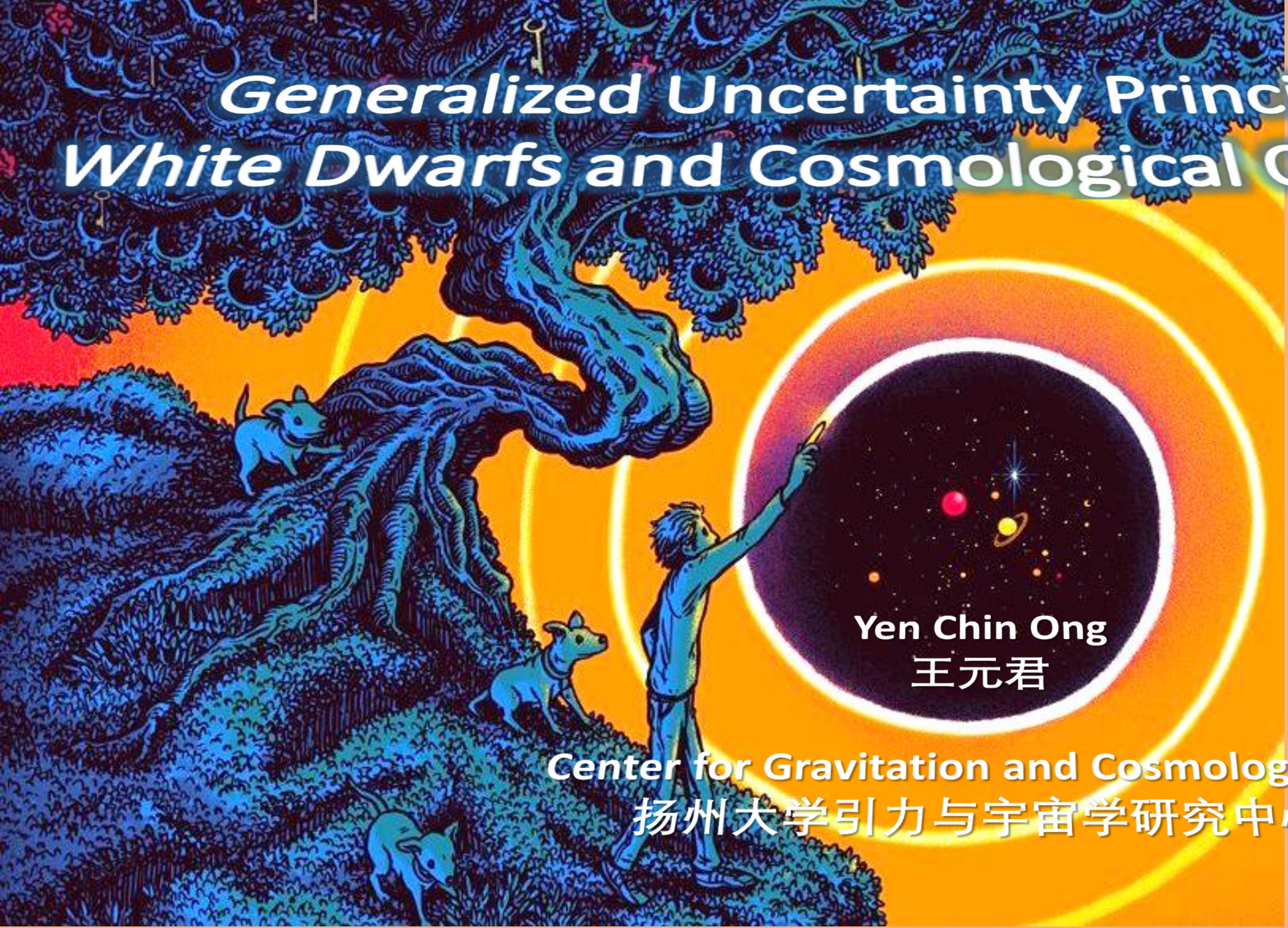
Generalized Uncertainty Principle, White Dwarfs and Cosmological Constant



Yen Chin Ong
王元君

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扬州大学引力与宇宙学研究中心

Talk Based On

- Yen Chin Ong, "Generalized Uncertainty Principle, Black Holes, and White Dwarfs: A Tale of Two Infinities", JCAP 09 (2018) 015, [arXiv:1804.05176 [gr-qc]].
- Yen Chin Ong, "An Effective Black Hole Remnant via Infinite Evaporation Time Due to Generalized Uncertainty Principle", JHEP 10 (2018) 195, [arXiv:1806.03691 [gr-qc]].
- Yen Chin Ong, Yuan Yao, "Generalized Uncertainty Principle and White Dwarfs Redux: How Cosmological Constant Protects Chandrasekhar Limit", Phys. Rev. D 98 (2018) 126018 [arXiv:1809.06348 [gr-qc]].
- Yuan Yao, Meng-Shi Hou, Yen Chin Ong, "A Complementary Third Law for Black Hole Thermodynamics", [arXiv:1812.03136 [gr-qc]].

Some Words On Rigor

"When one gets to the mathematical theories which are at the basis of quantum mechanics, one realizes that the attitude of certain physicists in the handling of these theories truly borders on the delirium. [...] One has to wonder what remains in the mind of a student who has absorbed this unbelievable accumulation of nonsense, a real gibberish! It should be to believe that today's physicists are only at ease in the vagueness, the obscure and the contradictory."

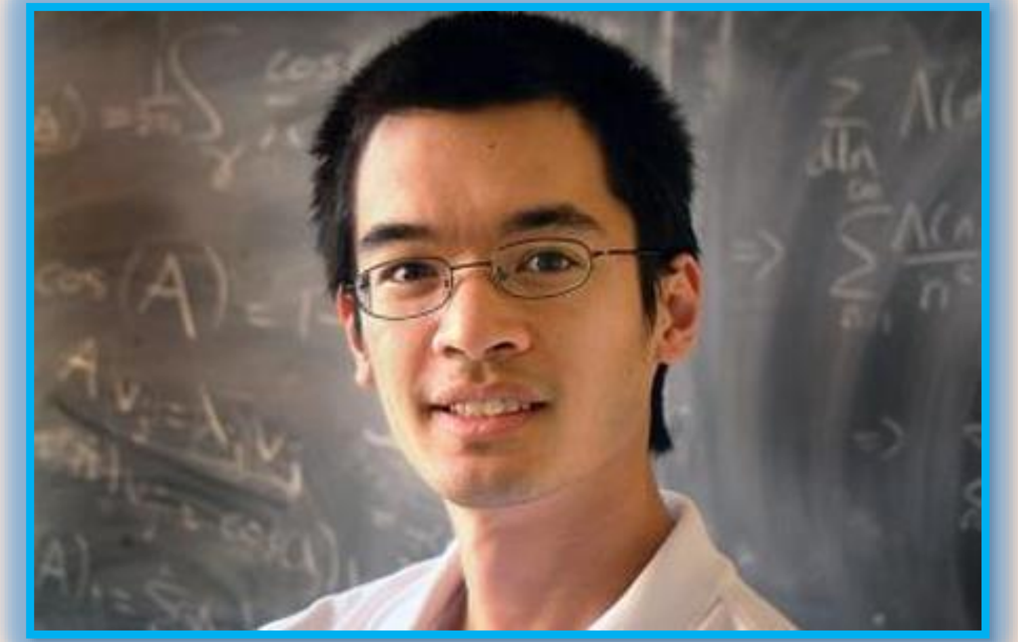
Jean Alexandre Eugène Dieudonné

J. DIEUDONNÉ: "*De la communication entre mathématiciens et physiciens*", dans "La Pensée Physique Contemporaine", S. Diner, D. Fargue et G. Lochak, éds., (Éditions A.Fresnel, Hiersac 1982) .

A Somewhat Different View

"I would like to discuss the *uncertainty principle*, that describes the dual relationship between physical space and frequency space. There are various concrete formalisations of this principle, most famously the Heisenberg uncertainty principle and the Hardy uncertainty principle – but in many situations, it is the *heuristic* formulation of the principle that is more useful and insightful than any particular rigorous theorem that attempts to capture that principle."
– Terence Chi-Sen Tao

[http://terrytao.wordpress.com/2010/06/25/the-uncertainty-principle/]

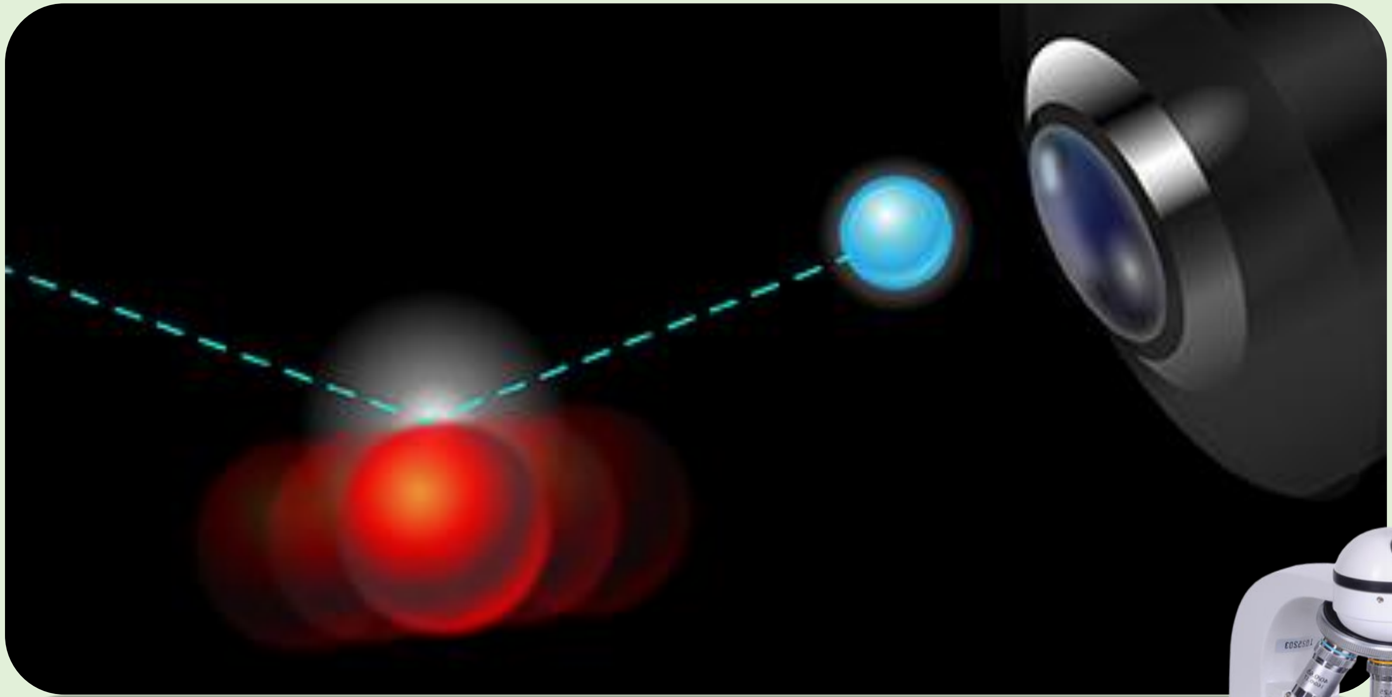


Field Medal (2006) "for his contributions to partial differential equations, combinatorics, harmonic analysis and additive number theory"

Generalized Uncertainty Principle

$$\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \frac{\alpha L_p^2 \Delta p^2}{\hbar}\right]$$

Heisenberg's Microscope



$$\Delta x \Delta p \sim \hbar$$

Heisenberg's Microscope with Gravitational Correction

Ronald J. Adler, "Six Easy Roads to the Planck Scale", Am. J. Phys. 78 (2010) 925, arXiv:1001.1205 [gr-qc].

Photon energy $E = h\nu$, so effective mass $M_{\mathrm{eff}} = h\nu / c^2 = h / c\lambda$ exerts force to accelerate particle: additional fuzziness!

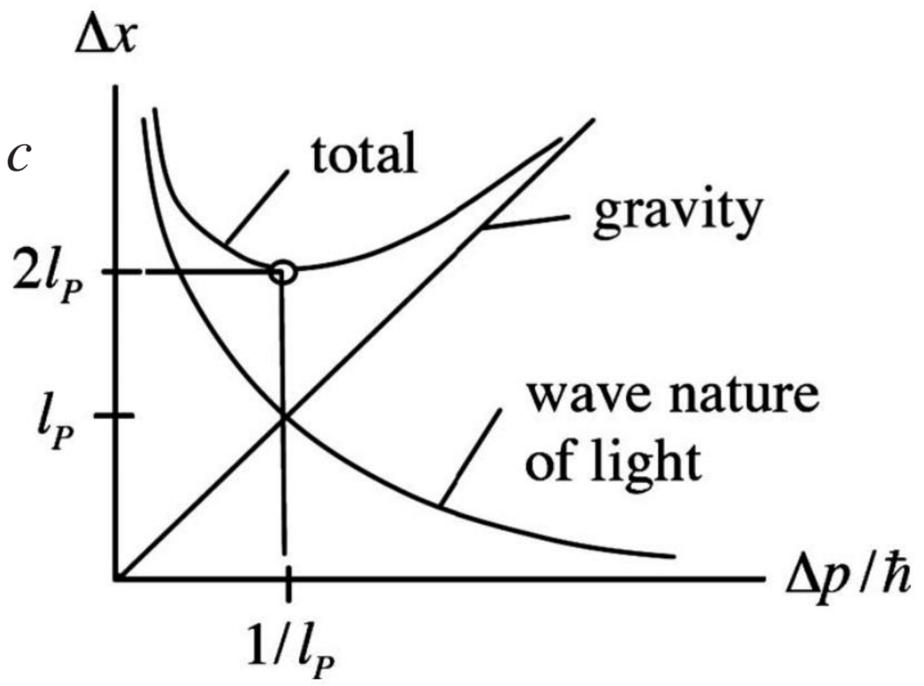
$$\Delta a_g \approx G M_{\mathrm{eff}} / r_{\mathrm{eff}}^2 = G(h/\lambda c)/r_{\mathrm{eff}}^2,$$

$$\Delta x_g \approx \Delta a_g t_{\mathrm{eff}}^2 \approx G(h/\lambda c)(t_{\mathrm{eff}}^2/r_{\mathrm{eff}}^2),$$

$r_{\mathrm{eff}}/t_{\mathrm{eff}} \approx c$

$$\Delta x_g \approx Gh/\lambda c^3 \approx (G\hbar/c^3)/\lambda = \ell_P^2/\lambda.$$

$$\Delta x \approx \left(\frac{\hbar}{\Delta p}\right) + \ell_P^2 \left(\frac{\Delta p}{\hbar}\right).$$



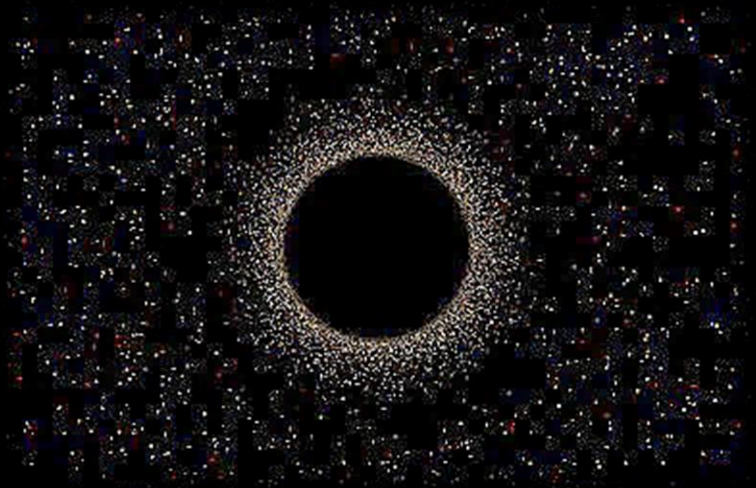
Also, black hole formation?

Heuristic Derivation of Bekenstein-Hawking Temperature of Black Holes via Uncertainty Principle

The Schwarzschild metric is

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

with the event horizon at $r_+ = 2GM/c^2$.



Heisenberg's Uncertainty Principle then yields, if we identify $\Delta x \sim r_+$,
the following approximation

$$\boxed{\Delta p \approx \frac{\hbar}{2\Delta x} \approx \frac{\hbar}{2r_+} = \frac{\hbar c^2}{4GM}}$$

Suppose now we interpret the quantum uncertainty in the kinetic energy of the emitted particles as due to thermal agitation, the uncertainty in the energy of photons emitted during Hawking evaporation is identified with

$$\Delta E = \Delta p c \approx \frac{\hbar c^3}{4GM} \sim k_B T \Rightarrow T = 2\pi \left[\frac{\hbar c^3}{8\pi k_B GM}\right]$$

where

$$\boxed{T_{BH} = \frac{\hbar c^3}{8\pi k_B GM}}$$

is the Bekenstein-Hawking temperature of the Schwarzschild black hole.

Hawking Radiation Does Not Originate from Near Horizon!

de Broglie wavelength：

$$\lambda = \frac{2\pi\hbar}{p}$$

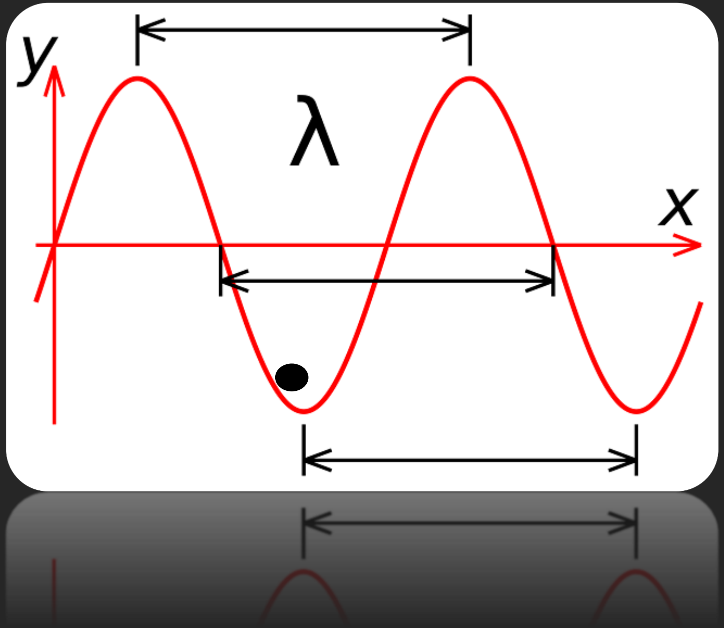
Quantum Mechanics

$$E = pc$$

thermodynamics

$$E = k_B T$$

Wavelength of Hawking Particle：



$$\lambda_T = \frac{2\pi\hbar}{k_B T/c} = \frac{2\pi\hbar c}{k_B} \cdot \frac{8\pi k_B G M}{\hbar c^3}$$
$$= 8\pi^2 \cdot \frac{2GM}{c^2} = 8\pi^2 r_h \approx 79\, r_h$$

S. B. Giddings, "Hawking radiation, the Stefan Boltzmann law, and unitarization," Phys. Lett. B754 (2016) 39, 1511.08221.

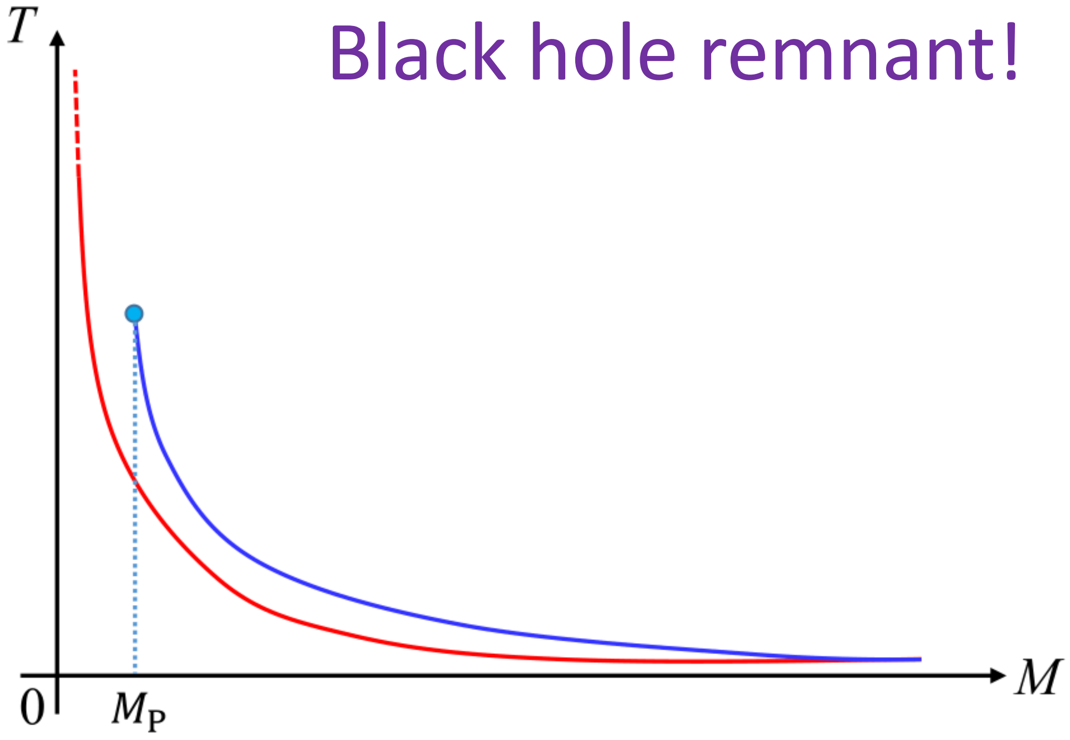
Hawking Evaporation with GUP

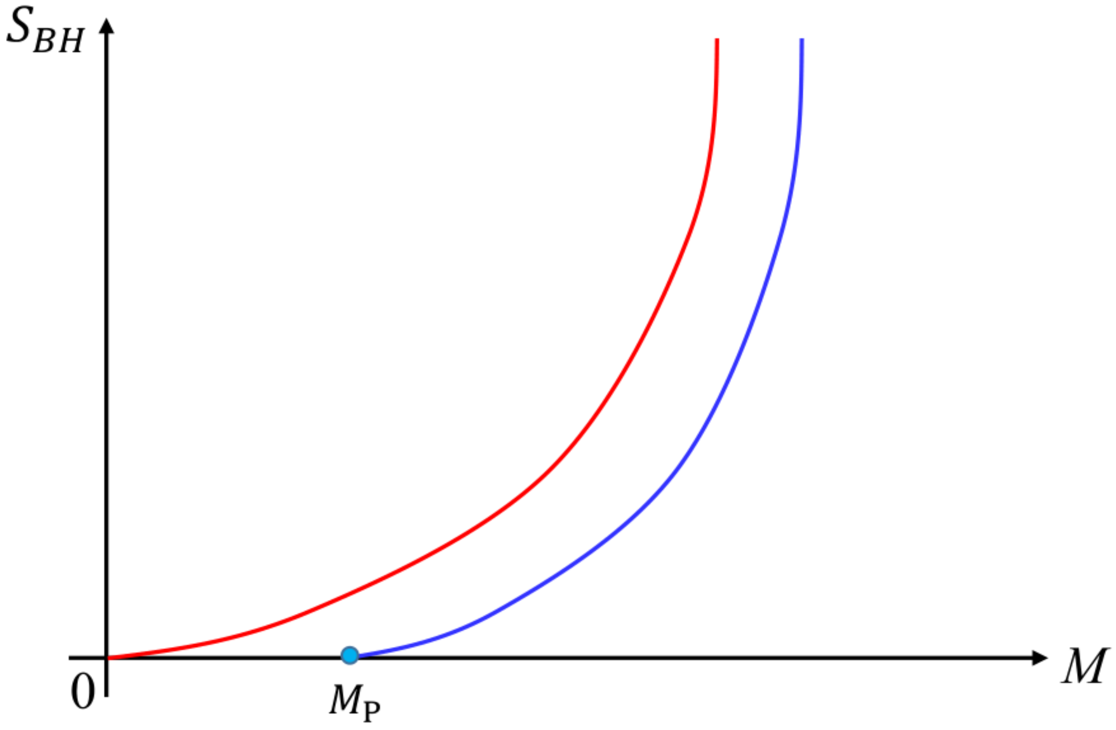
$$T = \frac{Mc^2}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha\hbar c}{GM^2}}\right)$$

Ronald J. Adler, Pisin Chen, David I. Santiago, "The Generalized Uncertainty Principle and Black Hole Remnants", Gen. Rel. Grav. 33 (2001) 2101, arXiv:gr-qc/0106080

$$\begin{aligned} S[\alpha] &= \int \frac{1}{T}\mathrm{d}M \\ &= 2\pi\left[M^2 + M\sqrt{M^2-\alpha} - \alpha\ln(M + \sqrt{M^2-\alpha})\right] \\ &\quad + \text{const.} \end{aligned}$$

Black hole remnant!

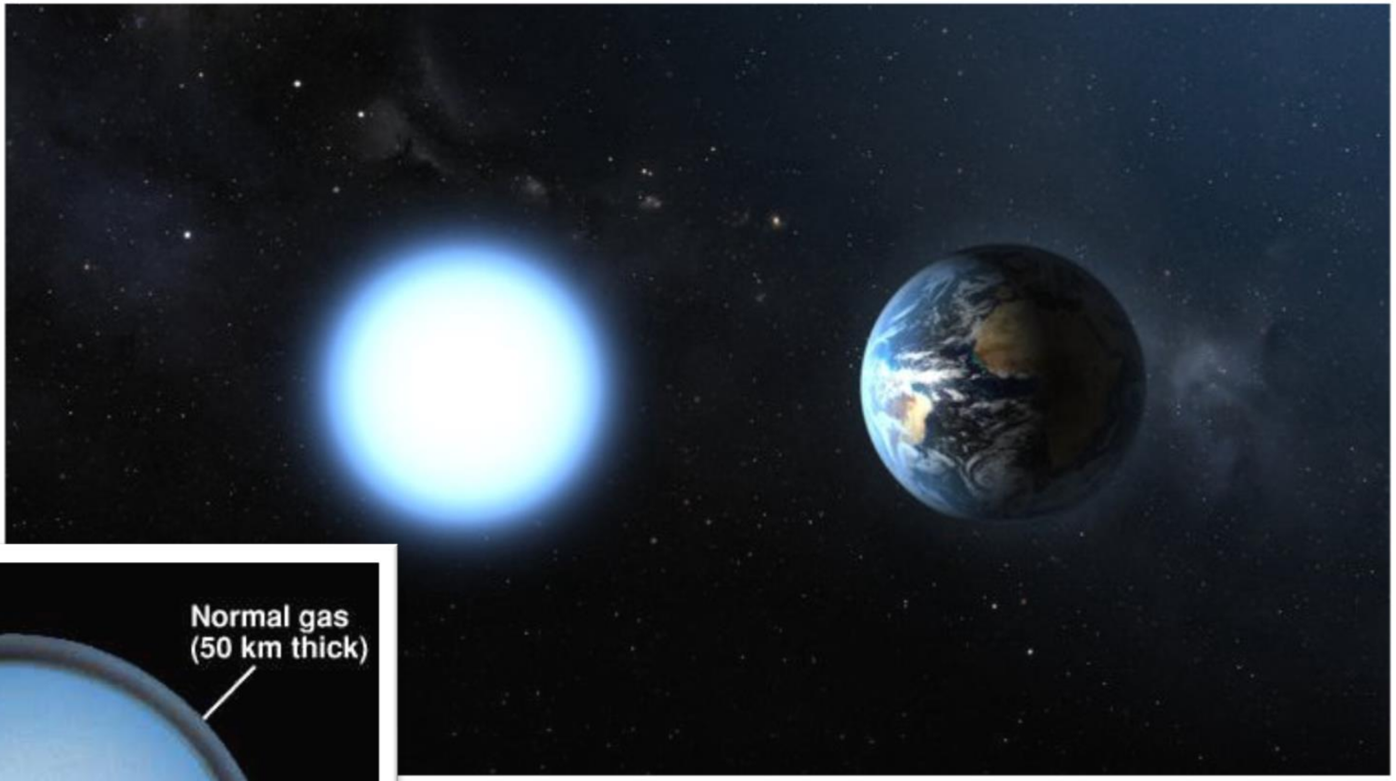


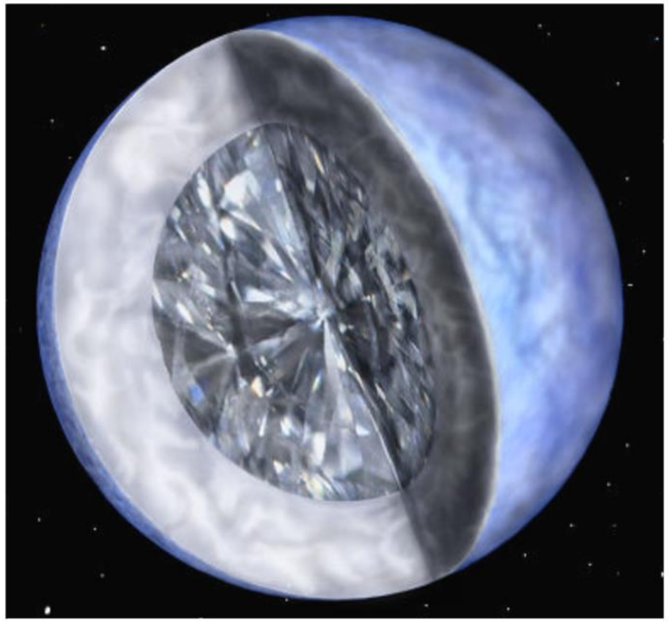


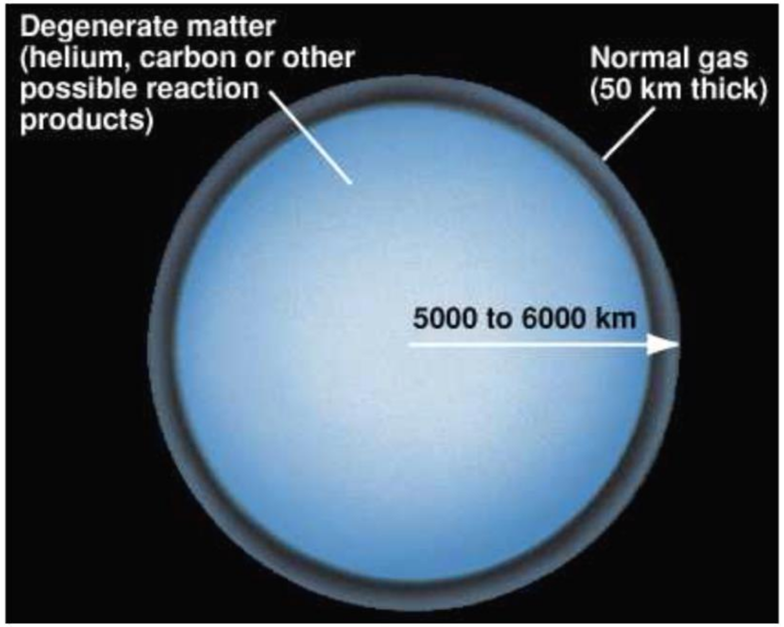
Let us start with white dwarfs

End stage of solar-mass stars

Supported by electron degenerative pressure







Heuristic Derivation of White Dwarf Properties

Textbook material:

Total kinetic energy of a non-relativistic white dwarf

Number of electron

$$E_k = \frac{N\Delta p^2}{2m_e} \sim \frac{N\hbar^2}{(\Delta x)^2 2m_e}$$

Assumption:
"electron star"!

Number density, $n=N/V$

$$\Delta x \sim n^{-\frac{1}{3}} = \left(\frac{V}{N}\right)^{1/3}$$

$$\Delta x \Delta p \sim \hbar$$

$$E_k = \frac{N\hbar^2 n^{\frac{2}{3}}}{2m_e} = \frac{N\hbar^2}{2m_e}\left(M^{\frac{2}{3}} m_e^{-\frac{2}{3}} R^{-2}\right) = \frac{M^{\frac{5}{3}}\hbar^2}{2m_e^{\frac{8}{3}} R^2}.$$

Heuristic Derivation of White Dwarf Properties

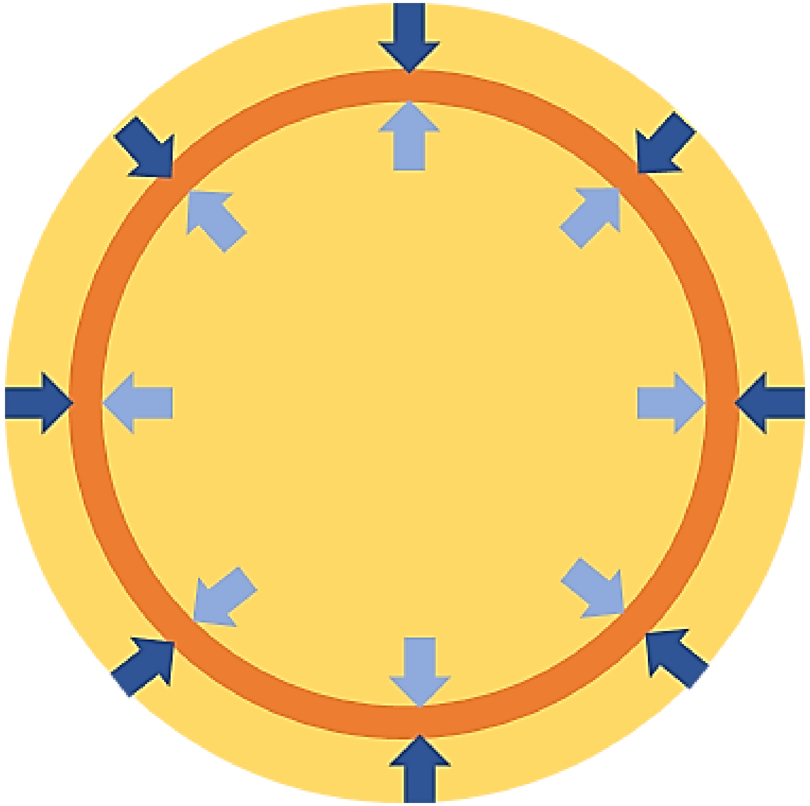
To withstand gravitational collapse, we must balance the kinetic energy with gravitational binding energy

$$|E_g| \sim \frac{GM^2}{R} = E_k$$

$$E_k = \frac{N\hbar^2 n^{\frac{2}{3}}}{2m_e} = \frac{N\hbar^2}{2m_e}\left(M^{\frac{2}{3}} m_e^{-\frac{2}{3}} R^{-2}\right) = \frac{M^{\frac{5}{3}}\hbar^2}{2m_e^{\frac{8}{3}} R^2}.$$

$$\boxed{R \sim \frac{\hbar^2}{2m_e^{\frac{8}{3}} G M^{\frac{1}{3}}}.}$$

Degenerative matter



(Ultra)Relativistic Case

$$E_k = N(\gamma - 1)m_e c^2, \quad p = \gamma m_e v.$$

$$v \sim c$$

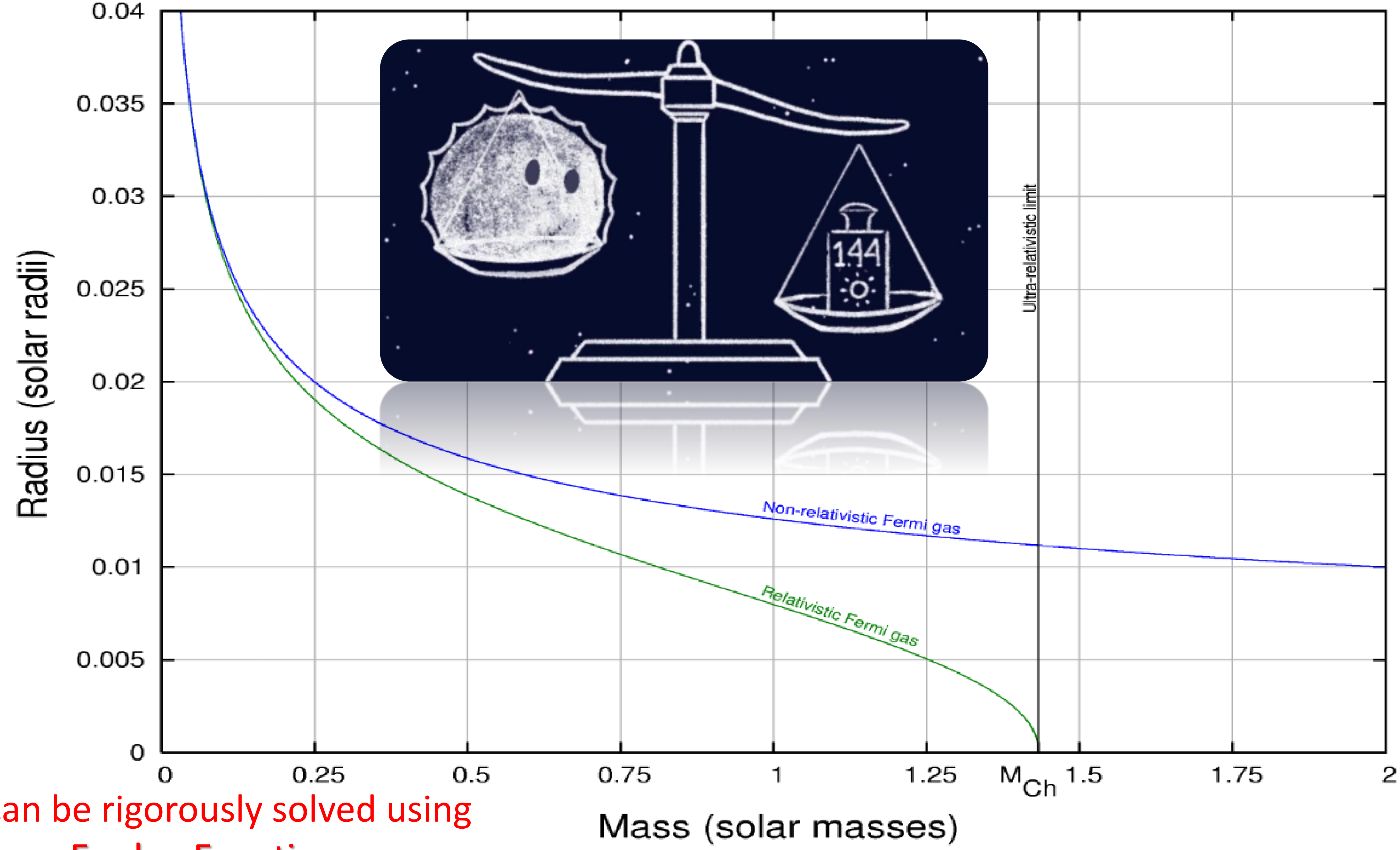
$$\Delta p c - m_e c^2 \sim \frac{\hbar c}{\Delta x} - m_e c^2 \sim \left(\frac{M}{m_e V}\right)^{\frac{1}{3}} \hbar c - m_e c^2.$$

$$\frac{GM^2}{R} \sim N\left[\left(\frac{M}{m_e V}\right)^{\frac{1}{3}} \hbar c\right] \sim \frac{M^{\frac{1}{3}} \hbar c}{m_e^{\frac{4}{3}} R}.$$

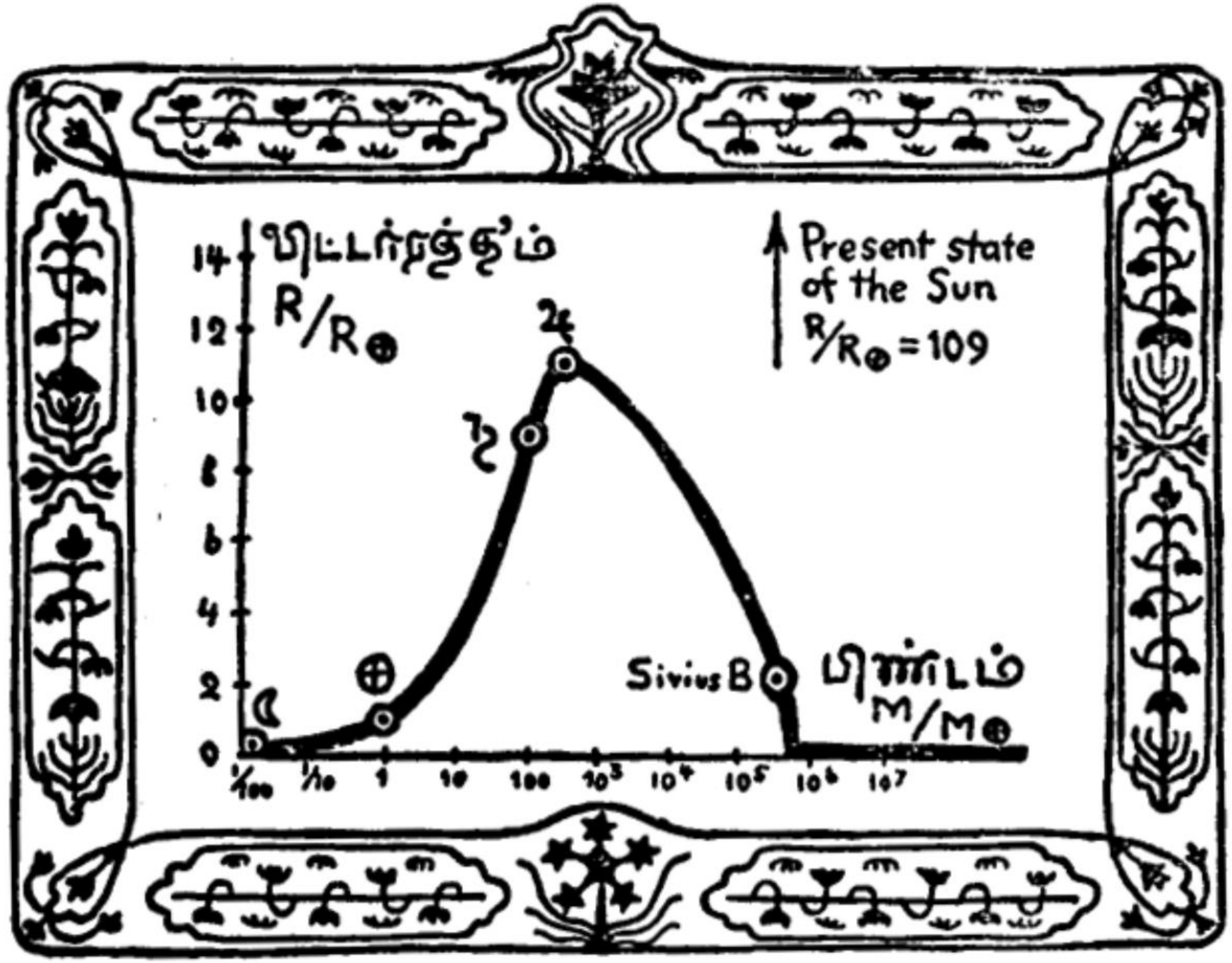
$$\boxed{M_{\mathrm{Ch}} \sim \frac{1}{m_e^2}\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}.}$$

Chandrasekhar limit
/Chandrasekhar mass
(1930)
1983 Nobel Prize in Physics

Subrahmanyan
Chandrasekhar
(1910-1995)

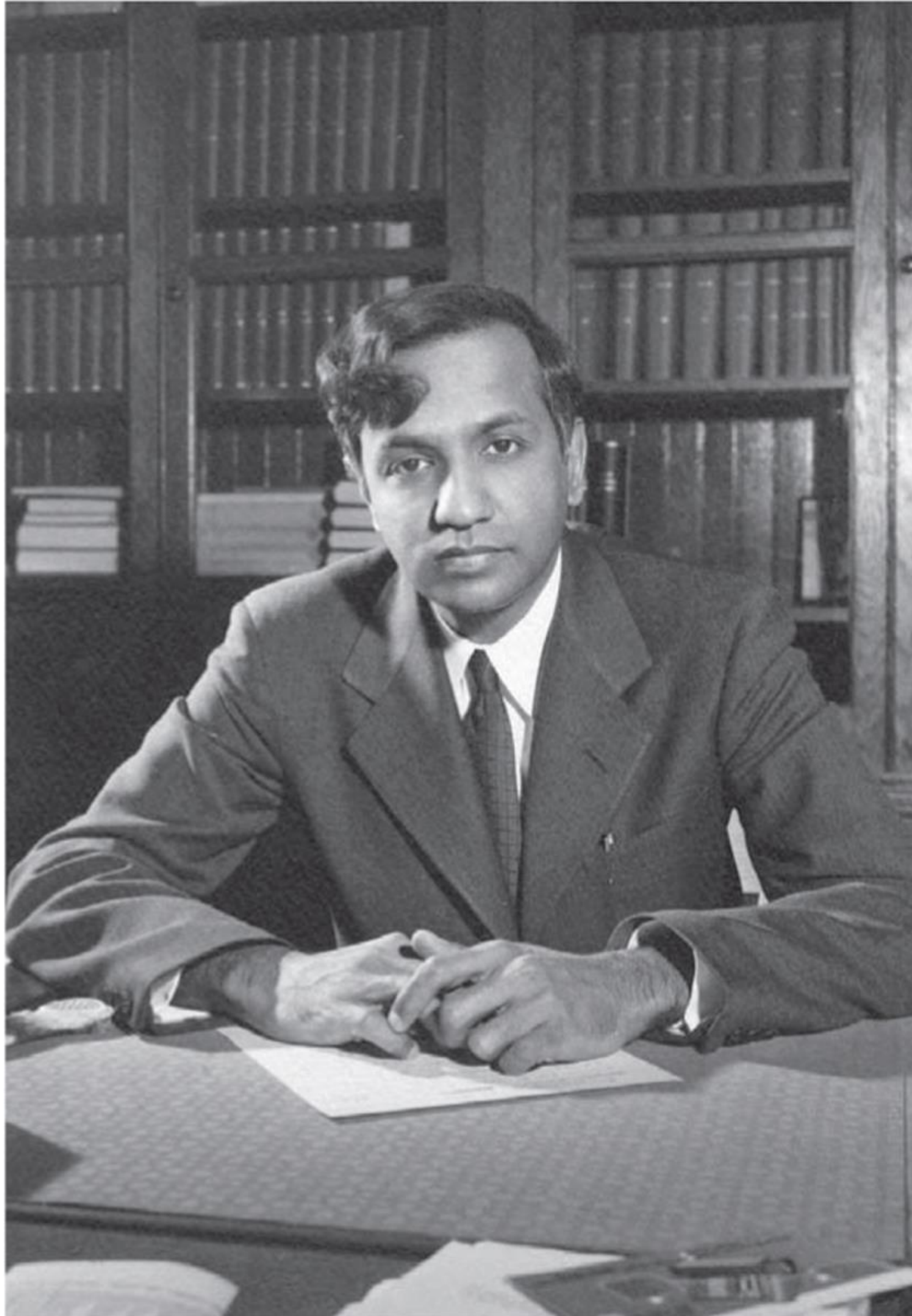


Can be rigorously solved using
Lane-Emden Equation



Ill. 62. The relationship between the radii and the masses of cold stellar bodies, according to the calculations of the Indian astrophysicist S. Chandrasekhar. The symbols ☽, ⊕, ♄, and ♃ respectively represent the Moon, the Earth, Saturn, and Jupiter. Note that for masses greater than 460,000 times the mass of the Earth, the radius becomes zero! The words for mass and radius are in Dr. Chandrasekhar's original Tamil.

(Chandrasekhar's original diagram, from *A Star Called the Sun* by George Gamow.)



White Dwarfs with GUP

$$\Delta x \Delta p \sim \hbar + \alpha L_p^2 \Delta p^2 / \hbar \qquad \Rightarrow \qquad \Delta p \sim \frac{\hbar \Delta x}{2\alpha L_p^2}\left[1 \pm \sqrt{1 - \frac{4\alpha L_p^2}{\Delta x^2}}\right]$$

Non-relativistic case:

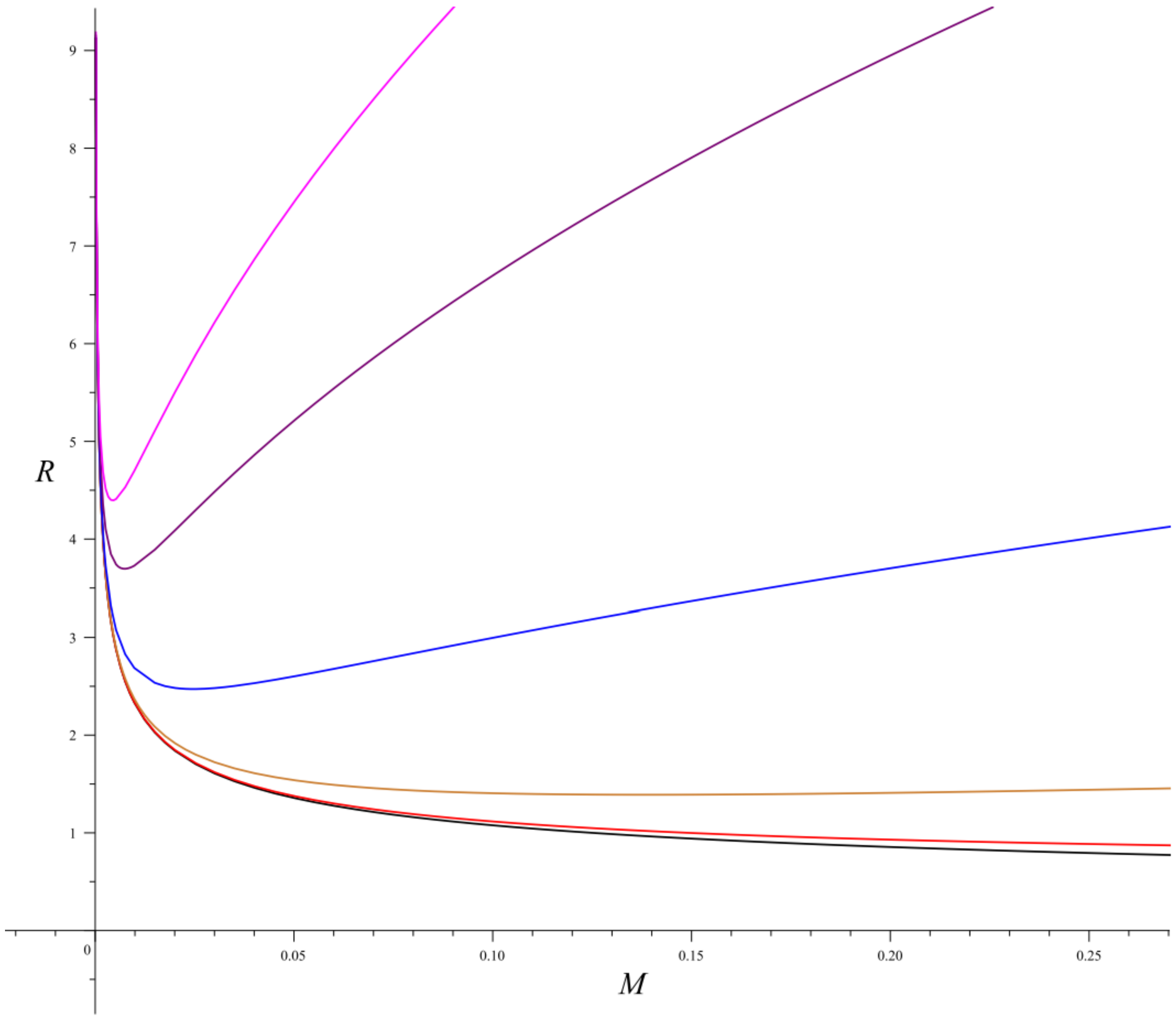
$$M^{\frac{5}{3}} \sim \frac{\hbar^2}{8 G m_e^{\frac{4}{3}} \alpha^2 L_p^4} R^3 \left(1 \pm \sqrt{1 - \frac{4\alpha L_p^2 M^{\frac{2}{3}}}{m_e^{\frac{2}{3}} R^2}}\right)^2$$

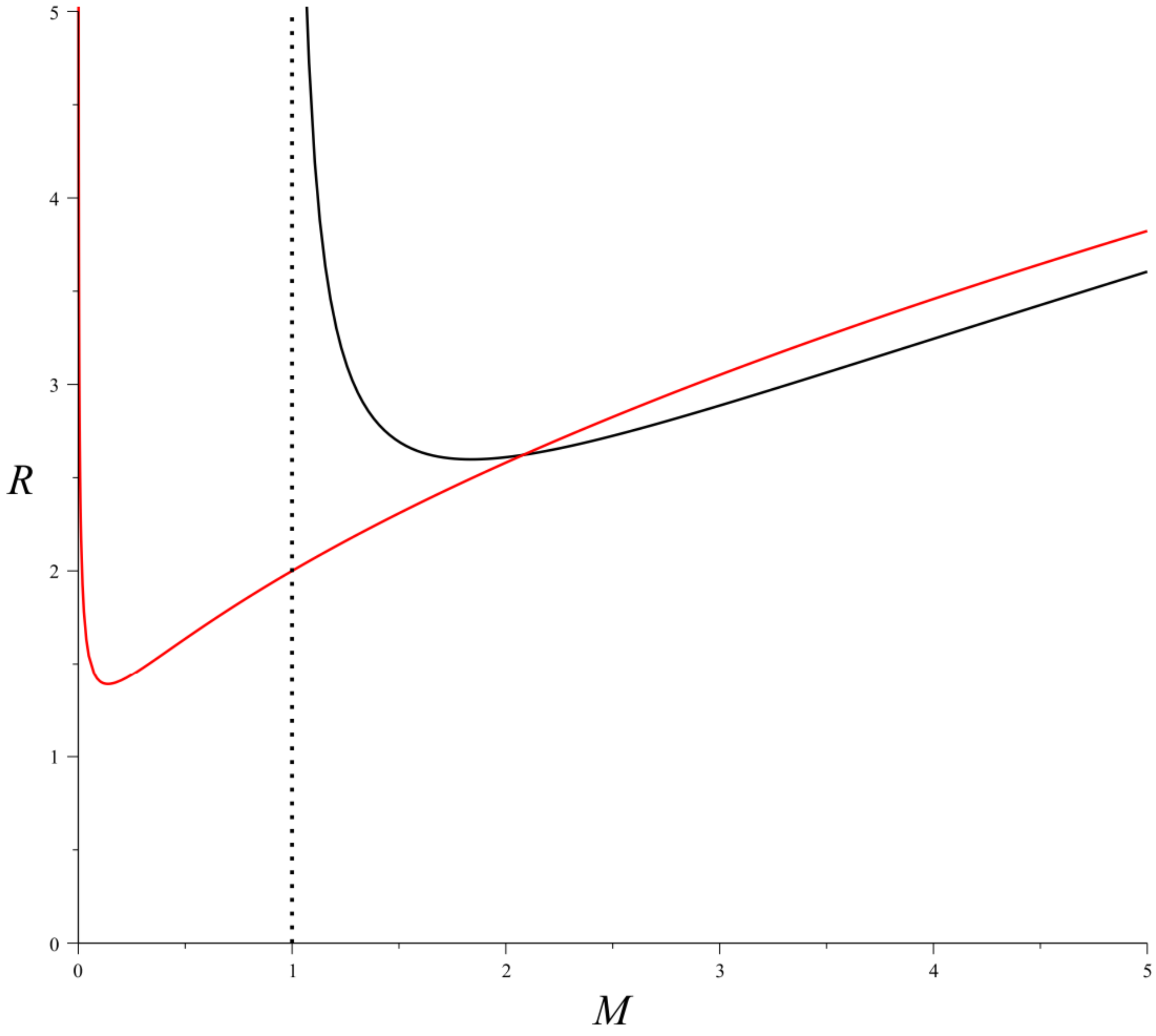
Function Property

A bounce eventually happens,

no matter how small α is!

$$\Delta x \Delta p \sim \hbar + \alpha L_p^2 \Delta p^2 / \hbar$$





Both non-relativistic and (ultra)relativistic case diverge at large M.

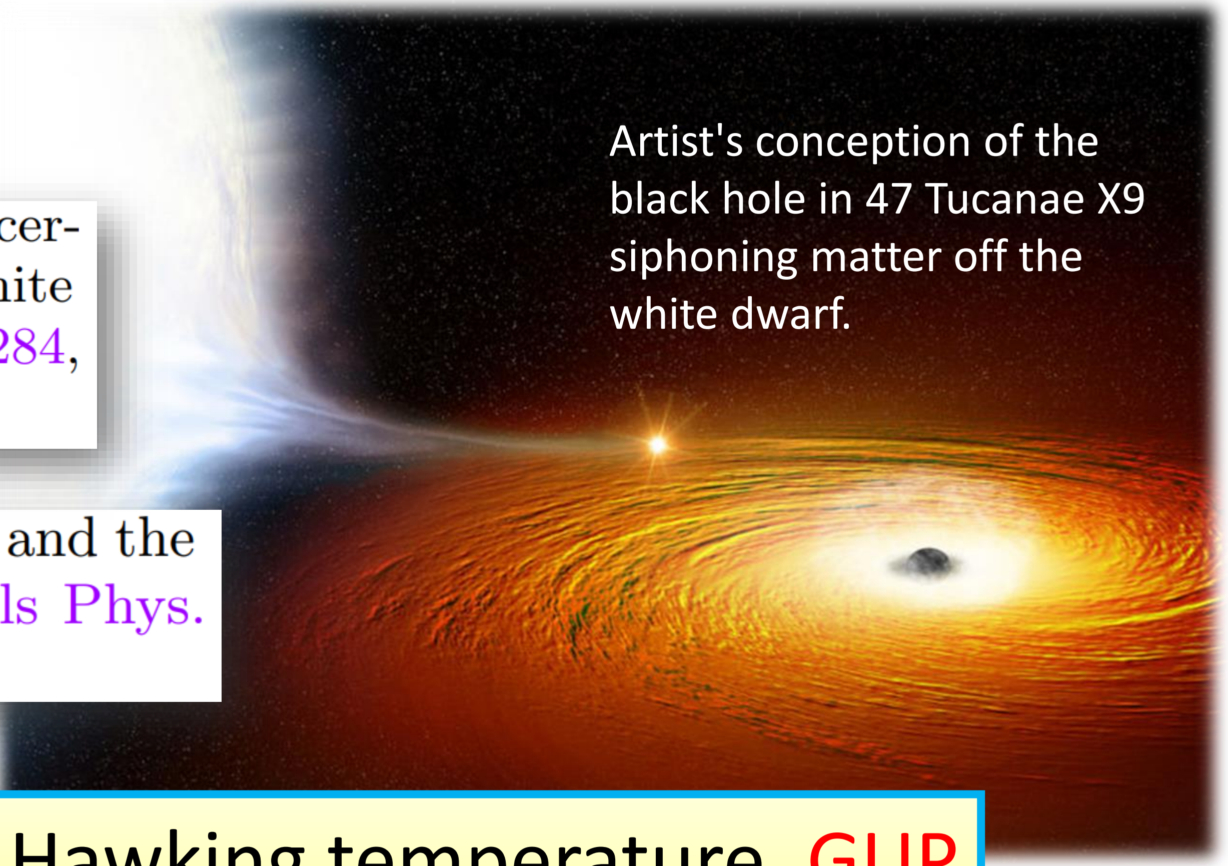
Arbitrarily large white dwarfs (?)

White Dwarfs with GUP: No Chandrasekhar Limit?

Previously already pointed out in

Mohamed Moussa, "Effect of Generalized Uncertainty Principle on Main-Sequence Stars and White Dwarfs", Adv. High Energy Phys. **2015** (2015) 343284, arXiv:1512.04337 [physics.gen-ph].

Reza Rashidi, "Generalized Uncertainty Principle and the Maximum Mass of Ideal White Dwarfs", Annals Phys. 374 (2016) 434, [arXiv:1512.06356 [gr-qc]].



Artist's conception of the black hole in 47 Tucanae X9 siphoning matter off the white dwarf.

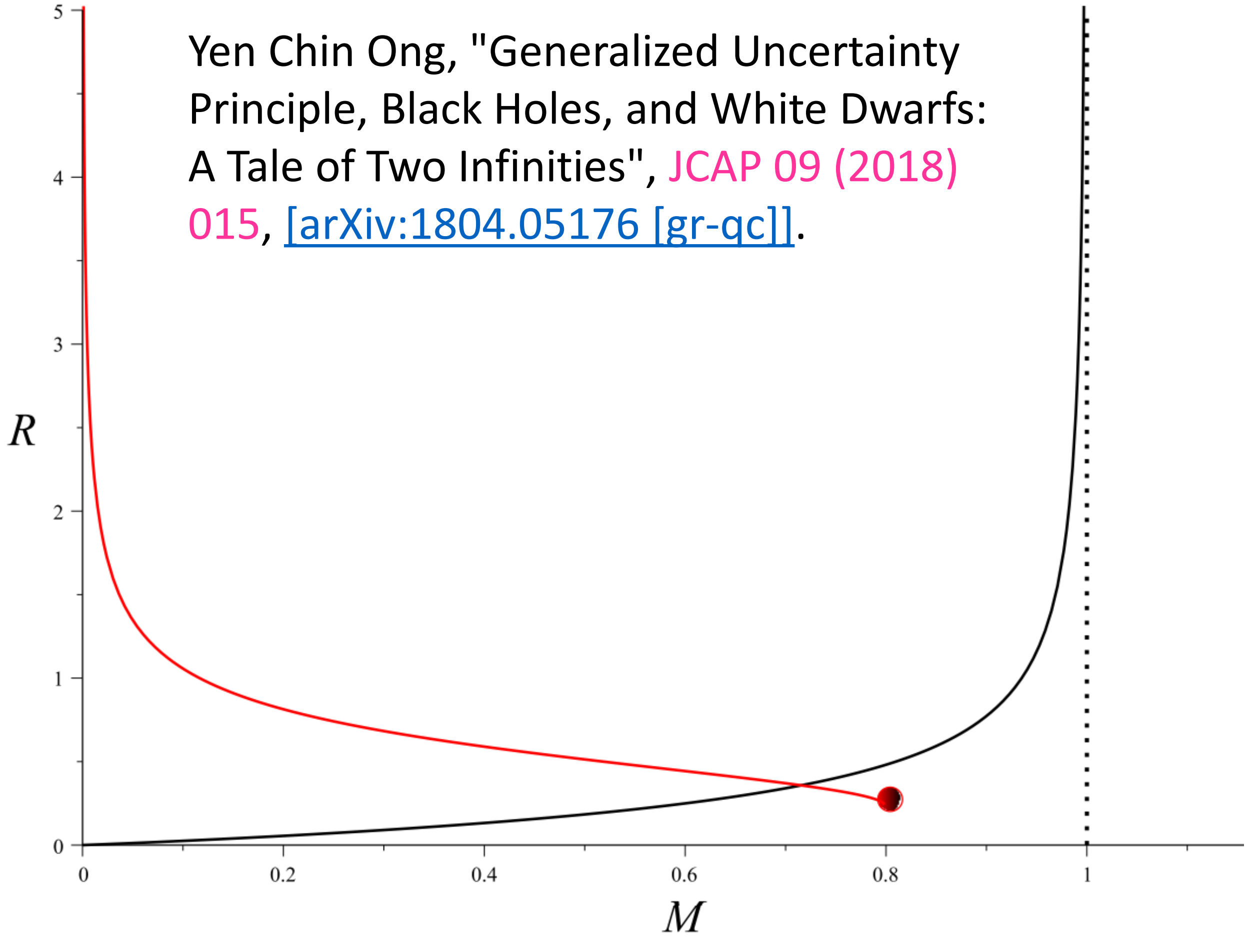
While preventing divergence in Hawking temperature, GUP *INTRODUCES* another infinity in white dwarf physics!

How to resolve this?

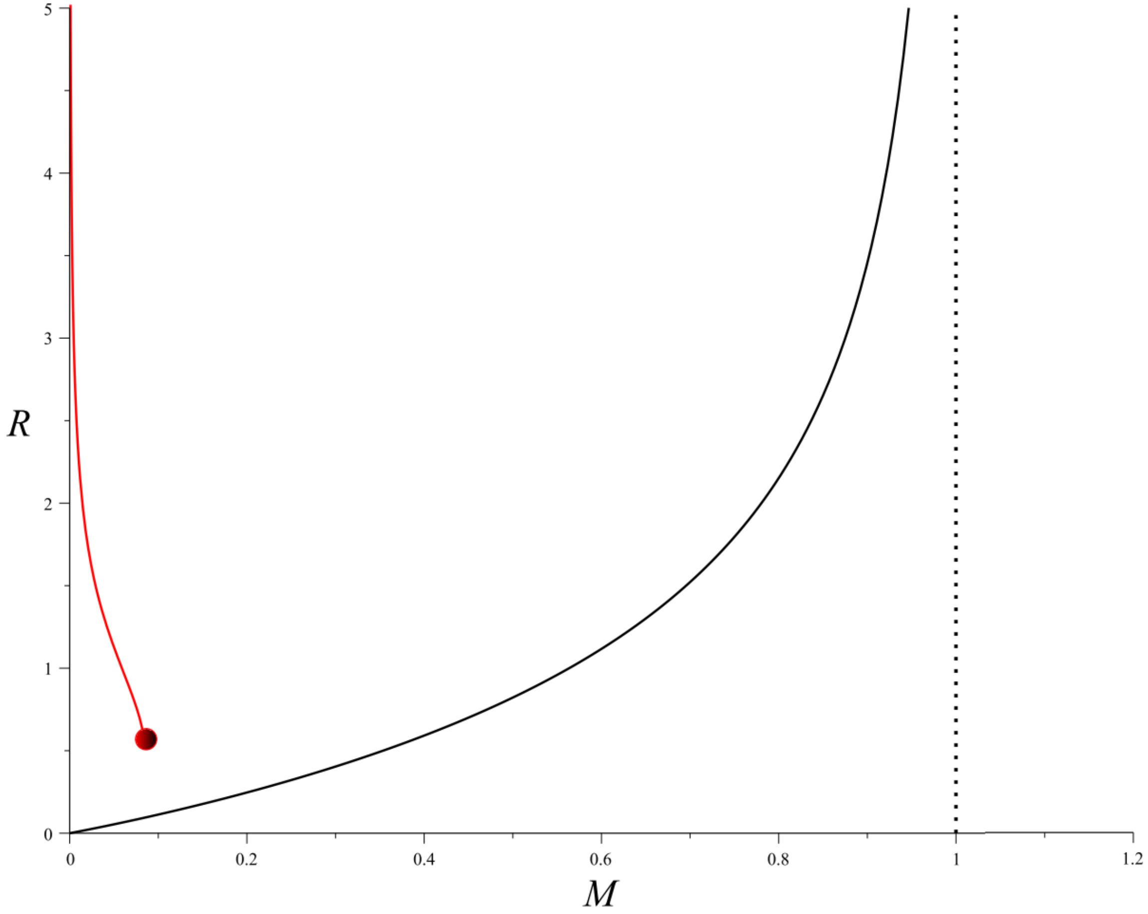
[A few possibilities]

A simple fix! Simply Choose $\alpha < 0$

$$\Delta x \Delta p \sim \hbar + \alpha L_p^2 \Delta p^2/\hbar$$



Yen Chin Ong, "Generalized Uncertainty Principle, Black Holes, and White Dwarfs: A Tale of Two Infinities", JCAP 09 (2018) 015, [arXiv:1804.05176 [gr-qc]].



Negative GUP parameter

Previously suggested in

Petr Jizba, Hagen Kleinert, Fabio Scardigli, "Uncertainty Relation on World Crystal and its Applications to Micro Black Holes", Phys. Rev. D **81** (2010) 084030, [arXiv:0912.2253 [hep-th]].

Also, if one takes the generalized Hawking temperature, $T = \frac{Mc^2}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha\hbar c}{GM^2}}\right)$

and make the reasonable assumption that one should be able to obtain it from Wick-rotating a deformed static Schwarzschild metric with metric coefficient

$$g_{tt} = -\left(1 - \frac{2M}{r} + \varepsilon\frac{M^2}{r^2}\right)$$

then for $|\varepsilon| \ll 1$, we have

$$\alpha = -4\pi^2\varepsilon^2\left(\frac{M}{2M_p}\right)^2 < 0.$$

Fabio Scardigli, Roberto Casadio, "Gravitational tests of the Generalized Uncertainty Principle", Eur. Phys. J. C **75** (2015) 425, arXiv:1407.0113 [hep-th].

How to Understand Negative Alpha Correction?

$$\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \frac{\alpha L_p^2 \Delta p^2}{\hbar}\right]$$

At large enough energy, RHS becomes smaller: vanishes at Planck scale!

$$\frac{\alpha L_p^2 \Delta p^2}{\hbar} = \frac{\alpha\left(\frac{\hbar G}{c^3}\right)\Delta p^2}{\hbar}$$

$$\sim \frac{\alpha G}{c^3}\left(\frac{E_p}{c}\right)^2 =$$

$$\frac{\alpha G}{c^3}\left(\frac{1}{c^2}\right)\frac{\hbar c^5}{G} = \alpha\hbar$$

Planck Scale Physics Becomes Classical!

Petr Jizba, Hagen Kleinert, Fabio Scardigli;
Bernard J. Carr, Jonas Mureika, Piero Nicolini

Previously in Literature:

- $\hbar$ as a dynamical field that goes to zero in the Planckian limit (Hossenfelder)
- Asymptotic Safe Gravity: If Planck mass is fixed, equivalent to zero G limit since $G = \hbar c / M_p^2$.
- Singularity of dilaton charged black hole (naïve but suggestive):

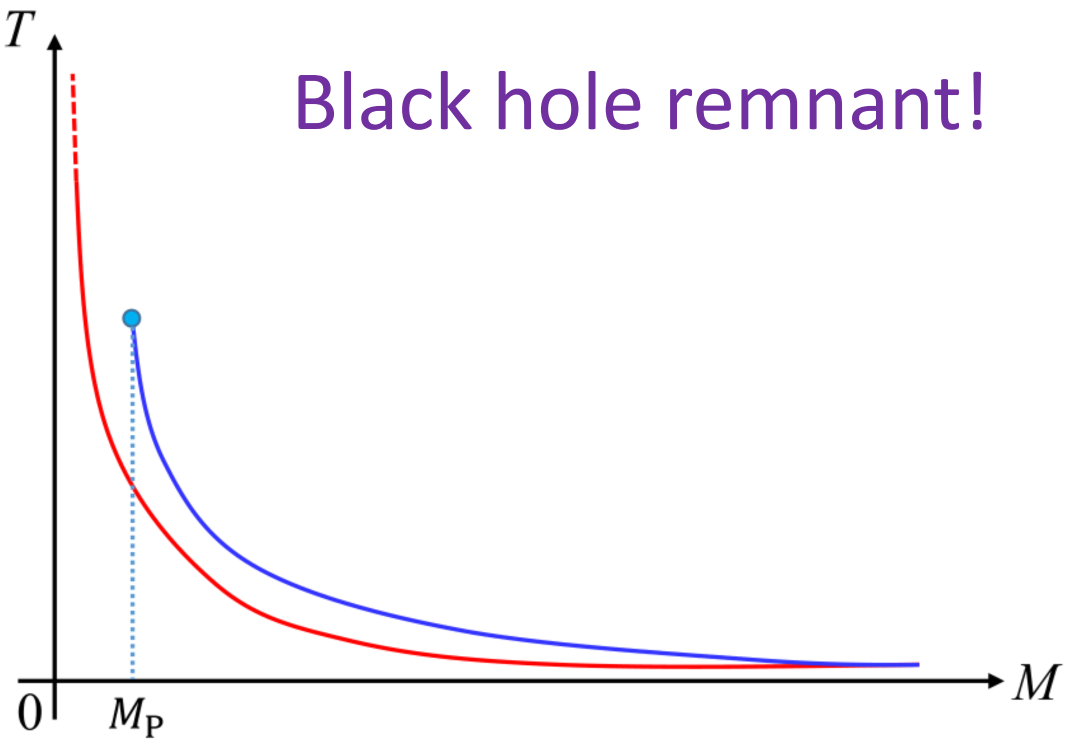
"[...] the string coupling is becoming very weak near the singularity. As we have discussed, we have no right to trust this solution near the singularity, but its difficult to resist speculating about what it might mean if the exact classical solution had a similar behavior. *It would suggest that, contrary to the usual picture of large quantum fluctuations and spacetime foam near the singularity, quantum effects might actually be suppressed.* The singularity would behave classically."

Gary T. Horowitz, "The Dark Side of String Theory: Black Holes and Black Strings", [arXiv:hep-th/9210119].

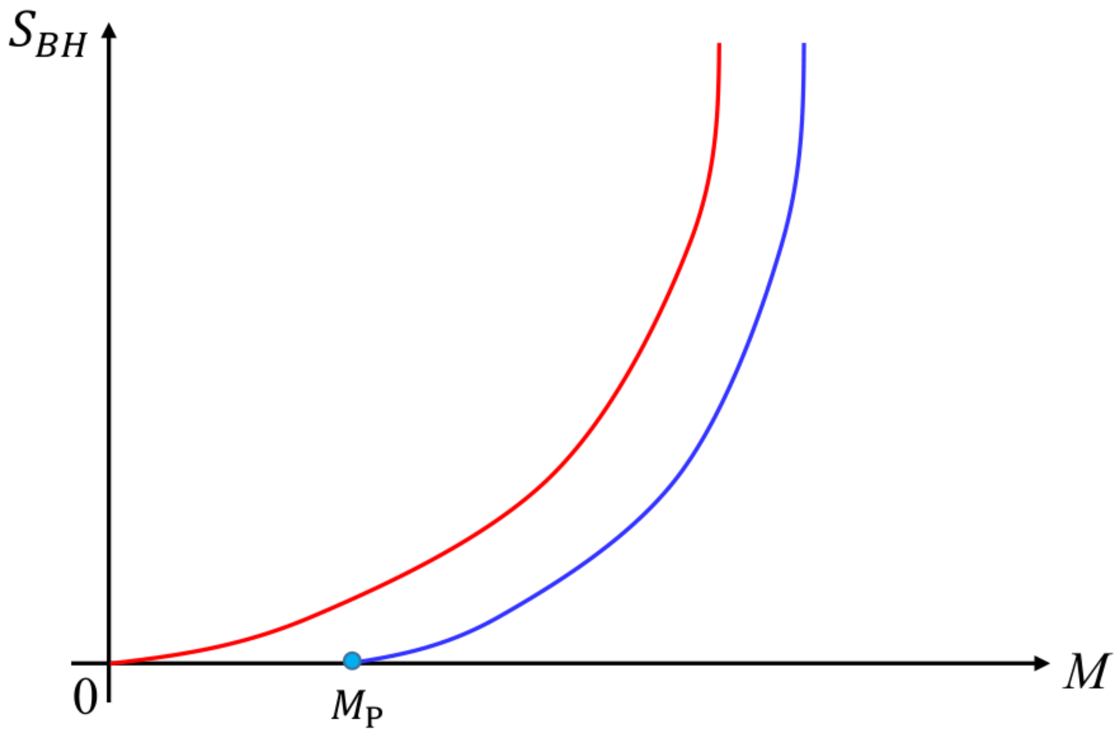
Hawking Evaporation with GUP

$$T = \frac{Mc^2}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha\hbar c}{GM^2}}\right)$$

Ronald J. Adler, Pisin Chen, David I. Santiago, "The Generalized Uncertainty Principle and Black Hole Remnants", Gen. Rel. Grav. 33 (2001) 2101, arXiv:gr-qc/0106080



Black hole remnant!



Does this Affect Black Hole Physics?

Yes, but OK!

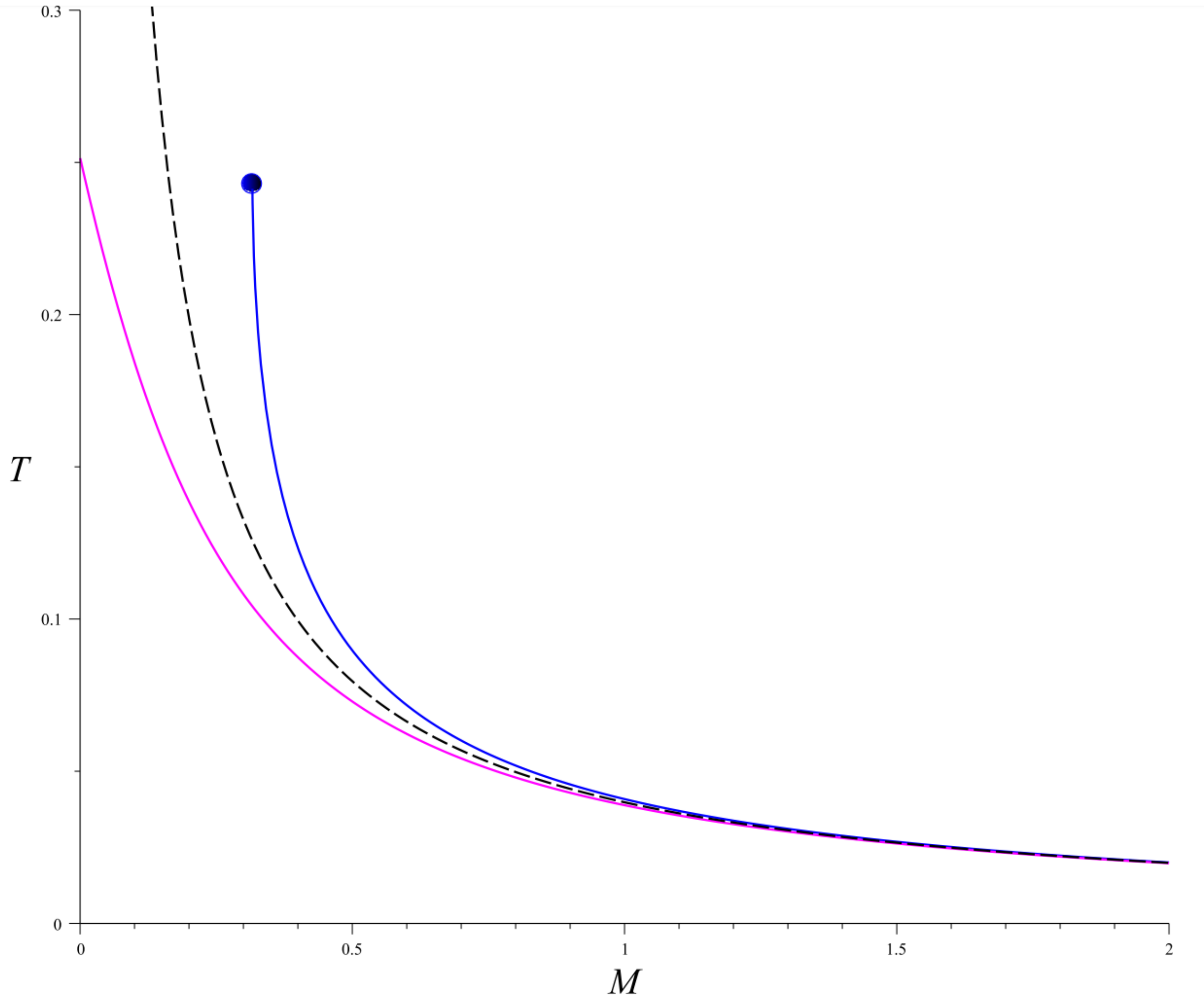
$$T[\alpha > 0] = \frac{M}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha}{M^2}}\right)$$

$$T[\alpha = 0] = \frac{1}{8\pi M}$$

$$T[\alpha < 0] = -\frac{M}{4|\alpha|\pi}\left(1 - \sqrt{1 + \frac{|\alpha|}{M^2}}\right)$$

Small mass limit:

$$T[\alpha < 0] \sim \frac{M}{4|\alpha|\pi}\frac{\sqrt{|\alpha|}}{M} = \frac{1}{4\pi\sqrt{|\alpha|}} < \infty$$



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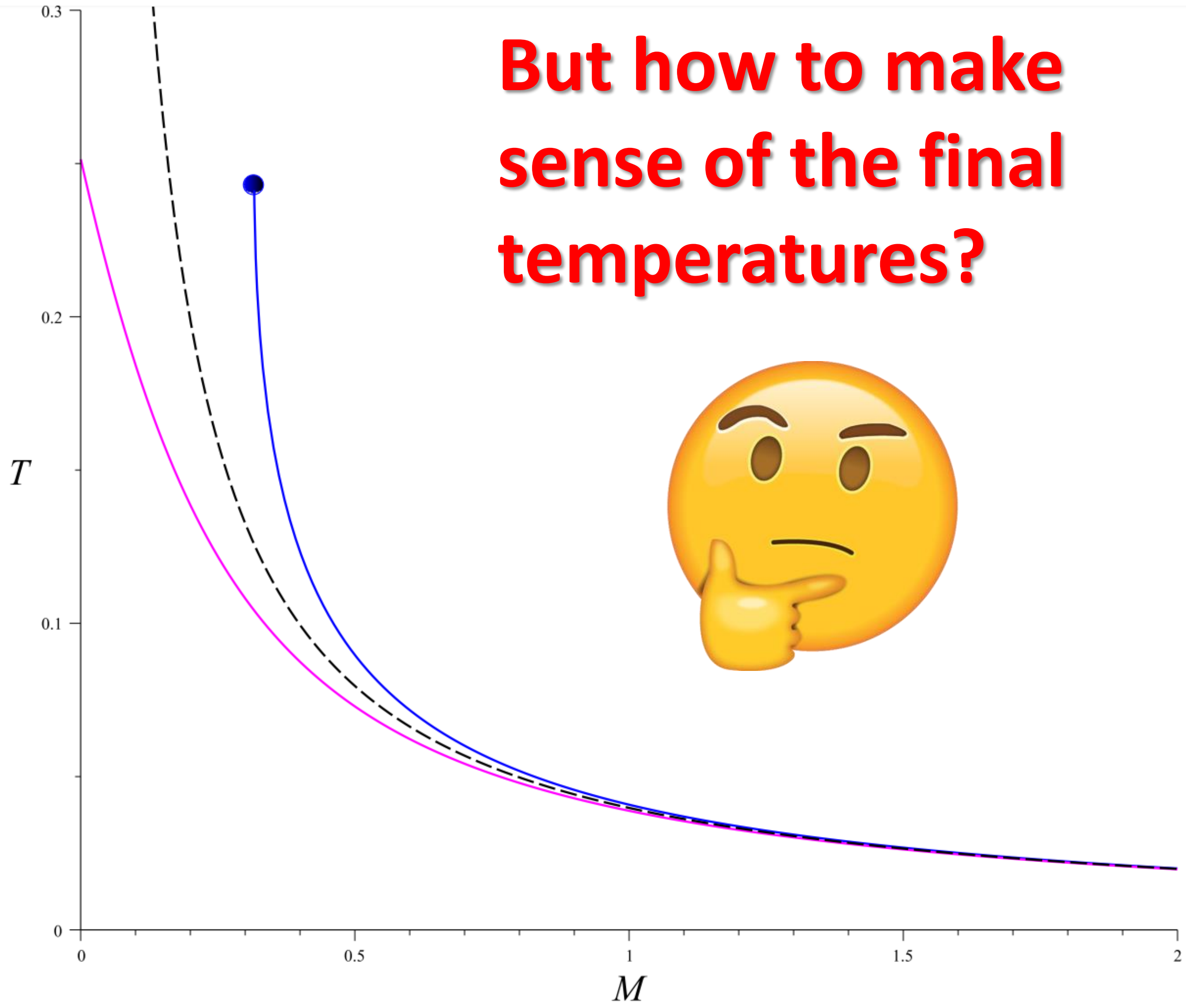
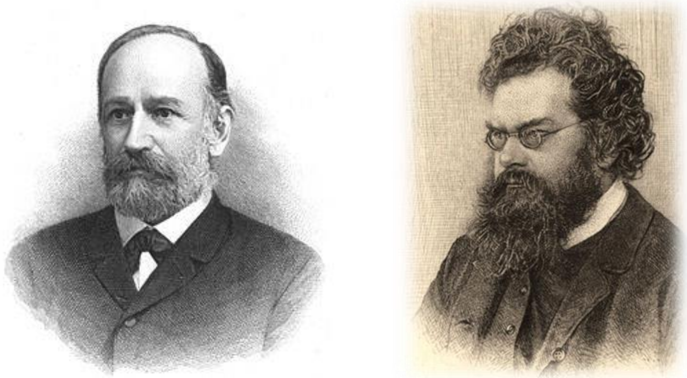


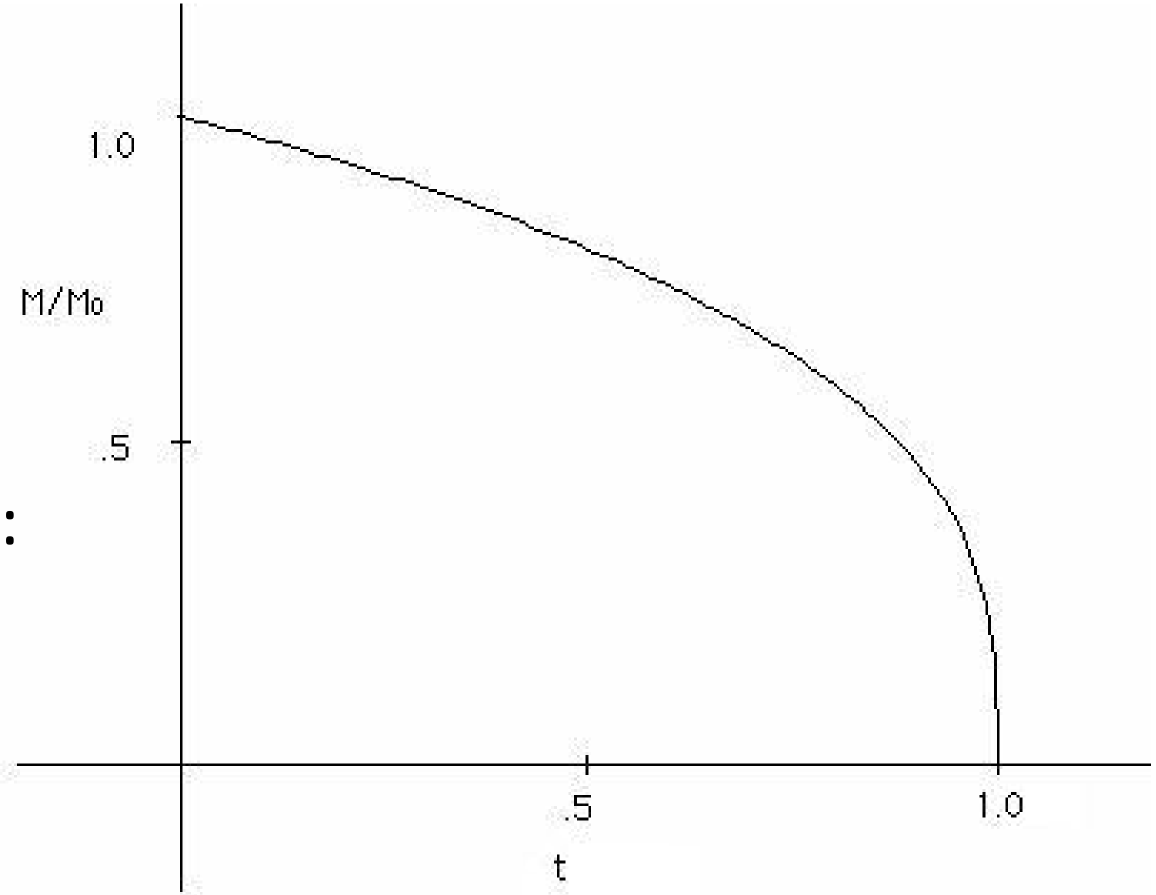
But how to make sense of the final temperatures?

Lifetime of a Black Hole

Hawking temperature:

<ins>Stefan-Boltzmann Equation:</ins>





Thermal Mass Loss (up to greybody factor):

$$\frac{dM}{dt} \sim -AT^4 \sim -\frac{1}{M^2}$$

Thus the lifetime of a black hole is of order M^3.

Lifetime for solar mass black hole = $O(10^{67})$ years

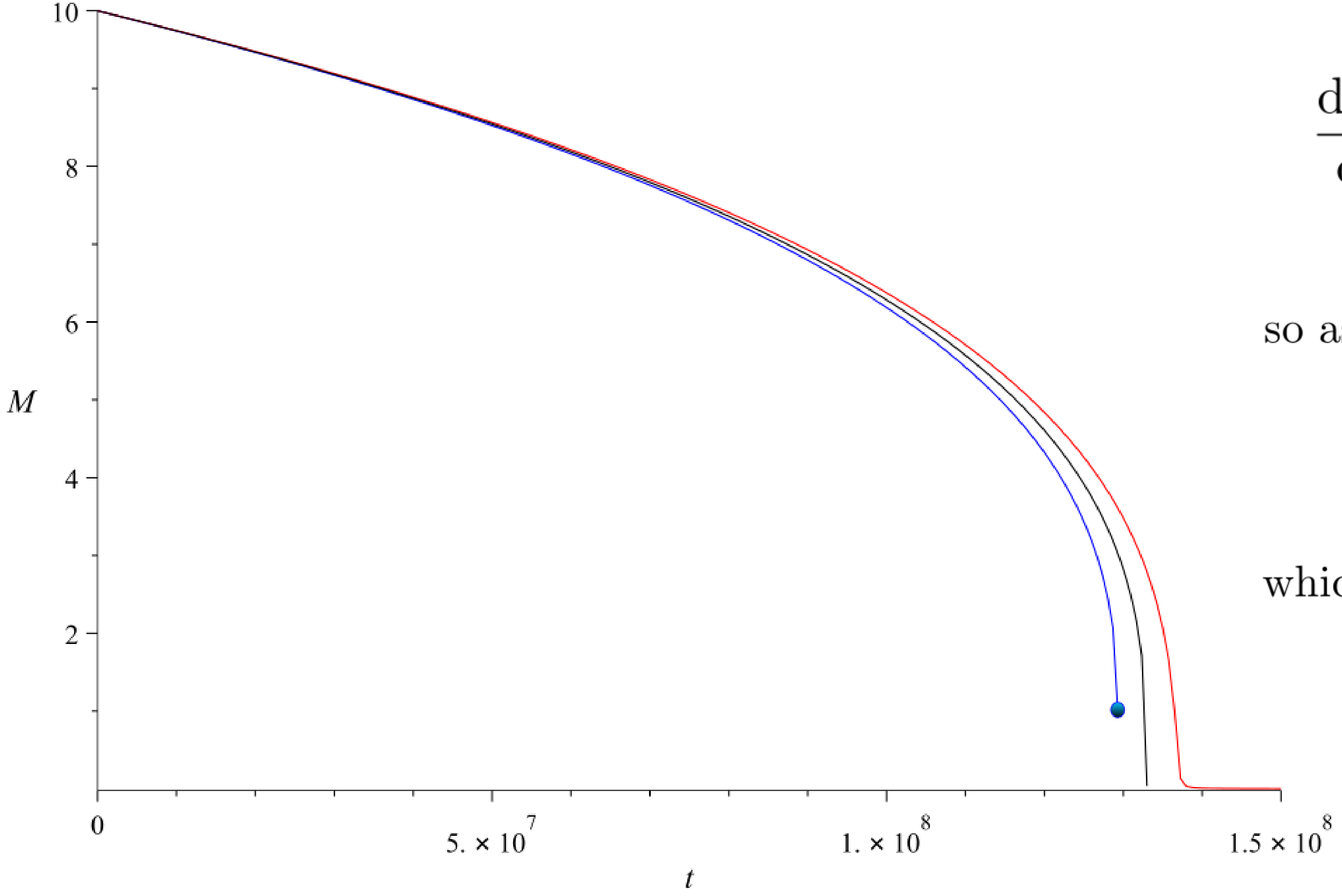


FIG. 2: Mass evolution of Schwarzschild black holes with no GUP correction (black, middle curve), positive GUP correction (blue, left curve), and negative GUP correction (red, right curve). The positive GUP correction leads to a remnant in finite time, while negative GUP correction yields infinite lifetime. These contrast with the usual case without GUP correction, in which the black hole completely evaporates in finite time.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{M^6}{(4|\alpha|\pi)^4}\left(1-\sqrt{1+\frac{|\alpha|}{M^2}}\right)^4.$$

so as M becomes sufficiently small, we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \sim -\frac{M^2}{(4\pi)^4\alpha^2},$$

which leads to

$$M = M_0\left(\frac{256\pi^4\alpha^2}{256\pi^4\alpha^2+M_0t}\right),$$

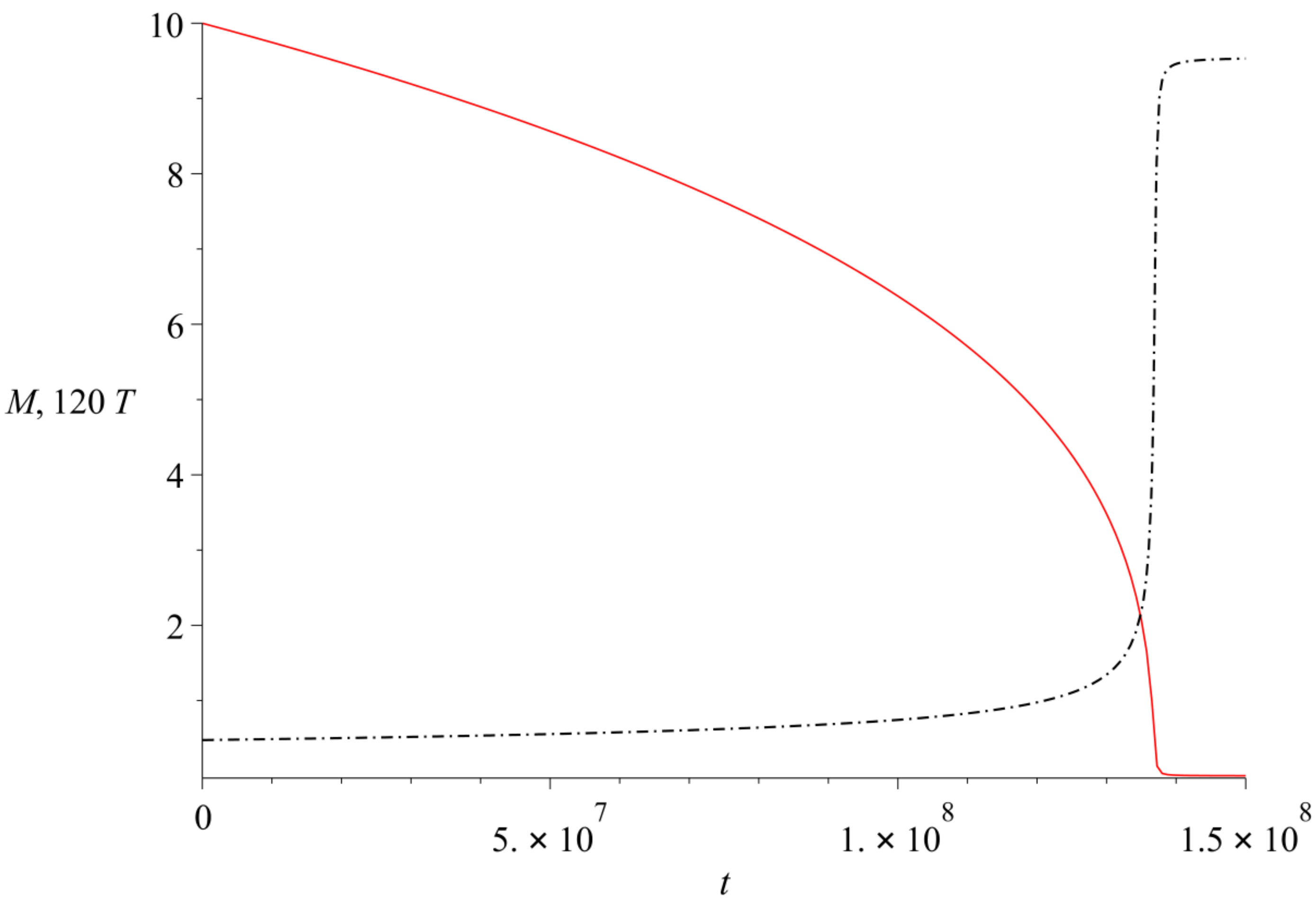


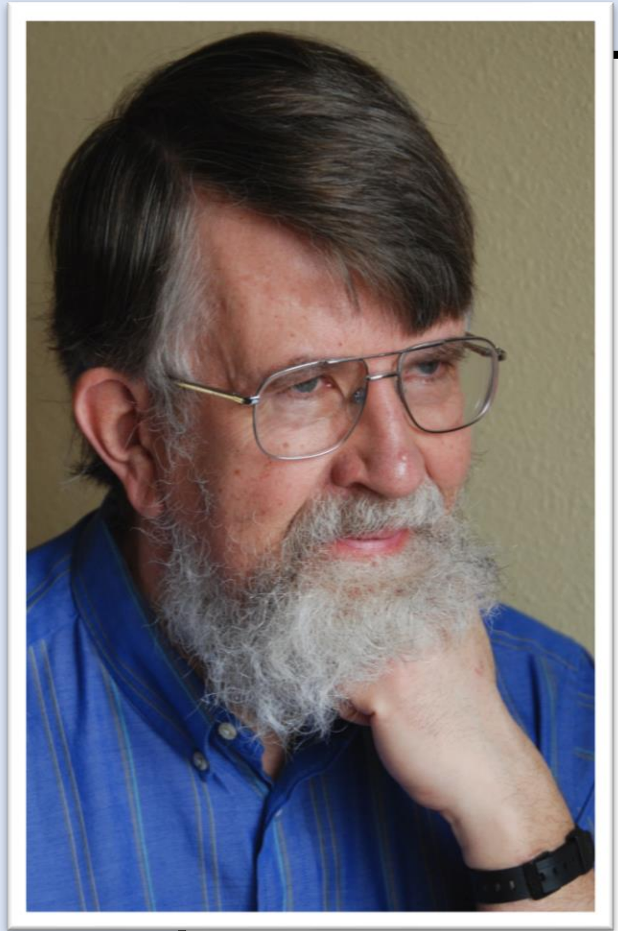
FIG. 2: The Hawking temperature of an asymptotically flat Schwarzschild black hole with $\alpha = -1$, here in black dash-dotted curve, as a function of time, shows that the temperature tends to a constant value. The mass of the black hole, in red solid curve, tends to zero asymptotically. In order to display both curves in the same diagram, we have multiplied the Hawking temperature by a factor of 120, so that the temperature curve tends to $120T^* = 120/4\pi \approx 9.549$.


Are Black Holes like
Burning Coal?

Black Hole Random Walk

Yasunori Nomura, Jaime Varela, Sean J. Weinberg, “Black Holes, Information, and Hilbert Space for Quantum Gravity, Phys. Rev. D.87 (2013) 084050

Don Page, “Is Black-Hole Evaporation Predictable?”, Phys. Rev. Lett. 44 (1980) 301



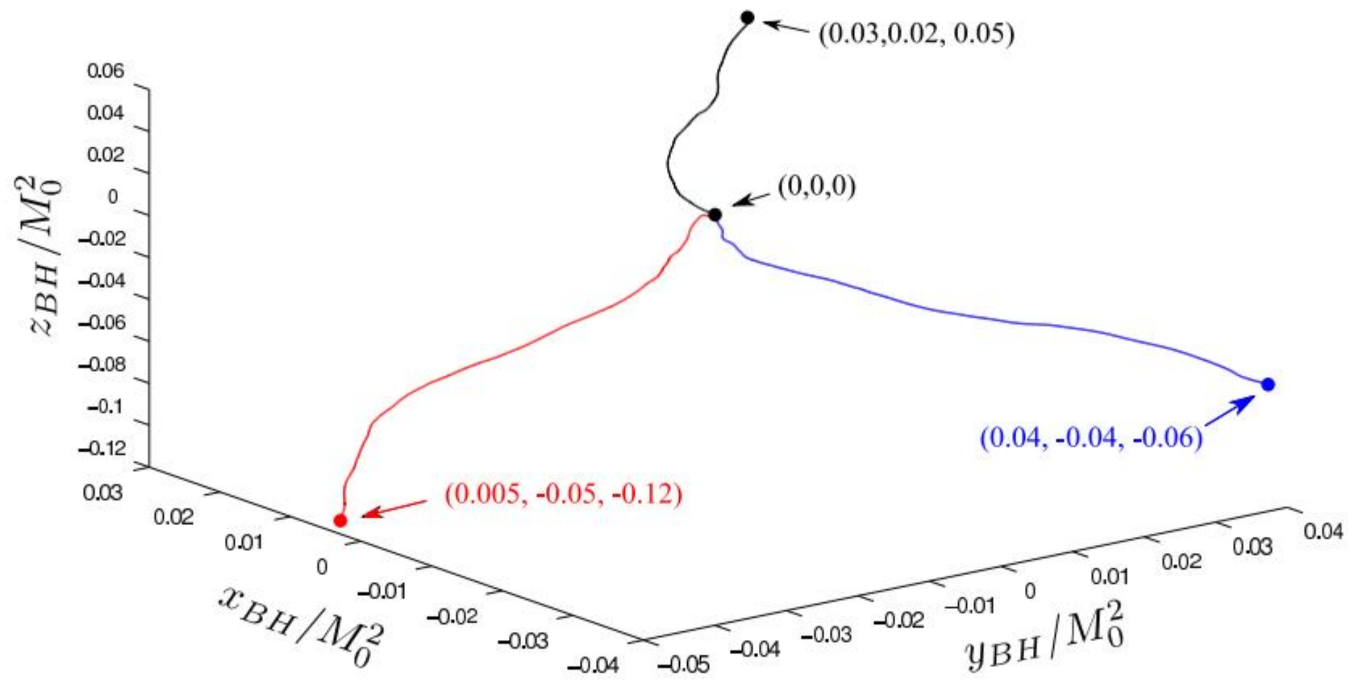
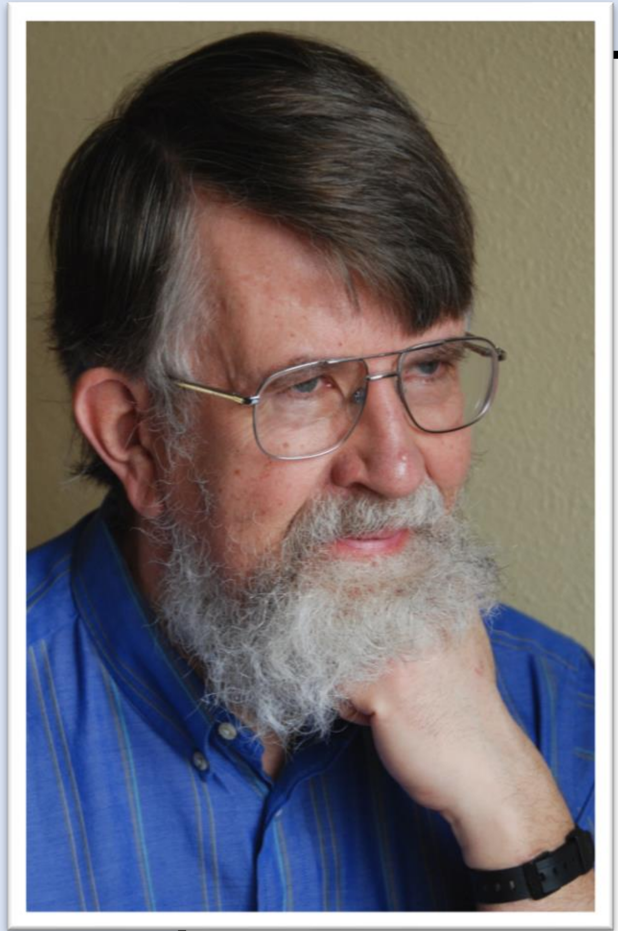


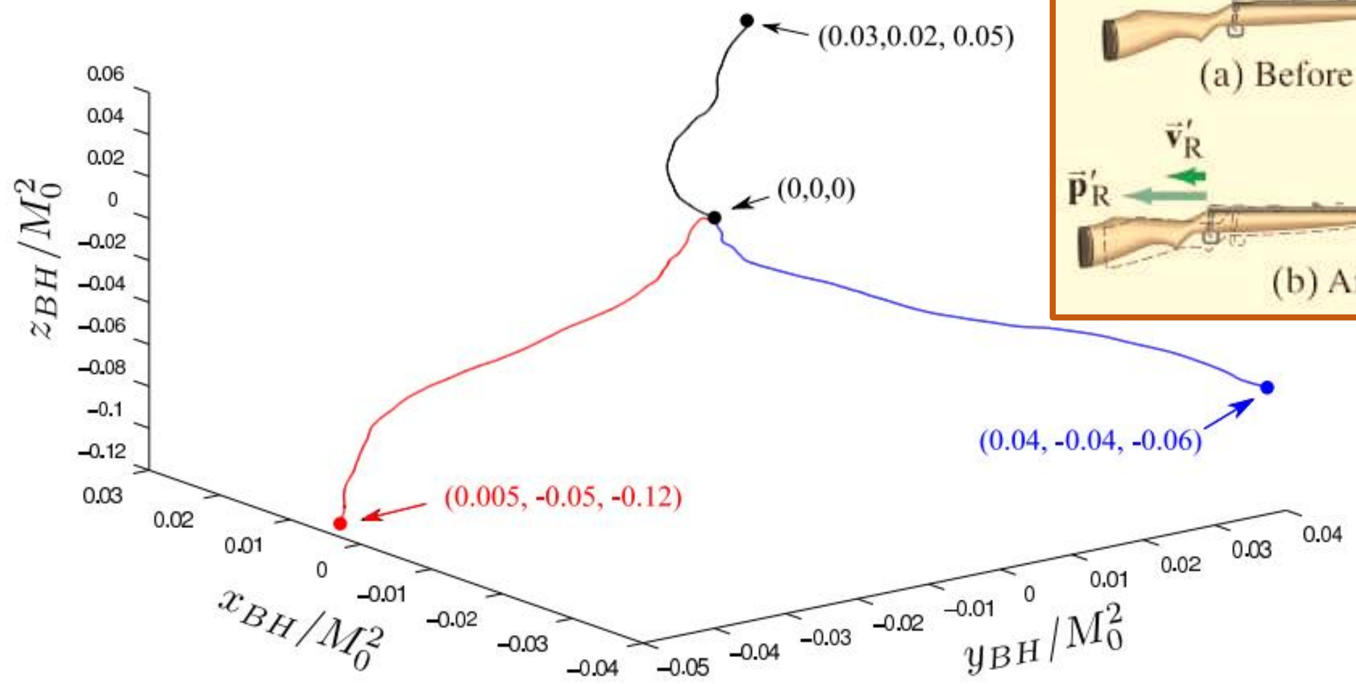
Figure 3: Typical paths of the black hole drifting in the three dimensional space $\mathbf{x}_{\rm BH} = (x_{\rm BH}, y_{\rm BH}, z_{\rm BH})$, normalized by M_0^2.

Black Hole Random Walk

Yasunori Nomura, Jaime Varela, Sean J. Weinberg, "Black Holes, Information, and Hilbert Space for Quantum Gravity, Phys. Rev. D.87 (2013) 084050

Don Page, "Is Black-Hole Evaporation Predictable?", Phys. Rev. Lett. 44 (1980) 301





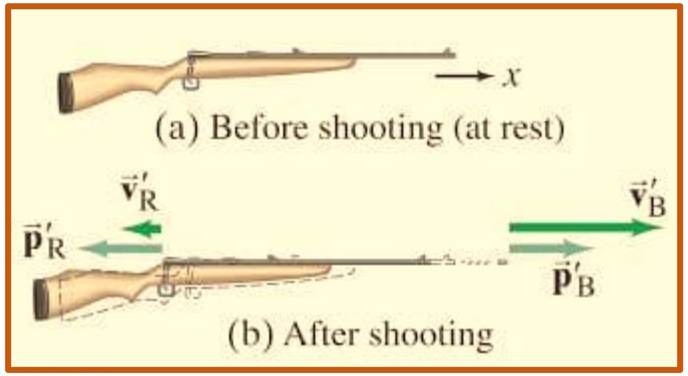


Figure 3: Typical paths of the black hole drifting in the three dimensional space $\mathbf{x}_{BH} = (x_{BH}, y_{BH}, z_{BH})$, normalized by M_0^2.

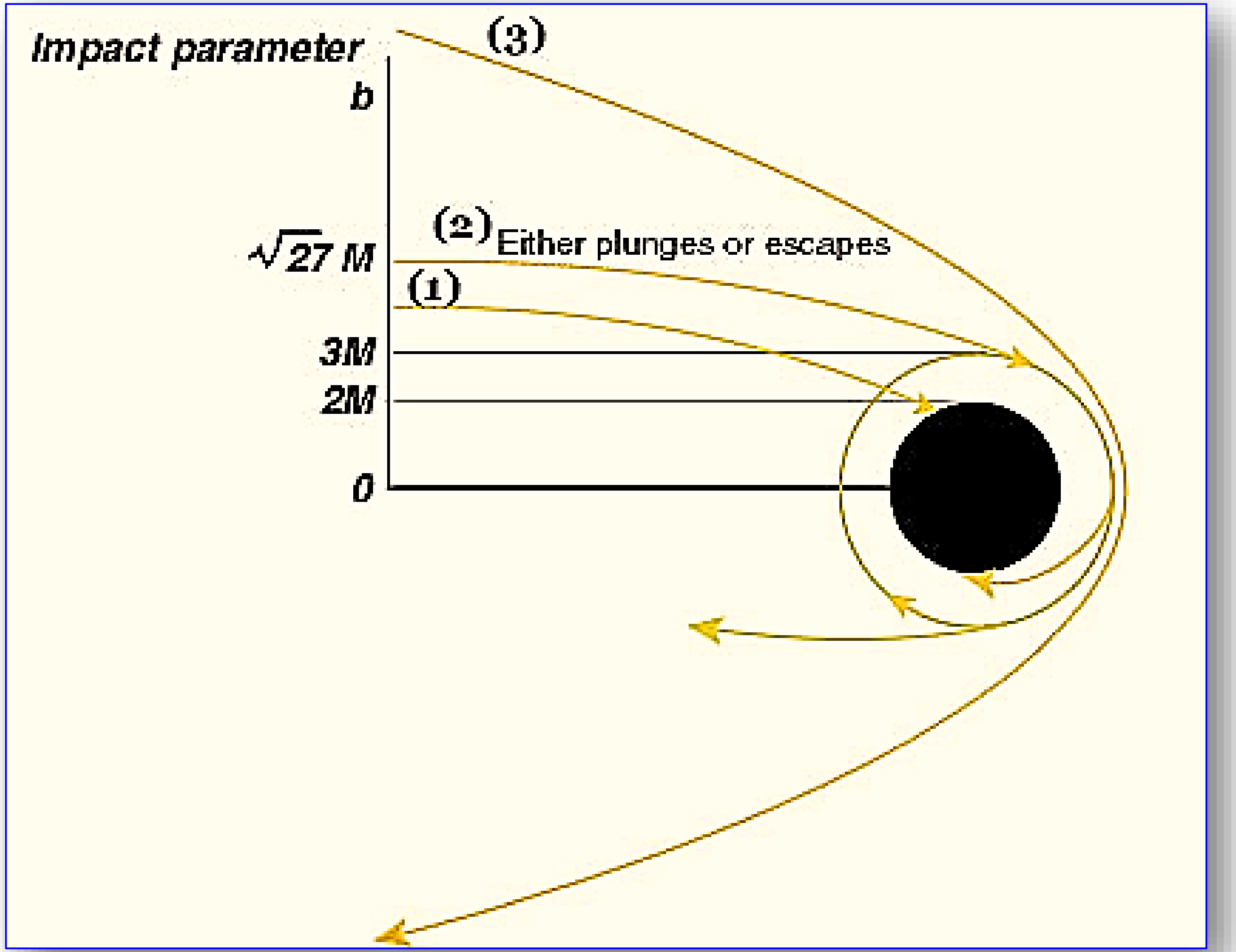
Sparsity

$$\eta[\alpha] := \frac{\lambda^2[\alpha]}{\sigma} = \frac{\lambda^2[\alpha]}{27\pi M^2}.$$

Finnian Gray, Sebastian Schuster, Alexander Van-Brunt, Matt Visser, "The Hawking Cascade from a Black Hole Is Extremely Sparse", Class. Quant. Grav. **33** (2016) 115003, [arXiv:1506.03975 [gr-qc]].

Matt Visser, Finnian Gray, Sebastian Schuster, Alexander Van-Brunt, "Sparsity of the Hawking Flux", in Proceedings of the MG14 Meeting on General Relativity (2017); pp. 1724-1729, [arXiv:1512.05809 [gr-qc]].

Wolfgang Mück, "Hawking Radiation is Corpuscular", Eur. Phys. J. C **76** (2016) 374, [arXiv:1606.01790 [hep-th]].



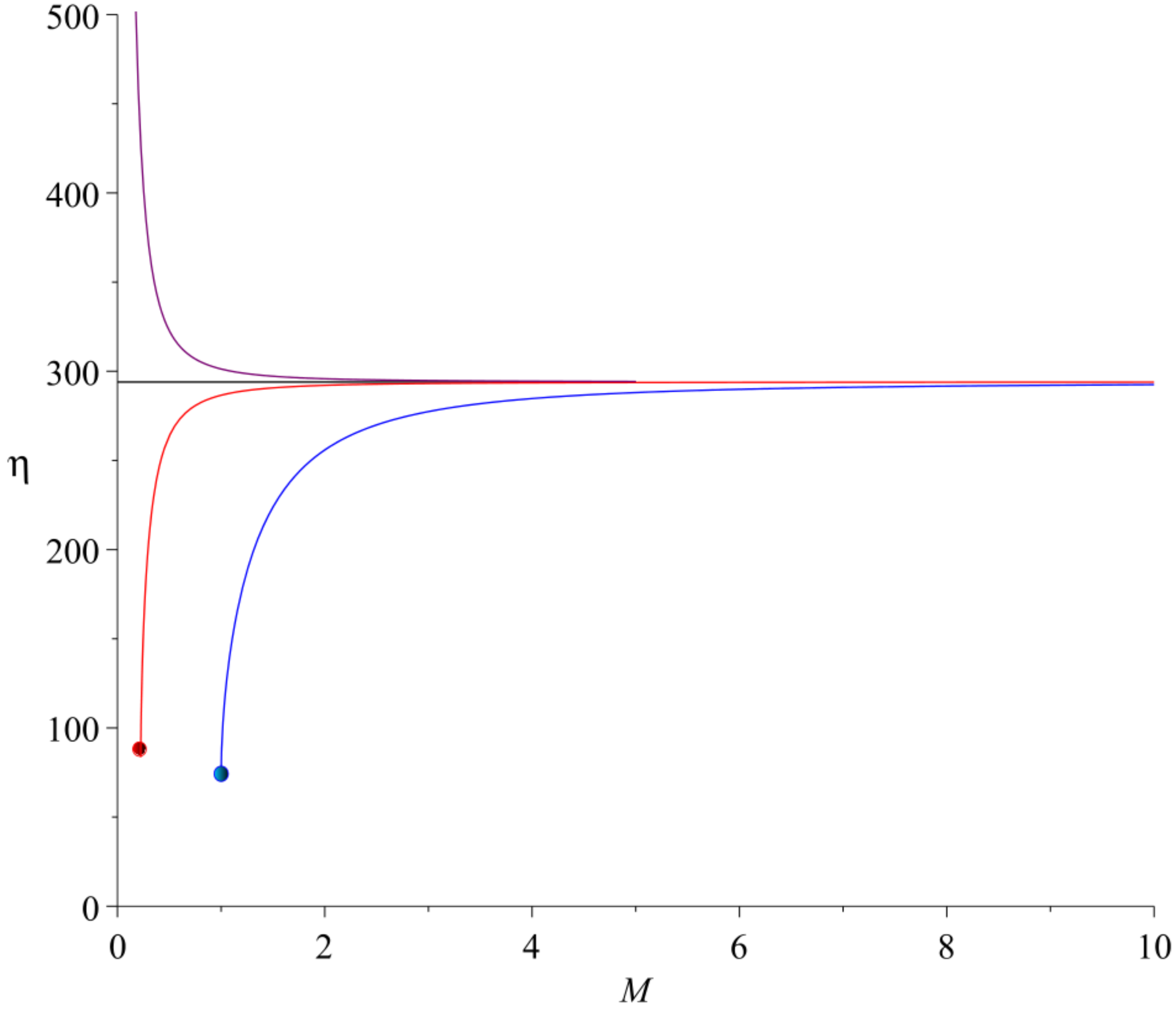


FIG. 4: The sparsity of Hawking radiation. The constant black line corresponds to the usual Hawking evaporation – the radiation remains sparse even towards the end of the evaporation. The bottom two curves are for $\alpha = 1$ (bottom right curve, blue) and $\alpha = 0.05$ (bottom left, red), respectively. For $\alpha > 0$: GUP correction leads to the decrease in η, and the radiation becomes less sparse towards the end. However, for $\alpha < 0$, we get an ever-increasing η, so the radiation becomes *extremely* sparse. Shown here (top, purple), is an example for which $\alpha = -0.05$.

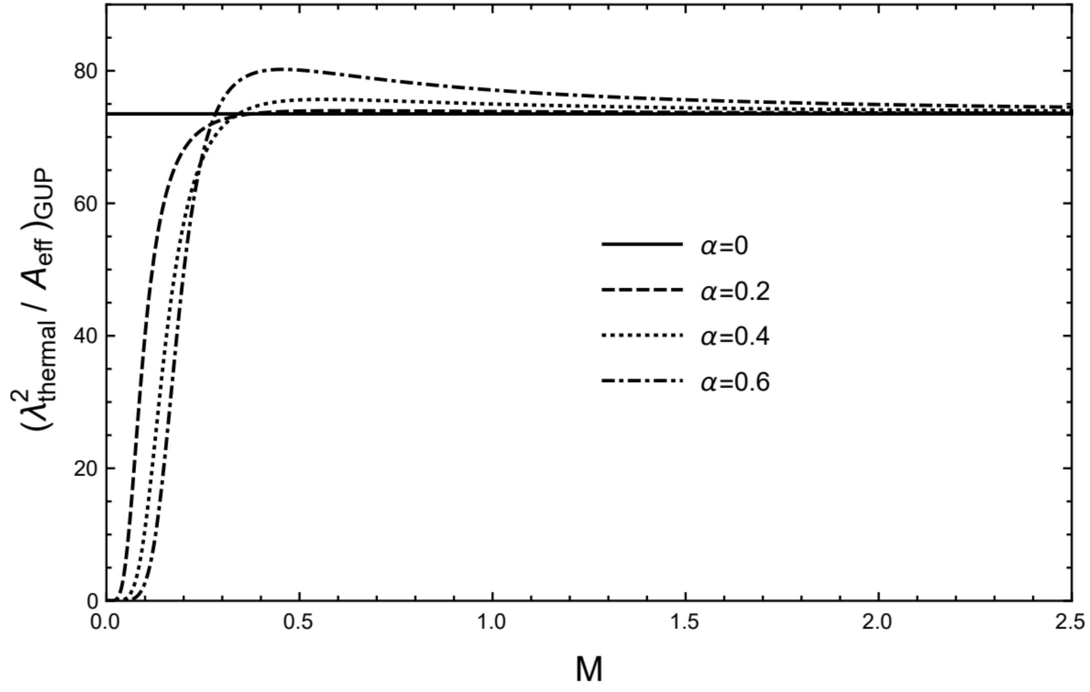


FIG. 2. GUP-corrected dimensionless parameter $\frac{\lambda^2_{thermal}}{A_{eff}}|_{GUP}$ as given by (28) versus M for different values of α. We have taken natural units $c = G = \hbar = 1$

Ana Alonso-Serrano, Mariusz P. Dabrowski, Hussain Gohar, Phys. Rev. D 97 (2018) 044029, arXiv:1801.09660 [gr-qc].

So finite temperature with vanishing mass is unattainable!

How generic is this?

Instead of a QG-motivated model, let us look at a *classical* modified gravity theory

Massive Gravity

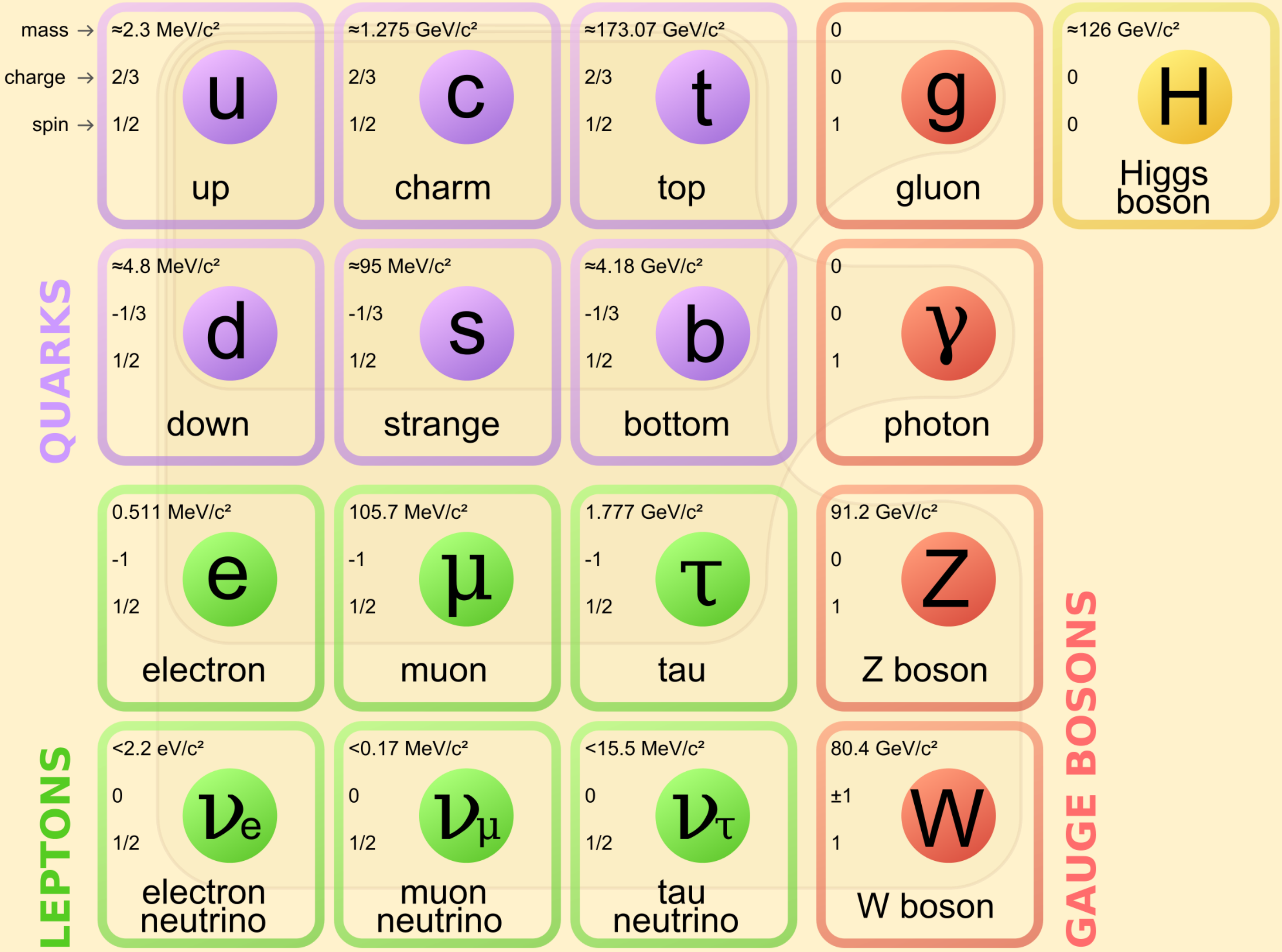
QFT point of view:

Gravity is a theory of spin-2 particles:

It is natural to ask:

(1) Could graviton have nonzero mass? After all, we thought neutrino is massless…

(2) How many types of gravitons?



Dyonic black hole solution in Massive Gravity

$$-g_{tt} = k + \frac{r^2}{l^2} - \frac{2m_0}{r} + \frac{q_E^2 + q_M^2}{r^2} + m^2\left(\frac{cc_1}{2}r + c^2 c_2\right)$$

$c > 0$, c_1, c_2 either sign OK

$$T = \frac{1}{4\pi}\left[\frac{k}{r_+} + \frac{3r_+}{l^2} - \frac{q_E^2 + q_M^2}{r_+^3} + m^2\left(cc_1 + \frac{c^2 c_2}{r_+}\right)\right].$$

S. H. Hendi, N. Riazi, S. Panahiyan, "Holographical Aspects of Dyonic Black Holes: Massive Gravity Generalization", Ann. Phys. (Berlin) 530 (2018) 1700211, arXiv:1610.01505 [hep-th].

In the absence of magnetic charge, the authors showed that by tuning the various parameters such that

$$m^2 c_2 c^2 + k - \Phi_E^2 = 0,$$

where Φ_E is the electric potential, one could have a solution with Hawking temperature of the form

$$T = 2r_h P + \frac{m^2 c c_1}{4\pi},$$

where $P = -\Lambda/(8\pi)$ is the pressure term in the extended black hole thermodynamics in an asymptotically locally anti-de Sitter spacetime. In the limit of vanishing horizon $r_h \to 0$, we see that $0 < T = m^2 c c_1/(4\pi) < \infty$.

“Remnant temperature: fluctuation?”

$$T = 2r_+ P + \frac{m^2 c c_1}{4\pi}, \tag{34}$$

which for the limit of $r_+ \longrightarrow 0$, it is non-zero.

Remembering that in the evaporation of black holes by the Hawking radiation mechanism, the horizon radius eventually vanishes. However, we see here that in this case, there will be a remnant for the temperature of black holes. This indicates that all the information regarding existence of the black holes is not vanished completely despite the statement of paradox information. In fact, a trace of existence of black holes will remain which presents itself as a fluctuation in temperature of the background spacetime where black hole was present. This specific property for the temperature is due to the existence of massive gravitons. This shows that generalization from massless gravitons to massive ones, introduces new properties to the thermodynamics of black holes which could solve and answer some long standing questions regarding the physics of black holes such as the information paradox. It is worthwhile to mention that existence of the remnant for temperature of the black holes, to our knowledge, so far was only reported for black holes in the context of massive gravity [63] and it is one of the unique properties of the massive gravity which make it different from other modified theories of the gravity.

S. H. Hendi, N. Riazi, S. Panahiyan, "Holographical Aspects of Dyonic Black Holes: Massive Gravity Generalization", Ann. Phys. (Berlin) 530 (2018) 1700211, arXiv:1610.01505 [hep-th].

A Constant Temperature Black Hole

$$\Phi_E = 0 \qquad \Lambda = 0 \qquad k = 1 \qquad c_2 = -1/(m^2c^2)$$

$$T \equiv m^2 c c_1/(4\pi)$$

Yen Chin Ong, Yuan Yao, "Generalized Uncertainty Principle and White Dwarfs Redux: How Cosmological Constant Protects Chandrasekhar Limit", Phys. Rev. D 98 (2018) 126018 [arXiv:1809.06348 [gr-qc]].

A Constant Temperature Black Hole

Set $m = c = 1$, so $c_2 = -1$.

The physical mass (the mass that appears in the first law of thermodynamics) is [32]

$$M = \frac{r_h}{2}\left[k + \frac{r_h^2}{l^2} + \frac{q_E^2 + q_M^2}{r^2} + m^2\left(\frac{cc_1}{2}r_h + c^2c_2\right)\right],$$

which, with our choice of the parameter values, reduces to $M = r_h^2/4$.

Evolution of Constant Temperature Black Hole

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{r_h}{2}\frac{\mathrm{d}r_h}{\mathrm{d}t} = -Cr_h^2\frac{1}{(4\pi)^4},$$

(?)

which yields, with $\tilde{C} = 2C/(4\pi)^4$,

$$\frac{\mathrm{d}r_h}{\mathrm{d}t} = -\tilde{C}r_h \Longrightarrow \int_{r_h(0)}^{\varepsilon}\frac{\mathrm{d}r_h}{r_h} = -\tilde{C}\int_0^{t^*}\mathrm{d}t,$$

where $r_h(0)$ is the initial horizon size, and t^* is the time at which the horizon has shrunk to ε. Integerating yields

$$\varepsilon = r_h(0)\exp\left[-\tilde{C}t^*\right].$$

Therefore, in order to shrink to zero size, $\varepsilon \to 0$, one must have an infinite evaporation time $t^* \to \infty$.

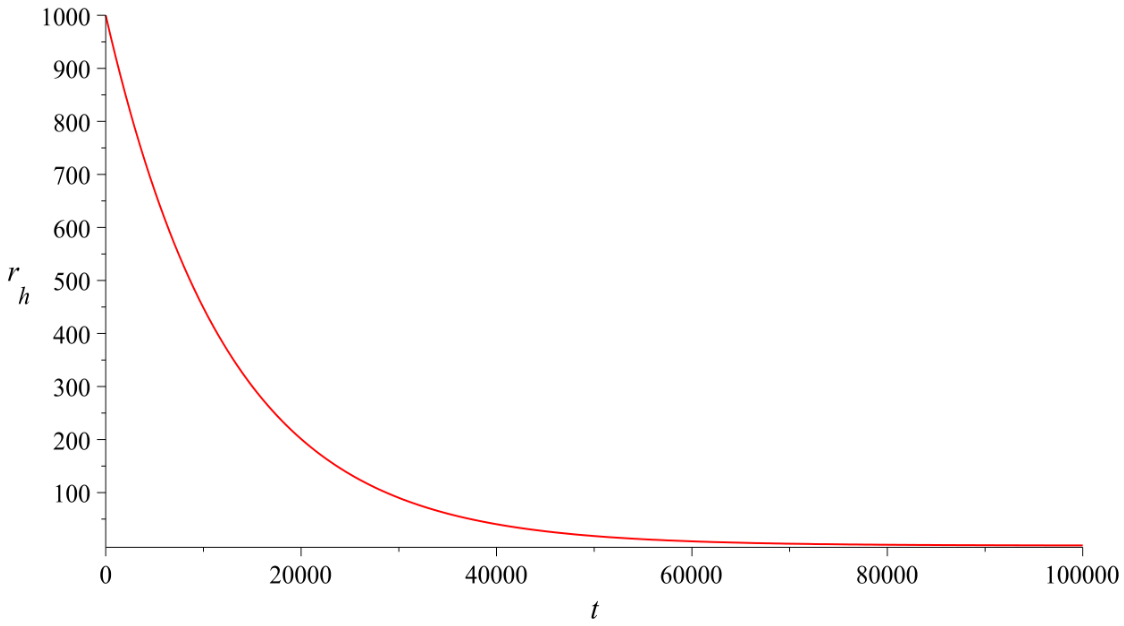


FIG. 3: The evolution of the massive gravity black hole event horizon radius as function of time. Here we choose $r_h(0) = 1000$. The black hole parameters are $m = k = c = 1$, $c_2 = -1$ The black hole asymptotes to zero size as time goes to infinity.

The Complementary Third Law

Theorem: *Consider an n-dimensional neutral static black hole spacetime, with areal radius r, and horizon at $r = r_h$. Assume that the Hawking temperature T and the black hole mass M are analytic functions of r_h. Suppose $\mathrm{d}M/\mathrm{d}t = -CAT^n$, where $C > 0$ is a constant, and $T \to T^* \in (0, \infty)$ as $r_h \to 0$, then $r_h \to 0$ only if $t \to \infty$, provided that the k-th derivative $M^{(k)}$, for $k < n-1$, do not all vanish when $r_h = 0$.*

In particular, this implies that in 4-dimensions, the evaporation is infinite if $M'(0)$ and $M''(0)$ do not both vanish.

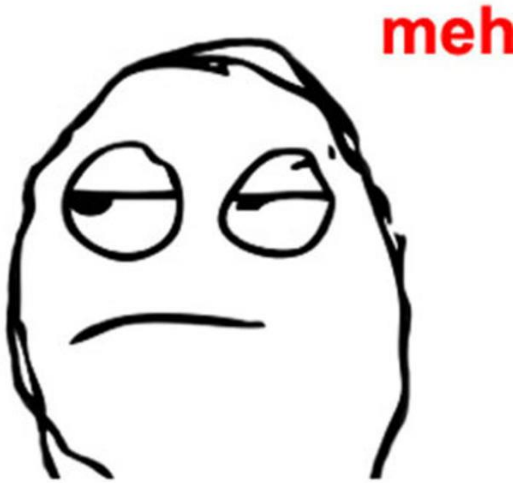
Third Law: Zero temperature (extremal) black hole, which is of nonzero size, is unattainable in finite number of steps.

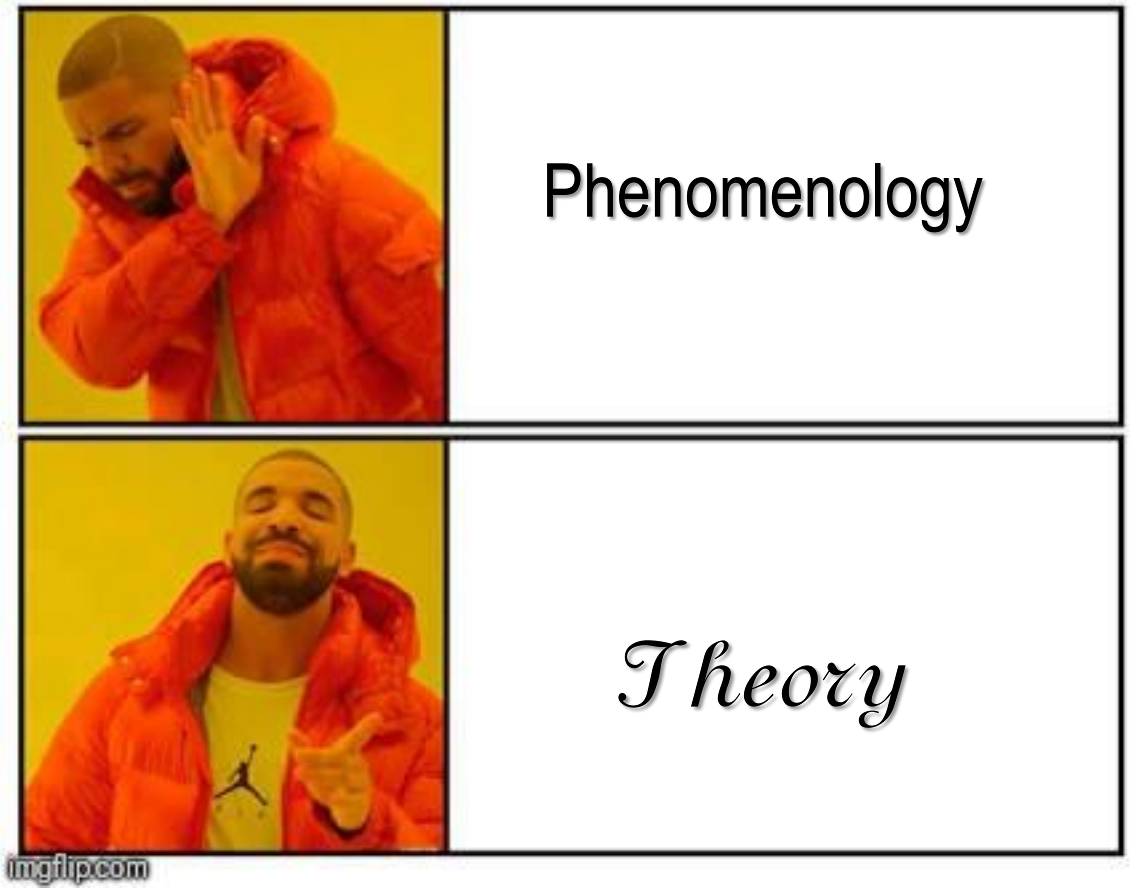
Here we have the opposite scenario, zero mass/size black hole is unattainable in finite time under Hawking evaporation if the temperature is nonzero.

Yuan Yao, Meng-Shi Hou, Yen Chin Ong, "A Complementary Third Law for Black Hole Thermodynamics", [arXiv:1812.03136 [gr-qc]].

Let us Return to GUP with negative α:

Despite its virtue in preventing arbitrarily large white dwarf, and being consistent with some models of quantum gravity, lacks theoretical derivation, of which some of us would go

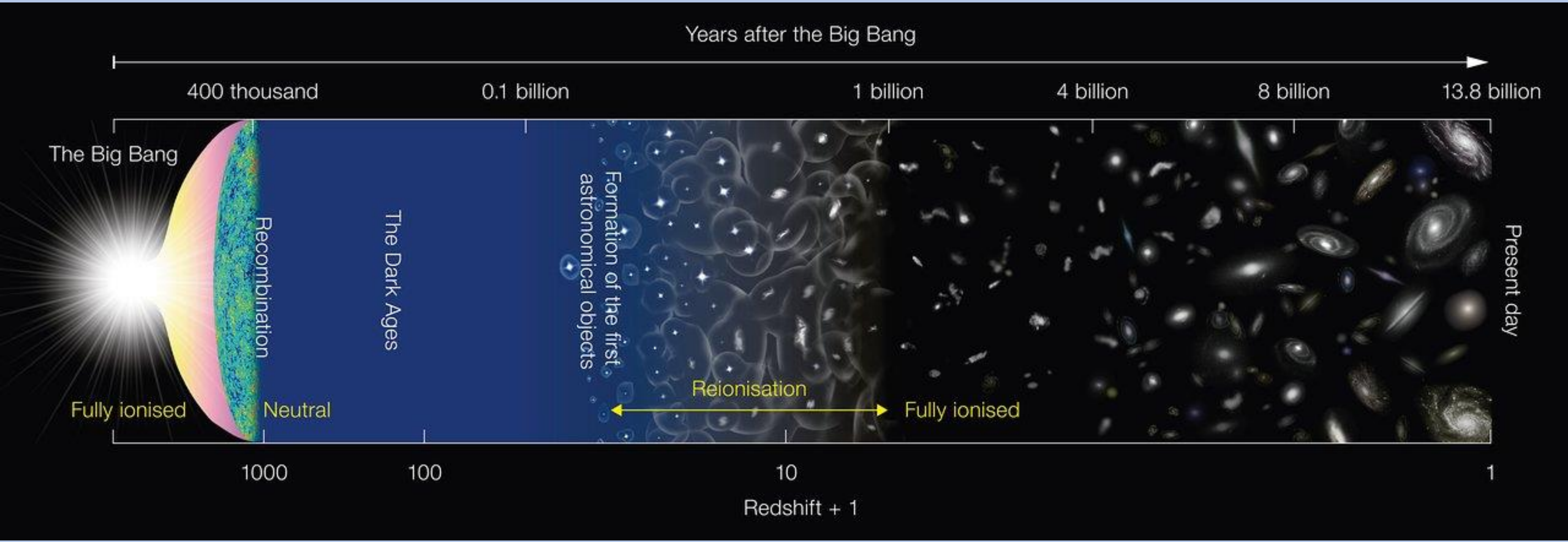




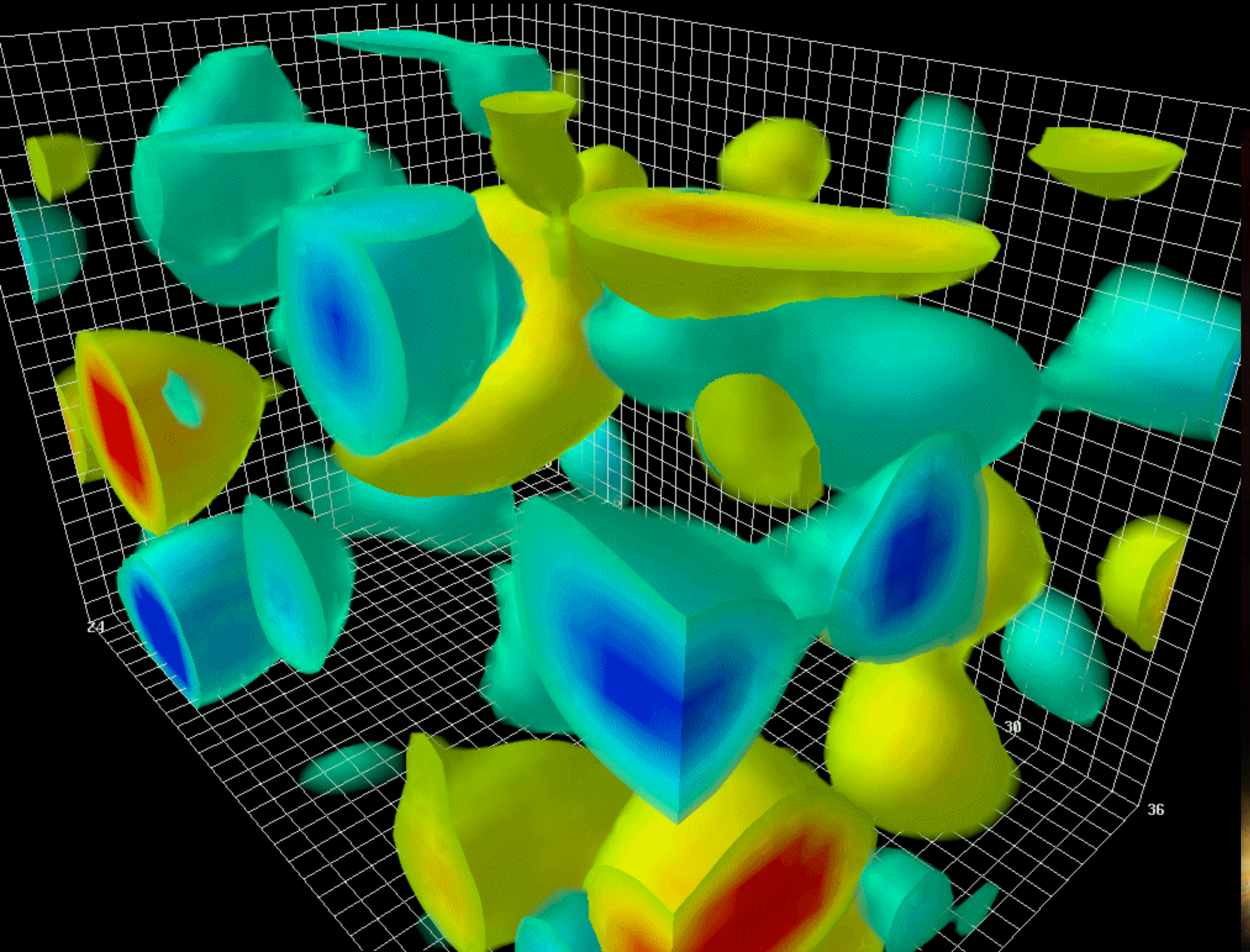
Can we resolve the white dwarf problem with another approach?

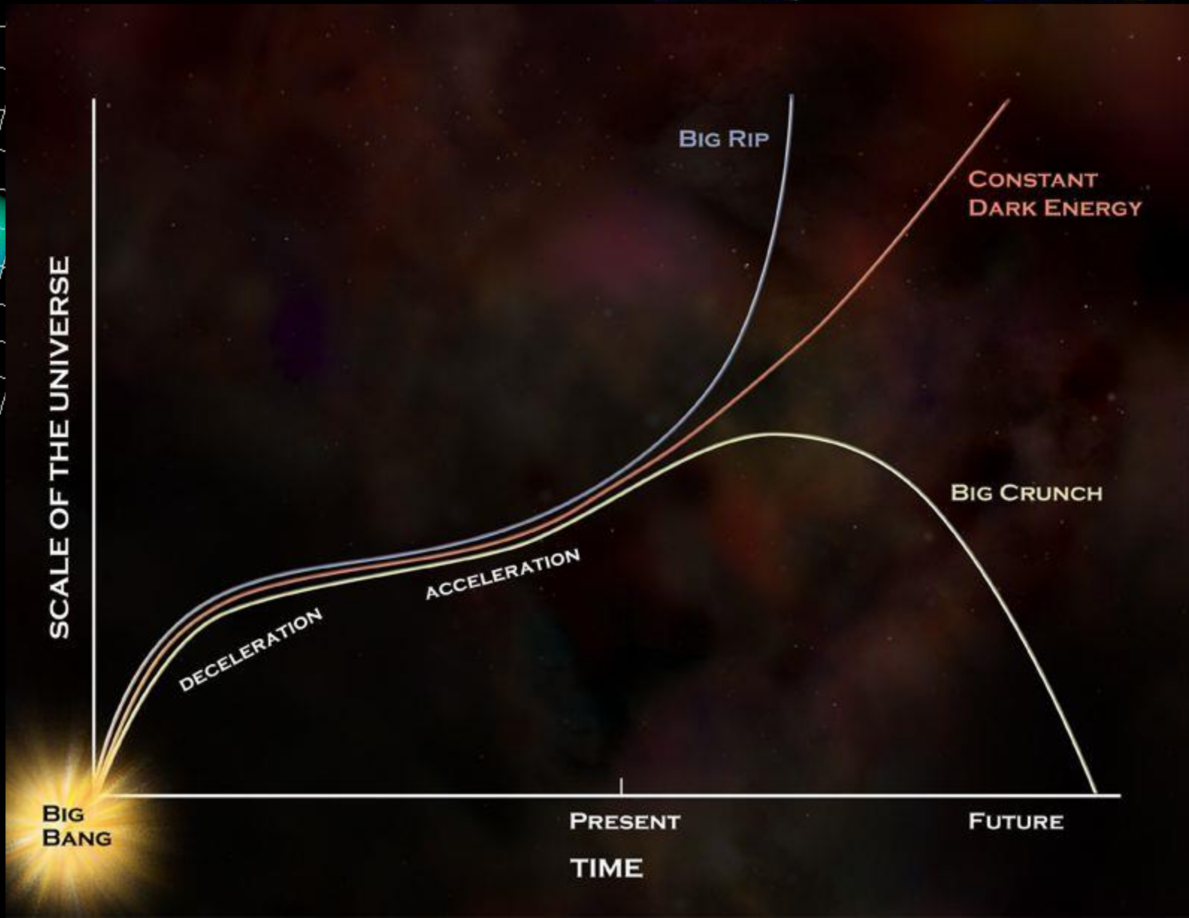
Let Us Consider the Actual Universe

Our Universe is undergoing an accelerated expansion



Cosmological Constant?





Extended Uncertainty Principle

$$\boxed{\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \beta \frac{\hbar (\Delta x)^2}{L^2}\right]}$$

Remark: To recover the correct black hole temperature via the heuristic method, $\beta = \pm 3$

c.f. for S^1: $\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 - C(\Delta x)^2\right)$

B. Bolen, M. Cavaglia, *(Anti-)de Sitter Black Hole Thermodynamics and the Generalized Uncertainty Principle*, Gen. Relativ. Grav. 37, 1255 (2005), arXiv:gr-qc/0411086v1.

M.I. Park*, The Generalized Uncertainty Principle in (A)dS Space and the Modification of Hawking Temperature from the Minimal Length*, Phys. Lett. B659, 698 (2008),
arXiv:0709.2307v4 [hep-th].

C. Bambi, F. R. Urban, *Natural Extension of the Generalised Uncertainty Principle*, Class.Quant.Grav.25:095006 (2008), arXiv:0709.1965v2 [gr-qc].

S. Mignemi, *Extended Uncertainty Principle and the Gometry of (Anti)-de Sitter Space*, Mod.Phys.Lett. A25 (2010) 1697-1703, arXiv:0909.1202v2 [gr-qc].

A Comparison: Uncertainty Principle on Unit Circle

1963: D. Judge published a single page, ultra-dense paper titled "*On the Uncertainty Relation for L_z and ϕ*" [Physics Letters, Vol. 5, No. 3, 1963]: (Details in 1964)

IL NUOVO CIMENTO VOL. XXXI, N. 2 16 Gennaio 1964

On the Uncertainty Relation for Angle Variables.

D. JUDGE

University College - Dublin

(ricevuto il 10 Luglio 1963)

Summary. — The uncertainty relation between the orbital angular momentum component L_z and the corresponding angle φ is discussed. The uncertainty for an angle variable is defined. The formula $\Delta L_z \cdot \Delta\varphi \geqslant \frac{1}{2}\hbar$, which is sometimes quoted, is shown to be incorrect, and an alternative relation, in full accord with the Heisenberg Uncertainty Principle, is derived.

$$\boxed{\frac{\Delta\phi \cdot \Delta L_z}{1 - \frac{3}{\pi^2}(\Delta\phi)^2} \geq \frac{\hbar}{2}}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 - C(\Delta x)^2\right)$$

Heisenberg's Uncertainty Principle: Modern Statement

Theorem: Assume that the self-adjoint operators A and B admits variances, then

H.P. Robertson, Phys. Rev. **34** (1929) 163.

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\langle\psi|\{\hat{A},\hat{B}\}|\psi\rangle^2 + \frac{1}{4}\langle\psi|\Gamma|\psi\rangle^2,$$

where $\hat{A} := A - \langle A\rangle; \hat{B} := B - \langle B\rangle; [\hat{A},\hat{B}] = i\Gamma$, and $\{\cdot,\cdot\}$ is the anti-commutator.

Corollary (Heisenberg's Uncertainty Principle): For the position operator X and momentum operator $P = -i\hbar\, \partial/\partial x$ in position space basis, we have $[X,P] = i\hbar$, and so

"Canonical" Commutation Relation

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\langle\{X,P\}\rangle^2 + \frac{\hbar^2}{4} \geq \frac{\hbar^2}{4}.$$

That is,

$$\boxed{\Delta X \cdot \Delta P \geq \frac{\hbar}{2},}$$

with equality attained if and only if

(1) $\hat{X}|\psi\rangle = c\hat{P}|\psi\rangle,\ c \in \mathbb{C}$, and

(2) $\langle\psi|\{\hat{X},\hat{P}\}|\psi\rangle = 0.$

See also E.H. Kennard, Z. Phys. **44** (1927) 326

$\boxed{\delta p \cdot \delta x \sim \hbar}$

Uncertainty Principle on Unit Circle

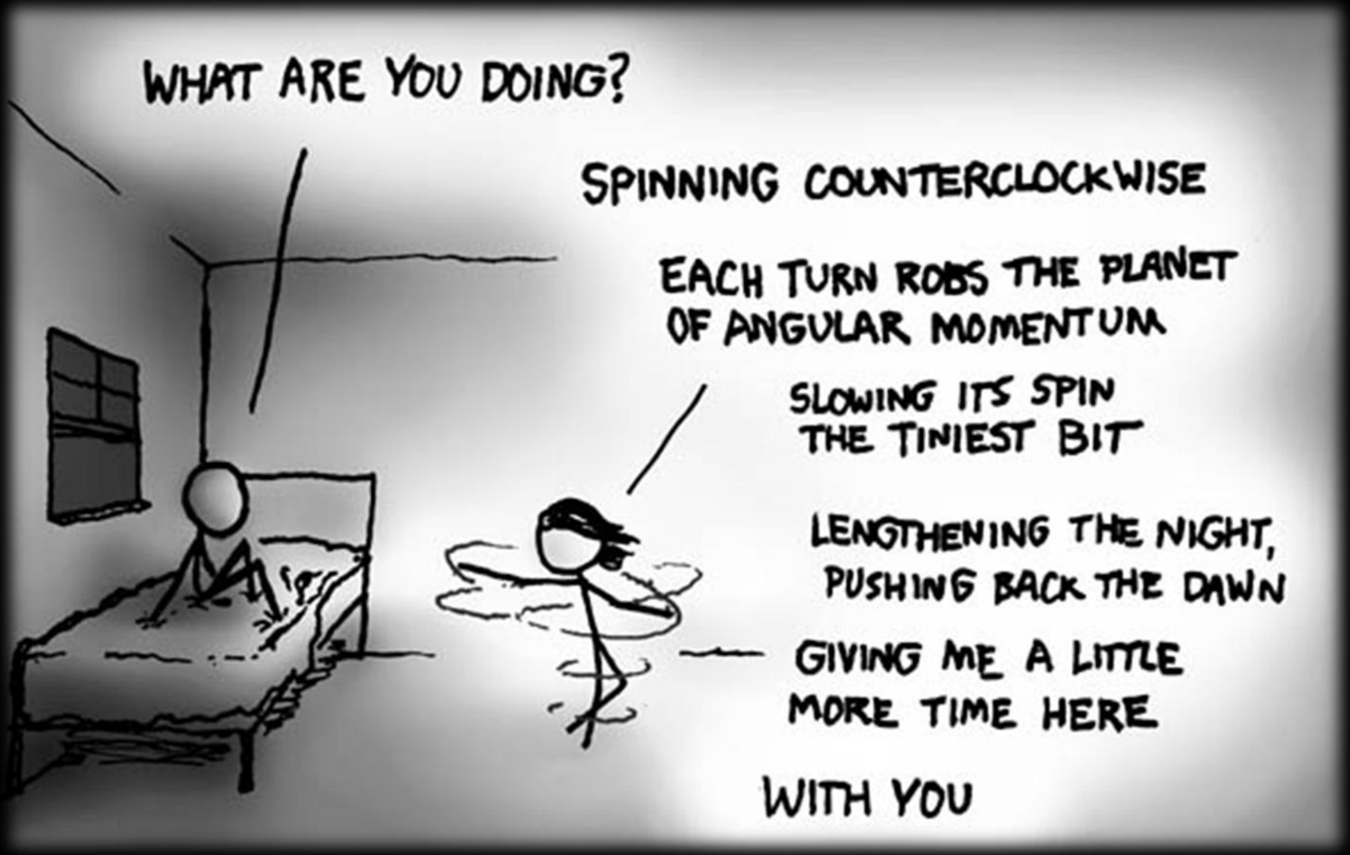
It can be shown that angular momentum and angular coordinate satisfies the canonical commutation relation:

$$[\phi, L_z] = i\hbar\,,\ L_z = -i\hbar\frac{\partial}{\partial\phi}$$

Yet can find states such that ΔL_z is sufficiently small, so that if

$$\Delta L_z \cdot \Delta\phi \geq \frac{\hbar}{2},$$

then $\Delta\phi \geq 2\pi.$ **Something is wrong!**



xkcd

Note: Commutation Relation does *not* imply [the form of] Uncertainty Principle

Recall that the uncertainty relation for any two operators A and B is usually written in the form:

$$(\Delta A)^2(\Delta B)^2 \geq \langle\psi|i[A,B]|\psi\rangle^2,$$

A very important notion that is not usually mentioned in quantum mechanics textbooks is the *domain of definition* of an operator. Like functions, an operator has domain.

Note: Commutation Relation does *not* imply [the form of] Uncertainty Principle

Definition: An *operator* A on the Hilbert space $\mathcal{H}$ is a linear map

$$A: \mathcal{D}(A) \longrightarrow \mathcal{H},$$
$$|\psi\rangle \longmapsto A|\psi\rangle,$$

where $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$, called the *domain of definition*, or simply, domain of A.

$$(\Delta A)^2(\Delta B)^2 \geq \langle\psi|i[A,B]|\psi\rangle^2,$$

LHS is defined on $|\psi\rangle \in \mathcal{D}(A) \cap \mathcal{D}(B)$, the subspace of $\mathcal{H}$ containing all states for which the uncertainties $\Delta A, \Delta B$ are well-defined [i.e. have physical meaning].

RHS is defined *only* for states on the subspace $\mathcal{D}([A,B]) = \mathcal{D}(AB) \cap \mathcal{D}(BA) \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$.

Thus commutation relation does not always allow us to derive the correct uncertainty principle!

F. Gieres, “Mathematical Surprises and Dirac’s Formalism in Quantum Mechanics”, Rep.Prog.Phys. 63 (2000) 1893, arXiv:quant-ph/9907069.

Note: Commutation Relation does *not* imply [the form of] Uncertainty Principle

Take $|\psi\rangle \in \mathcal{D}(A) \cap \mathcal{D}(B)$, define $\hat{A} = A - \langle A\rangle, \hat{B} = B - \langle B\rangle$,
for simplicity, denote $A|\psi\rangle = A\psi$. Then

$$\begin{aligned} |i\langle A\psi, B\psi\rangle - i\langle B\psi, A\psi\rangle| &= \left|i\langle \hat{A}\psi, \hat{B}\psi\rangle - i\langle \hat{B}\psi, \hat{A}\psi\rangle\right| \\ &\le \left|\langle \hat{A}\psi, \hat{B}\psi\rangle\right| + \left|\langle \hat{B}\psi, \hat{A}\psi\rangle\right| = 2\left|\langle \hat{A}\psi, \hat{B}\psi\rangle\right| \\ &\le 2\left|\hat{A}\psi\right| \cdot \left|\hat{B}\psi\right| = 2\Delta A \cdot \Delta B \end{aligned}$$

Triangle Inequality

Cauchy-Schwarz Inequality

$$\boxed{\Delta A \cdot \Delta B \ge \frac{1}{2}|i\langle A\psi, B\psi\rangle - i\langle B\psi, A\psi\rangle|}$$

Both sides are now defined on the *same domain*

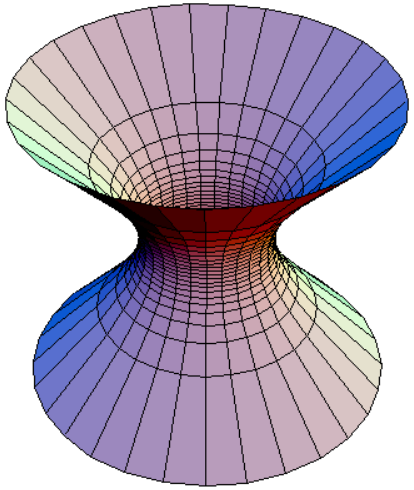
$$|\psi\rangle \in \mathcal{D}(A) \cap \mathcal{D}(B)$$

Thus the uncertainty principle is determined not by commutation relation, but by *Hermitian sesquilinear form*.

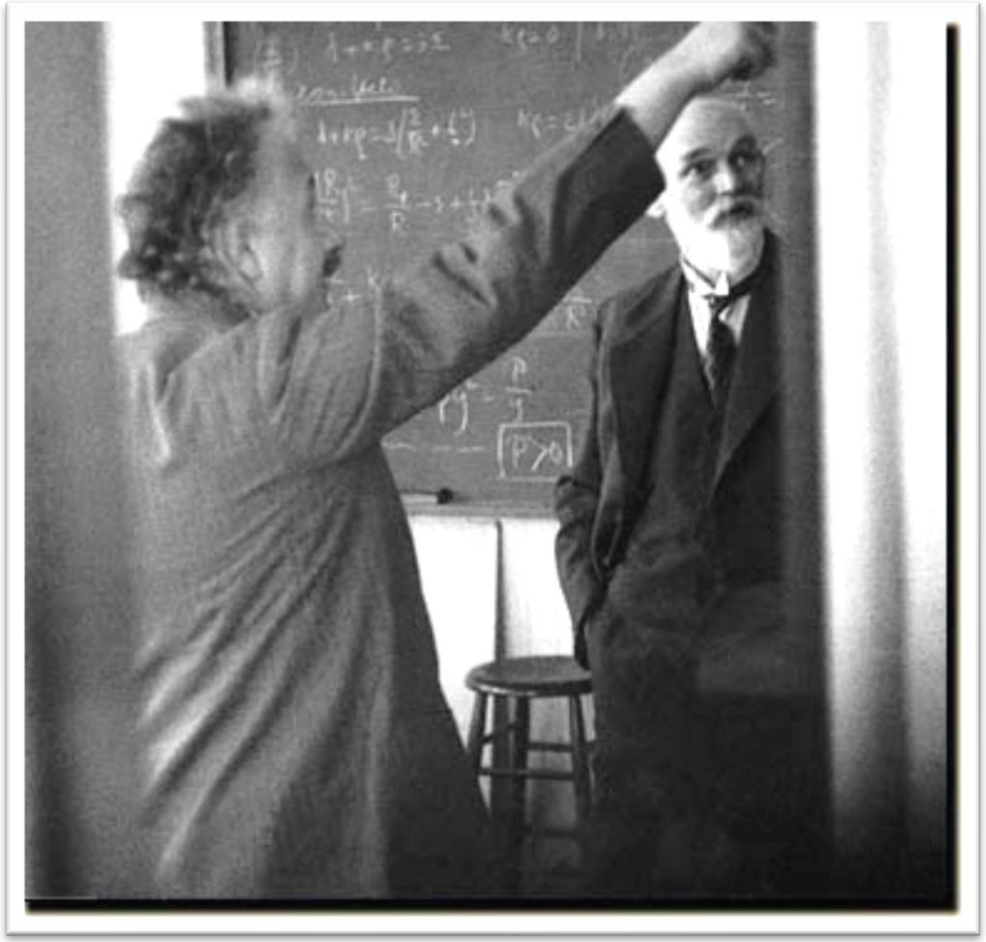
In Short:
Uncertainty Principle Depends on Geometry

The uncertainty principle concerns Fourier transforms of functions, which is nontrivial on curved manifolds.

Alexey Golovnev, Lev Vasil'evich Prokhorov, "Uncertainty Relations in Curved Spaces", J. Phys. A **37** (2004) 2765, [arXiv:quant-ph/0306080].
Thomas Schürmann, "Uncertainty Principle on 3-Dimensional Manifolds of Constant Curvature", Found. Phys. **48** (2018) 716, [arXiv:1804.02551 [quant-ph]].

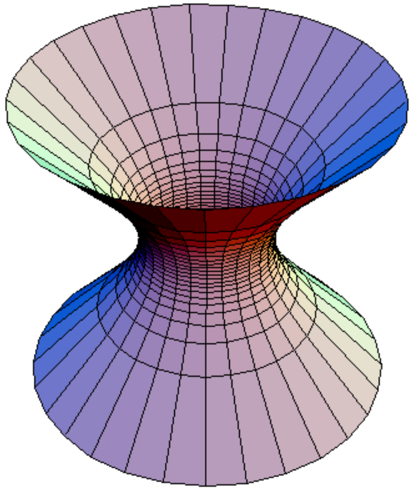


What happens in de Sitter space?



Extended Uncertainty Principle

$$\boxed{\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \beta \frac{\hbar (\Delta x)^2}{L^2}\right]}$$



Remark: To recover the correct black hole temperature via the heuristic method, $\beta = \pm 3$

c.f. for S^1: $\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 - C(\Delta x)^2\right)$

B. Bolen, M. Cavaglia, *(Anti-)de Sitter Black Hole Thermodynamics and the Generalized Uncertainty Principle*, Gen. Relativ. Grav. 37, 1255 (2005), arXiv:gr-qc/0411086v1.

M.I. Park, *The Generalized Uncertainty Principle in (A)dS Space and the Modification of Hawking Temperature from the Minimal Length*, Phys. Lett. B659, 698 (2008),
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C. Bambi, F. R. Urban, *Natural Extension of the Generalised Uncertainty Principle*, Class.Quant.Grav.25:095006 (2008), arXiv:0709.1965v2 [gr-qc].

S. Mignemi, *Extended Uncertainty Principle and the Gometry of (Anti)-de Sitter Space*, Mod.Phys.Lett. A25 (2010) 1697-1703, arXiv:0909.1202v2 [gr-qc].

Extended Generalized Uncertainty Principle

$$\boxed{\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \alpha \frac{L_p^2 (\Delta p)^2}{\hbar} + \beta \frac{\hbar (\Delta x)^2}{L^2}\right]}$$

$$M^{\frac{4}{3}} \sim \frac{c^4 R^2}{2\alpha m_e^{\frac{2}{3}} G^2}\left[1 - \sqrt{1 - \frac{4G\hbar\alpha}{c^3}\left(\frac{\beta}{L^2} + \frac{M^{\frac{2}{3}}}{m_e^{\frac{2}{3}} R^2}\right)}\right]$$

$$R_{1,2}(M) := \frac{\sqrt{2}}{2}\left[\frac{L^2 M^{\frac{4}{3}} m_e^{\frac{4}{3}} - L^2 M^{\frac{2}{3}} \pm \sqrt{\mathscr{F}(\alpha, \beta, M, L)}}{\beta m_e^{\frac{2}{3}}}\right]^{\frac{1}{2}},$$

where

$$\mathscr{F}(\alpha, \beta, M, L) := L^4 (M m_e)^{\frac{8}{3}} - 4L^2 (M m_e)^{\frac{8}{3}} \alpha\beta - 2L^4 M^2 m_e^{\frac{4}{3}} + L^4 M^{\frac{4}{3}}.$$

Extended Generalized Uncertainty Principle

$$\boxed{\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \alpha \frac{L_p^2 (\Delta p)^2}{\hbar} + \beta \frac{\hbar (\Delta x)^2}{L^2}\right]}$$

QG-correction

Classical geometry-correction

$$M^{\frac{4}{3}} \sim \frac{c^4 R^2}{2\alpha m_e^{\frac{2}{3}} G^2}\left[1 - \sqrt{1 - \frac{4G\hbar\alpha}{c^3}\left(\frac{\beta}{L^2} + \frac{M^{\frac{2}{3}}}{m_e^{\frac{2}{3}} R^2}\right)}\right]$$

$$R_{1,2}(M) := \frac{\sqrt{2}}{2}\left[\frac{L^2 M^{\frac{4}{3}} m_e^{\frac{4}{3}} - L^2 M^{\frac{2}{3}} \pm \sqrt{\mathscr{F}(\alpha, \beta, M, L)}}{\beta m_e^{\frac{2}{3}}}\right]^{\frac{1}{2}},$$

where

$$\mathscr{F}(\alpha, \beta, M, L) := L^4 (M m_e)^{\frac{8}{3}} - 4L^2 (M m_e)^{\frac{8}{3}} \alpha\beta - 2L^4 M^2 m_e^{\frac{4}{3}} + L^4 M^{\frac{4}{3}}.$$

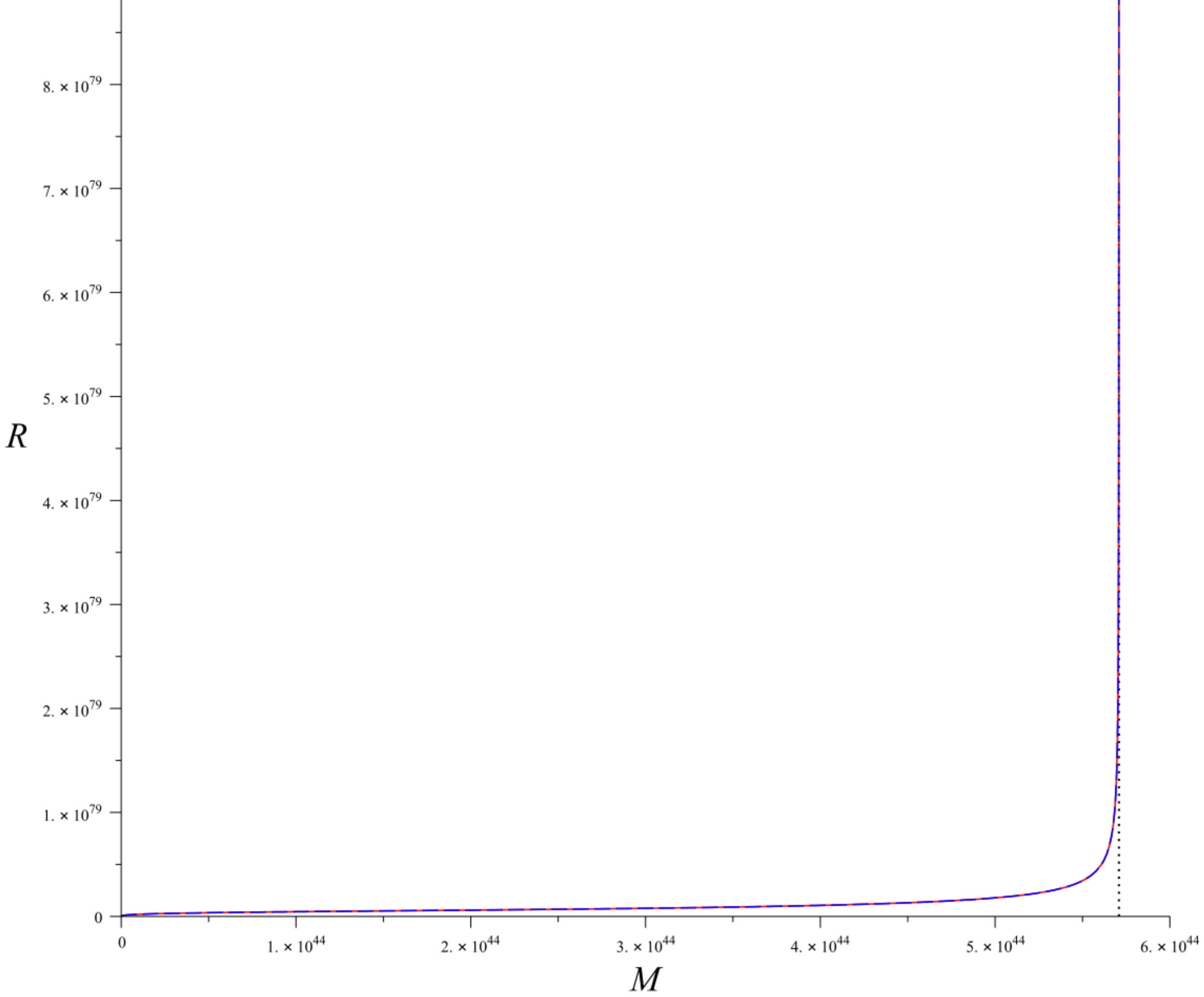


FIG. 2: The mass-radius relationship of an ultra-relativistic white dwarf with EGUP correction, $R_1(M)$. Without EGUP correction, it is simply the vertical Chandrasekhar limit. The effect of EGUP is to cause sufficiently small white dwarfs to deviate away from the Chandrasekhar limit, but note that no star can exist above the limit. Red curve and blue curve correspond to $\alpha = 1$ and $\alpha = -1$ respectively, they pretty much coincide with each other. Varying the magnitude of α up to 10^{11} does not change the result by much.

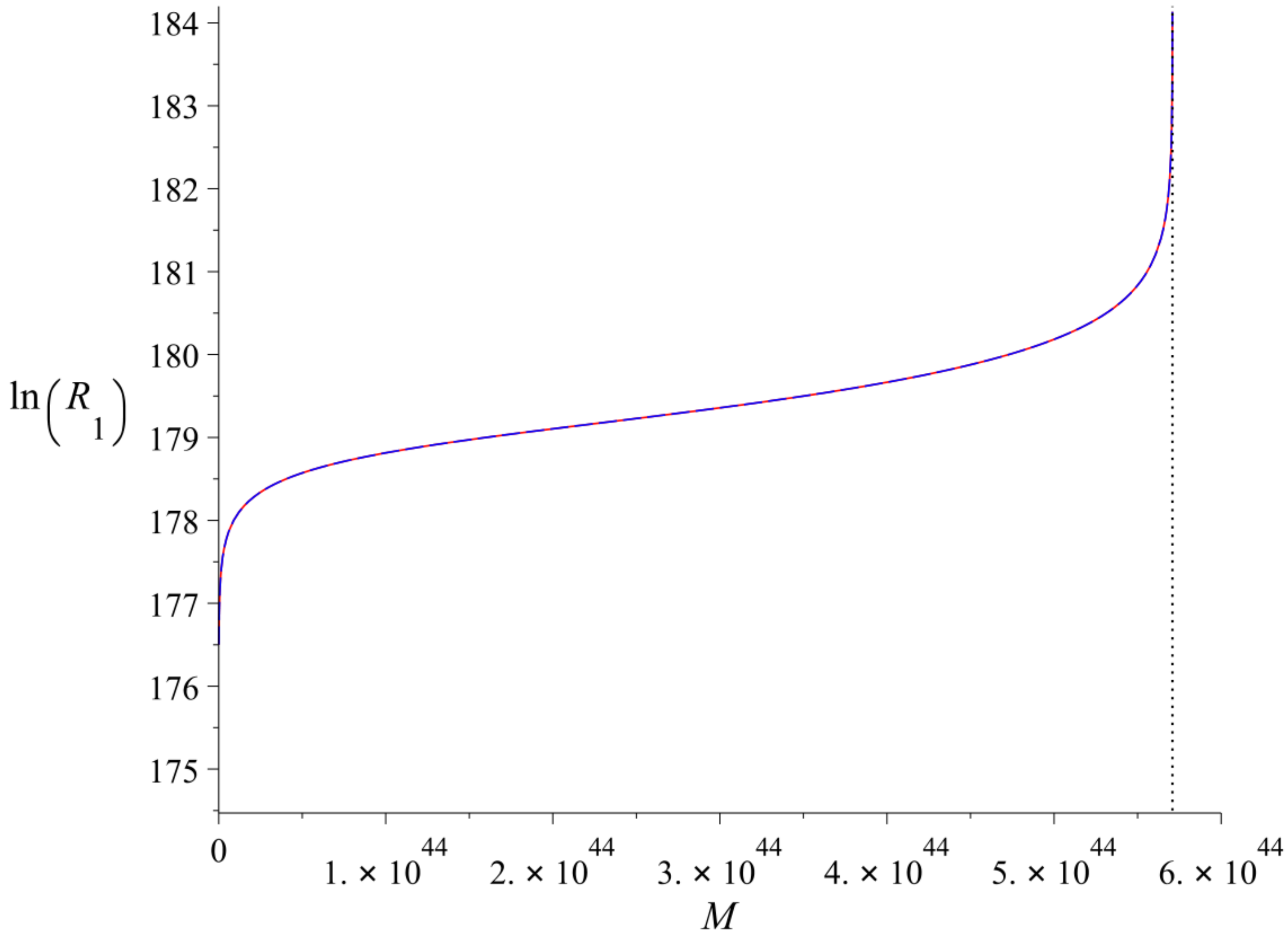


FIG. 4: The mass-radius relationship of an ultra-relativistic white dwarf with EGUP correction, $\log[R_1(M)]$. Red curve and blue curve correspond to $\alpha = 1$ and $\alpha = -1$ respectively, they are still indistinguishable even in log plot. The dashed vertical line corresponds to $M = M_{\rm Ch}$.

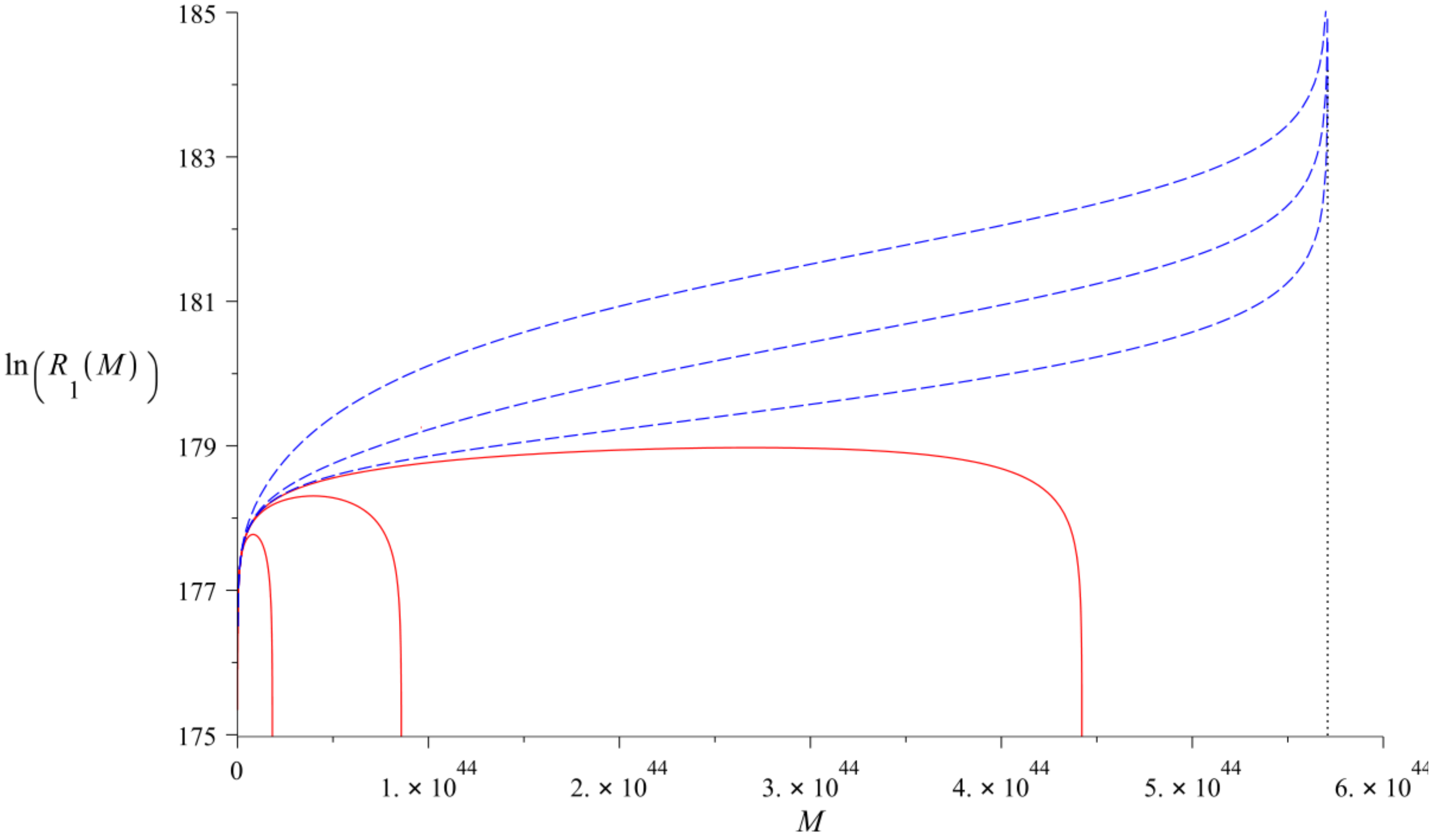


FIG. 5: The mass-radius relationship of an ultra-relativistic white dwarf with EGUP correction, $\log[R_1(M)]$. Solid curves are for $\alpha > 0$ and dashed curves are for $\alpha < 0$. The curves, from top to bottom, correspond respectively to $\alpha = -10^{113}, -10^{112}, -4 \times 10^{110}, 4 \times 10^{110}, 10^{112}, 10^{113}$, respectively. The dashed vertical line corresponds to $M = M_{\rm Ch}$.

A Bound On Cosmological Constant

$$\Lambda < \frac{M^{\frac{2}{3}}}{R^2 m_e^{\frac{2}{3}}} \lesssim \frac{1}{M_{\odot}^{\frac{4}{3}} m_e^{\frac{2}{3}}} = 10^{-36}$$

Still large compared to the required $\sim 10^{-122}$, but small compared to the "natural" value O(1).

Talk Based On

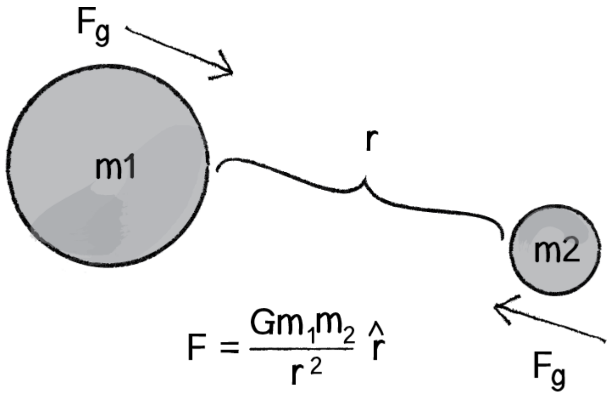
- Yen Chin Ong, "Generalized Uncertainty Principle, Black Holes, and White Dwarfs: A Tale of Two Infinities", JCAP 09 (2018) 015, [arXiv:1804.05176 [gr-qc]].
- Yen Chin Ong, “An Effective Black Hole Remnant via Infinite Evaporation Time Due to Generalized Uncertainty Principle”, JHEP 10 (2018) 195, [arXiv:1806.03691 [gr-qc]].
- Yen Chin Ong, Yuan Yao, “Generalized Uncertainty Principle and White Dwarfs Redux: How Cosmological Constant Protects Chandrasekhar Limit”, Phys. Rev. D 98 (2018) 126018 [arXiv:1809.06348 [gr-qc]].
- Yuan Yao, Meng-Shi Hou, Yen Chin Ong, “A Complementary Third Law for Black Hole Thermodynamics”, [arXiv:1812.03136 [gr-qc]].

Maximum Force/Tension Conjecture

(1) <u>Maximum Force Conjecture (Strong Form):</u>
In 4-dimensions, forces are bounded from above by $F_{\max} = c^4/(4G)$.

(2) <u>Maximum Force Conjecture (Weak Form):</u>
In 4-dimensions, there exists a positive number $K < \infty$, such that forces are bounded from above by $F_{\max} = c^4 K/G$. It is also possible that K is a supremum instead of a maximum.

$$\approx 3.25 \times 10^{43} \text{ Newtons}$$
$$\approx \quad 3 \times 10^{39} \text{ Tonnes}$$



Garry Gibbons, "The Maximum Tension Principle in General Relativity", Found. Phys., 32 (2002) 1891, [arXiv:hepth/0210109].

Christoph Schiller, "General Relativity and Cosmology Derived From Principle of Maximum Power or Force", Int. Jour. Theo. Phys. 44 (2005) 1629, [arXiv:physics/0607090 [physics.gen-ph]].

John Barrow, Garry Gibbons, "Maximum Tension: With and Without a Cosmological Constant", Mon. Not. Roy. Astron. Soc. 446 (2014) 3874-3877, [arXiv:1408.1820 [gr-qc]].

“Thermodynamics Force”

$$\boxed{F = T\frac{\mathrm{d}S}{\mathrm{d}r_h} = \frac{\mathrm{d}M}{\mathrm{d}r_h} = \frac{c^4}{2G}}$$

Essentially just black hole thermodynamics (1st Law).

Mariusz P. Dąbrowski, Hussain Gohar, “Abolishing the Maximum Tension Principle”, Phys. Lett. B **748** (2015) 428, [arXiv:1504.01547 [gr-qc]].

Claimed GUP-corrected Schwarzschild black hole contradicts the maximum force conjecture.

$$\begin{aligned} T[\alpha] &= \frac{1}{8\pi M} + \frac{1}{32}\frac{\alpha}{\pi M^3} + \frac{1}{64}\frac{\alpha^2}{\pi M^5} + \cdots \\ &= T\left(1 + \frac{\alpha}{4M^2} + \frac{\alpha^2}{8M^4} + \cdots\right) \\ &= T\left(1 + 16\pi^2\alpha T^2 + 512\pi^4\alpha^2 T^4 + \cdots\right), \end{aligned}$$

$$\begin{aligned} S[\alpha] &= S\left(1 - \frac{\alpha}{2M^2}\ln M + \frac{\alpha^2}{16M^4} + \cdots\right) \\ &= S\left(1 - \frac{\pi\alpha}{S}\ln S + \frac{\alpha^2\pi^2}{S^2} + \cdots\right), \end{aligned}$$

$$F[\alpha] = F\left(1 - \frac{\alpha\pi}{S} + 16\pi^2\alpha T^2 + \cdots\right)$$

$$T = T[\alpha = 0]$$

$$T[\alpha] = \frac{M}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha}{M^2}}\right)$$

$$\begin{aligned} S[\alpha] &= \int \frac{1}{T}\mathrm{d}M \\ &= 2\pi\left[M^2 + M\sqrt{M^2-\alpha} - \alpha\ln(M + \sqrt{M^2-\alpha})\right] \\ &\quad + \text{const.} \end{aligned}$$

$$\frac{\mathrm{d}S[\alpha]}{\mathrm{d}r_h} = \frac{2\pi(M^2 + M\sqrt{M^2-\alpha} - \alpha)}{\sqrt{M^2-\alpha}}$$

The apparent divergence comes from taking the limit $S \to 0$ and $T \to \infty$. This is inconsistent. In fact, $F \equiv F[\alpha], \forall\alpha.$

$$T[\alpha]\,\frac{\mathrm{d}S[\alpha]}{\mathrm{d}r_h} \equiv \frac{1}{2}$$

The History of Massive Gravity: Phase 1

- Fierz-Pauli Theory [1939]: Construction of theory of massive spin-2 particle . DOF = 5 = 2s+1.
- van Dam-Veltman-Zakharov (vDVZ) Discontinuity [1970]: massless limit does not recover GR; light-bending prediction off by *whopping* 25%.
- Vainshtein Mechanism [1972]: Force the theory to recover the correct limit that matches linearized general relativity.
- Bolware-Deser ghost [1972]: Non-linearity introduced by Vainshtein mechanism excites a 6th DOF – a ghost mode arises.

dGRT (de Rham, Gabadadze, Tolley) Non-Linear Massive Gravity [2010]:

Exorcise BD ghost by adding *even more non-linearity.*

Is everything ok now?

$$S = \frac{{M_{pl}}^2}{2} \int d^4x \sqrt{-g} \left[R - 2\Lambda + m_g^2 (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4) \right]$$

$$\mathcal{L}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2],$$

$$\mathcal{L}_3 = \frac{1}{3}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]),$$

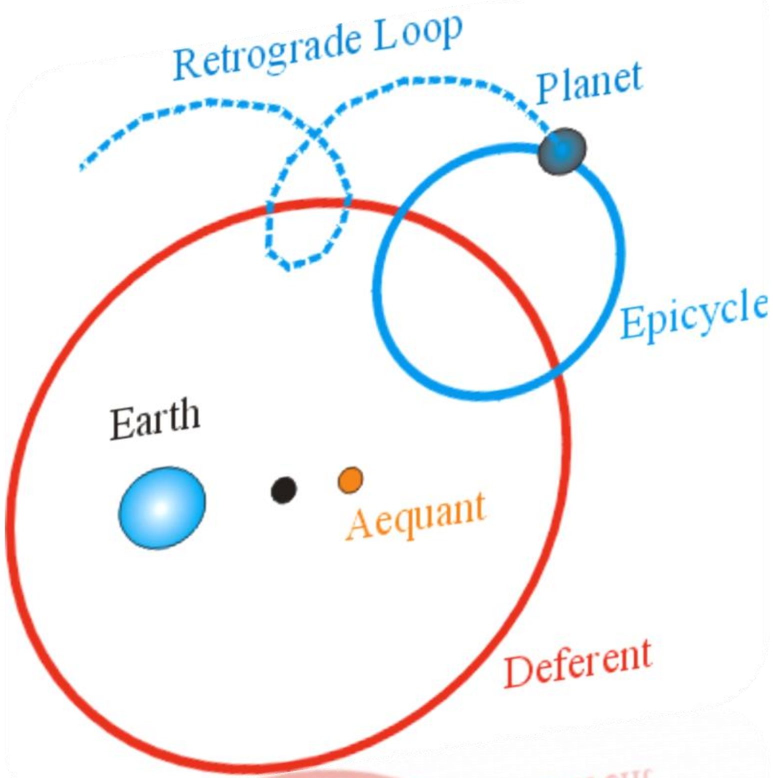
$$\mathcal{L}_4 = \frac{1}{12}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4])$$

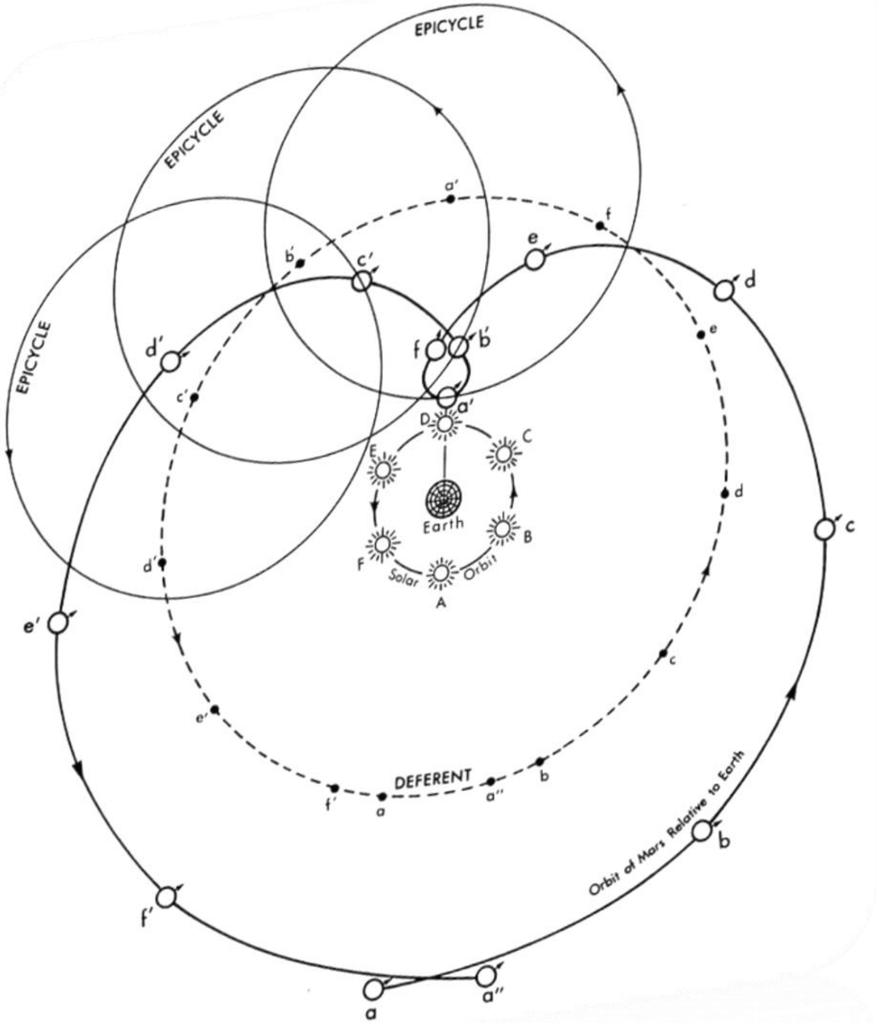
$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_\nu$$

$g_{\mu\nu}$ = physical metric

$f_{\mu\nu}$ = fiducial metric

A Flavor of Ptolemy's Epicircles!





Claudius Ptolemy,
100 – 168

Hints of Problems

Various hints of superluminal propagations:

Gruzinov, [1106.3972 [hep-th]];

Burrage, de Rham, Heisenberg, Tolley, JCAP 1207 (2012) 004, [1111.5549 [hep-th]];

de Fromont, de Rham, Heisenberg, Matas, JHEP 1307 (2013) 067, [1303.0274 [hep-th]];

Chiang, Izumi, Chen, JCAP 12 (2012) 025, [1208.1222v2 [hep-th]].

Instability in Cosmological Solutions:

De Felice, Gumrukcuoglu, Mukohyama, Phys. Rev. Lett. 109 (2012) 171101, [1206.2080 [hep-th]];

De Felice, Gumrukcuoglu, Lin, Mukohyama; JCAP 1305 (2013) 0351, [1303.4154 [hep-th]]; Class.Quant.Grav. 30 (2013) 184004, [1304.0484 [hep-th]];

Kuhnel, Phys.Rev. D 88 (2013) 064024, [1208.1764 [gr-qc]].

Massive Gravity: Further Analysis

Deser & Waldron [2012]: Found 2nd order superluminal shock waves, using eikonal approximation.

Due to *non-linearity that exorcizes BD Ghost!* Analysis is not complete.

Augustin-Louis Cauchy, 1789 – 1857

Izumi & Ong [2013]: Analyzed the structure of first order shocks, using full PDE analysis [Cauchy-Kovalevskaya Theorem] First careful and correct analysis of characteristic of non-linear massive gravity.

Sofia Kovalevskaya, 1850 – 1891

Deser, Izumi, Ong, Waldron [2013]:

Proof of existence of first order superluminal shock, also improved analysis via spin connection. Existence for *local* acausality established.