Einstein gravity with matter.

Conclusions

The majority of supergravities, when cast into the $f(R)$ frame, have irrational power of the scalar coupling. Few examples, including all $D = 10$ supergravities have nice integer powers. Another example we find is $\mathcal{N} = 2$, $D = 5$ gauged supergravity with a vector multiplet. Such theories have a natural $f(R)$ description.

The original theory and the $f(R)$ counterpart do not share the same solution space.

The $f(R)$ frame is related to the Kaluza-Klein circle reduction and it implies that the breathing mode is auxiliary. This could hold even with appropriate higher-order curvature terms. If this is true for M-theory, it suggests some special property of the string coupling dilaton field.

Conclusions

The fact that $\mathcal{N} = 1$ $D = 10$ supergravity can be expressed as

$$e^{-1}\mathcal{L}_{10} = \sqrt{-\tfrac{1}{3}RF_{(3)}^2} - \tfrac{1}{4}\alpha'\Big(\mathrm{tr}(F_{(2)}^2) + \mathrm{Riem}^2\Big)$$

suggests that there can exist very unusual types of coupling between the curvature tensors and the matter form fields. In this formulation, there is no string loop coupling constant and the theory is strictly non-perturbative.

Such a construction of supergravities and such types of coupling between gravity and form fields were not considered previously.

Our works seem to raise more questions than answer them.

AdS and Lifshitz Black Holes in Conformal and Einstein-Weyl Gravities

A Quick Review

The most general action of gravity up to quadratic curvature invariants is

$$I = \frac{1}{2\kappa^2}\int \sqrt{-g}\, d^4x (R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2 + \gamma E_{\mathrm{GB}})\,.$$

The theory admits one AdS vacuum with $R_{\mu\nu} = -\Lambda\, g_{\mu\nu}$, and the AdS Schwarzschild black hole.

The linear spectrum consists of a massless graviton, a massive scalar and a ghost-like massive spin-2 mode.

The massive scalar decouples if we set $\beta = -\alpha/3$, and the massive spin-2 mode disappears and it is replaced by the log modes at the critical point $\alpha = -1/2$.

The AdS Schwarzschild black hole has a temperature, but vanishing energy, entropy and free energy, and hence can be viewed as the "thermalized vacuum."

Einstein-Weyl gravity and conformal gravity

When $\beta = -\alpha/3$, and $\gamma = \alpha/2$, the theory is simply Einstein-Weyl gravity, with

$$I = \frac{1}{2\kappa^2} \int \sqrt{-g}\, d^4x (R - 2\Lambda + \tfrac{1}{2}\alpha |\text{Weyl}|^2)\,.$$

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - 2\alpha(2\nabla^\rho \nabla^\sigma + R^{\rho\sigma}) C_{\mu\rho\sigma\nu} = 0\,.$$

Conformal Gravity is like to take $\alpha \to \infty$:

$$I = \frac{1}{2\kappa^2} \int \sqrt{-g}\, d^4x (\tfrac{1}{2}\alpha |\text{Weyl}|^2)\,.$$

$$-2\alpha(2\nabla^\rho \nabla^\sigma + R^{\rho\sigma}) C_{\mu\rho\sigma\nu} = 0\,.$$

Lifshitz vacua

$$ds^2 = \frac{dr^2}{\sigma\, r^2} - r^{2z} dt^2 + r^2(dx^2 + dy^2)\,,$$

with

$$\alpha = \frac{z^2 + 2z + 3}{4z(z-4)}\,.$$

The Existence of such a solution may not be so surprising since the previously-known Liftshitz solutions are constructed with a massive vector:

$$e^{-1}\mathcal{L} = R - 2\Lambda - \tfrac{1}{4}F_{(2)}^2 - \tfrac{1}{2}c^2 A_{(1)}^2\,.$$

$$ds^2 = \ell^2\Big(- r^{2z}dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{r^2}\Big)\,, \qquad A_{(1)} = q r^z dt\,,$$

$$c^2 = \frac{2z}{\ell^2}\,, \qquad \Lambda = -\frac{z^2 + z + 4}{2\ell^2}\,, \qquad q^2 = \frac{2\ell^2(z-1)}{z}\,.$$

Thus it is natural expect Lifshitz solutions in theories with massive spin-2 modes.

Schrödinger vacua

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2(-2dtdx + dy^2)\,,$$

with

$$\alpha = \frac{1}{2z(1-2z)}\,.$$

Focus of the talk

Are there new spherically (T^2, H^2)-symmetric black holes beyond Schwarzschild-AdS black holes?

Are there new black holes that are asymptotic to the Lifshitz vacua?

Ansatz

$$ds^2 = -a(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{2,k}\,.$$

Equations of motion:

$$a'' = \frac{r^2 f a'^2 + 4a^2(k + 6r^2 - f - rf') - raa'(4f + rf')}{2r^2 af}\,.$$

Note that the trace equation is independent of α.

$$\begin{aligned} f'' \;=\; & \frac{1}{2r^2a^2 f(ra' - 2a)}\Big(\frac{4r^2a^2}{\alpha}(a(k + 3r^2 - f) - rfa') + r^3 f^2 a'^3 + 2r^2a^2 \\ & -r^2afa'^2(3f + rf') - a^3(48r^4 - 16r^2 f + 8f^2 - 24r^3 f' + 4rff' + 3 \\ & -4ka^3(4r^2 - 2f - rf')\Big)\,. \end{aligned}$$

We cannot solve these two equations.

Still some properties can be extracted.

Noether charge for $k = 0$

$$ds^2 = -a(r)\,dt^2 + \frac{dr^2}{f(r)} + b(r)(dx^2 + dy^2),.$$

The metric is invariant under the scaling $a \to a/\lambda^2$, $b \to b\lambda$, $t \to \lambda$, $(x, y) \to (x, y)/\sqrt{\lambda}$. Thus the curvature polynomial invariants are invariant under this scaling. Furthermore $\sqrt{-g}$ is invariant. Thus the Lagrangian has this additional global symmetry. Gauging this global symmetry gives rise to Noether current and the charge:

$$\begin{aligned}\lambda \;=\; & \frac{1}{\sqrt{a^3 f}\,(ra' - 2a)}\Big(2ra(18ra^2 - 10a^2 f - 2rafa' - r^2 fa'^2) \\ & -\alpha(4ra - fa' - af')(36r^2a^2 - 8a^2 f - rafa' - 2r^2 fa'^2 - 9ra^2 f')\Big)\,.\end{aligned}$$

Physical meaning of the Noether charge

In two-derivative gravities, the Noether charge has been shown with examples to have the following properties

$$E = -\frac{\lambda\omega_2}{16\pi\,(z+2)} = \frac{2}{(z+2)}T\,S\,.$$

This is not longer in general true for solutions in higher-derivative gravities.

Solutions in conformal gravity

Without explicit solutions, it is very difficult to demonstrate whether new black holes with massive spin-2 hair could emerge.

Fortunately, conformal gravity provides an answer. Up to an conformal factor, the most general spherically symmetric black hole was known, given by

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{2,k}^2 \,, \qquad f = br^2 + \frac{c^2 - k^2}{3d} r + c + \frac{d}{r} \,,$$

The constant b is the cosmological constant of the asymptotic AdS. If $c = k$, the solution becomes the usual Schwarzschild-AdS black hole. Thus it has one extra parameter, associated with the massive spin-2 hair.

Is this a black hole?

Surprisingly, the global structure of this solution was not studied previously.

Thermodynamics

Schwarzschild-AdS: $f = -\frac{1}{3}\Lambda\rho^2 + k - 2M/\rho$.

$$M = \tfrac{1}{6}\rho_+(3k - \Lambda\rho_+^2)\,.$$

$$T = \frac{k - \Lambda\rho_+^2}{4\pi\rho_+}\,, \qquad S = \tfrac{1}{6}\alpha(3k - \Lambda\rho_+^2)\omega_2\,,$$

$$F = -\frac{\alpha\omega_2}{32\pi}\int_{r_+}^{\infty} r^2 dr\,|\mathrm{Weyl}|^2 = -\frac{\alpha(3k - \Lambda\rho_+^2)^2\omega_2}{72\pi\rho_+}\,.$$

$$E = \frac{\alpha\Lambda\rho_+(-3k + \Lambda\rho_+^2)\omega_2}{36\pi} = \frac{\alpha(-\Lambda)\omega_2}{6\pi}\,M\,.$$

Thermodynamic relations

$$dE = TdS + \Theta\, d\Lambda\,, \qquad F = E - TS\,, \qquad \Theta = -\frac{\alpha\rho_+(3k - \Lambda\rho_+^2)\omega_2}{72\pi}\,.$$

New Smarr relation:

$$E = 2\Theta\Lambda$$

Thermodynamics for new black holes

$$f = r^2 + \frac{c^2 - k^2}{3d}\, r + c + \frac{d}{r}\,,$$

Set $d = -\tilde{d} r_+$, we have

$$r_+^2 = -c + \frac{c^2 - k^2}{3\tilde{d}} + \tilde{d} > 0\,.$$

$$T = \frac{(3\tilde{d} - c)^2 - k^2}{12\pi r_+ \tilde{d}}\,, \qquad S = \tfrac{1}{6}\alpha(k + 3\tilde{d} - c)\omega_2\,.$$

$$F = -\frac{\alpha\omega_2\Big((c-k)^2 - 3(c-k)\tilde{d} + 3\tilde{d}^2\Big)}{24\pi r_+}\,.$$

Additional thermo qunatities?

How to handle the extra parameter? One might expect the following

$$dE = TdS + XdY\,, \qquad F = E - TS - XY$$

In this relation, we have three unknowns (E, X, Y). We need an independent way of calculating these quantities.

All the previously-known methods (Deser-Tekin or AMD) of calculating the energy yield divergent result. We adopt a new method, and find the energy.

Thermalized vacuum

It is instructive to look at the solution with $d = 0$ first:

$$f = r^2 + \Xi r + k\,.$$

This solution has a curvature singularity at $r = 0$, which can be shielded by an horizon at $r = r_0$ provided that Ξ is chosen so that $\Xi^2 \geq 4k$. The temperature is given by

$$T_0 = \frac{r_0^2 - k}{4\pi r_0}\,.$$

However, we find that the entropy and free energy both vanish, suggesting that the energy should vanish also. Thus the solution can be viewed as a "thermalized vacuum." In a Deser-Tekin or AMD calculation, this thermalized vacuum will generate a divergence in the evaluation of the mass, and it should be subtracted.

The thermalized vacuum is locally conformal to the de Sitter space and the horizon is mapped to the cosmic horizon.

Energy of new black holes

Adopting a new method, we find that

$$E = \frac{\alpha\,\omega_2}{4\pi}(-d + m)\,,$$

where

$$m \equiv \frac{(c-k)(c^2-k^2)}{18d}\,.$$

Note that when $c = k$, it reproduces the energy for the Schwarzschild-AdS black hole. When $d = 0$, it is necessary that $c \to k$ with $\Xi = (c^2 - k^2)/(3d)$ held fixed. In this limit, the quantity m vanishes, and hence we see that the thermalized vacuum indeed has zero energy.

Thermodynamical relations

It turns out that

$$F = E - TS\,.$$

But clearly, $dE \neq TdS$. How we introduce the new (X, Y). It is natural to think that the thermalized vacuum, it is a true vacuum, can be held fixed when we vary thermodynamical variables, and indeed, if we hold Ξ fixed, we have $dE = TdS$. This leads to

$$dE = TdS + \Psi d\Xi\,, \qquad \Psi = \frac{\alpha\omega_2(c-k)}{24\pi}\,.$$

The most general thermodynamical relations

$$f = -\tfrac{1}{3}\Lambda r^2 + \Xi\, r + c + \frac{d}{r}\,, \qquad \text{with} \qquad 3\Xi\, d = c^2 - k^2\,.$$

Letting r_+ be the radius of the outer horizon, and defining $d = -r_+\tilde{d}$, we have

$$T = \frac{(3\tilde{d} - c)^2 - k^2}{12\pi r_+ \tilde{d}}\,, \qquad S = \tfrac{1}{6}\alpha\omega_2(k + 3\tilde{d} - c)\,,$$

$$\Psi = \frac{\alpha\omega_2(c - k)}{24\pi}\,, \qquad \Theta = \frac{\alpha\omega_2 d}{24\pi}\,,$$

$$F = -\frac{\alpha\omega_2\Big((c-k)^2 - 3(c-k)\tilde{d} + 3\tilde{d}^2\Big)}{24\pi r_+}\,, \qquad E = 2\Theta\,\Lambda + \Psi\,\Xi\,.$$

These thermodynamic quantities satisfy the relations

$$dE = TdS + \Theta\, d\Lambda + \Psi\, d\Xi\,, \qquad F = E - T\,S\,.$$

The entropy of the general black hole can be decomposed as

$$S = \tfrac{1}{2}\alpha w_2\, k + \tfrac{1}{6}\alpha\,(-\Lambda)A + 8\pi\Psi + \tfrac{1}{2}\alpha\omega_2\Xi r_+\,,$$

$z = 4, 0$ **Lifshitz(-like) black holes**

$$ds^2 = -r^8 f dt^2 + \frac{4dr^2}{r^2 f} + r^2 d\Omega^2_{2,k}\,, \qquad f = 1 + \frac{c}{r^2} + \frac{c^2 - k^2}{3r^4} + \frac{d}{r^6}\,.$$

$$ds^2 = -f dt^2 + \frac{4dr^2}{r^2 f} + r^2 d\Omega^2_{2,k}\,, \qquad f = 1 + \frac{c}{r^2} + \frac{c^2 - k^2}{3r^4}\,.$$

Thermodynamics can also be worked out for these two solutions.

AdS and Lifshitz black holes in Einstein-Weyl gravity

Near-horizon structure:

$$
\begin{aligned}
a(r) &= (r - r_0) + a_2\,(r - r_0)^2 + a_3\,(r - r_0)^3 + a_4\,(r - r_0)^4 + \cdots\,, \\
f(r) &= f_1(r - r_0) + f_2\,(r - r_0)^2 + f_3\,(r - r_0)^3 + f_4\,(r - r_0)^4 + \cdots\,.
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= \frac{3r_0^3 + 5f_1\,r_0^2 - 2f_1^2\,r_0 + k\,r_0 + k\,f_1}{f_1^2\,r_0^2} - \frac{(3r_0^2 - f_1\,r_0 + k)}{4\alpha\,f_1^2\,r_0}\,, \\
f_2 &= \frac{(f_1 - 3r_0)(3r_0^2 - 2f_1\,r_0 + k)}{f_1\,r_0^2} + \frac{3(3r_0^2 - f_1\,r_0 + k)}{4\alpha\,f_1\,r_0}\,.
\end{aligned}
$$

Schwarzschild-AdS black hole $f_1 = 3r_0 + k/r_0$.

Numerical analysis: an example

We find that if α lies in the region $-\infty < \alpha < -\frac{1}{2}$, then defining

$$f_1 = 3r_0 + k/r_0 + \delta\,,$$

there is a range for δ, with $\delta_- < \delta < \delta_+$, for which the numerical solutions indicate the occurrence of asymptotically AdS black holes. The lower limit δ_- is negative, while the upper limit δ_+ is positive. If the value of δ is fine-tuned to be *equal* to δ_- or δ_+, then the asymptotic behaviour of the black hole changes from AdS to Lifshitz. The value of z in the asymptotically Lifshitz case is given previously. If the parameter δ is chosen to lie outside the range $\delta_- \le \delta \le \delta_+$, then the numerical analysis indicates that the solution becomes singular.

As an example, let us consider $\alpha = -\frac{11}{16}$, which implies that there should exist asymptotically Lifshitz solutions with $z = 2$. Taking $k = 0$ and choosing $r_0 = 10$, we find that the limiting values for δ are

$$\delta_- \approx -11.596956988\,, \qquad \delta_+ \approx 62.826397763\,.$$