Subsystem Trace Distance in Conformal Field Theory

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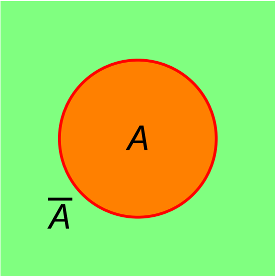
Based on works with Paola Ruggiero and Pasquale Calabrese
Phys. Rev. Lett. 122, 141602 (2019) [1901.10993] and work in progress

11th June 2019 @ ICTS-USTC

Outline

Motivation

- Emerging interests in subsystems rather than the entire system
- Entanglement entropy in
  - many-body quantum systems Vidal-Latorre-Rico-Kitaev 0211074, Kitaev-Preskill 0510092, Levin-Wen 0510613, $\cdots$
  - CFTs Calabrese-Cardy 0405152, $\cdots$
  - holographic theories and gravity Ryu-Takayanagi 0603001, $\cdots$
  - $\cdots$
- Entanglement is one of the most intriguing character in a quantum system that is different from a classical one
- Entire system in state ρ
- Subsystem reduced density matrix (RDM) $\rho_A = \mathrm{tr}_{\bar{A}}\rho$
- von Neumann entanglement entropy $\boxed{S_A = -\mathrm{tr}_A(\rho_A \log \rho_A)}$
- Rényi entanglement entropy $S_A^{(n)} = -\frac{\log \mathrm{tr}_A \rho_A^n}{n-1}$ for integer n
- $S_A = \lim_{n \to 1} S_A^{(n)}$



- Thermalization of subsystems in nonequilibrium evolutions of isolated quantum systems Calabrese-Cardy 0503393, Rigol-Dunjko-Olshanii 0708.1324
  - The initial state is a pure state $|\psi(0)\rangle$ that is not an energy eigenstate of the Hamiltonian H
  - The entire system is always in a pure state $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\psi(0)\rangle$
  - In long time the subsystem "feels" likes in a thermal state

$$\lim_{t\to\infty} \rho_A(t) = \rho_{A,\mathrm{therm}}$$

  - For nonintegral/chaotic systems the thermal state is the ordinary canonical ensemble state
  - For integrable system it is the generalised Gibbs ensemble Rigol-Dunjko-Yurovsky-Olshanii 0604476
  - The system acts as its own thermal bath
- Eigenstate thermalization hypothesis Deutsch 1991, Srednicki 1994
  - Local measures cannot distinguish a highly excited energy eigenstate from a thermal state

- Black hole information paradox Bekenstein 1974, Hawking 1975
  - ◇ Matter in a pure state collapses to a black hole
  - ◇ Classically there seems nothing wrong
  - ◇ Quantum effect leads to a paradox
  - ◇ The black hole has Bekenstein-Hawking entropy
  - ◇ It gives out Hawking radiation which is thermal
  - ◇ The black hole appears thermal for a local observer
  - ◇ After the black hole evaporates out, the initial information seems lost
  - ◇ Semiclassical gravity cannot distinguish the high energy energy eigenstate and thermal state
- Knowing the entanglement entropy is not enough
- In all the above examples one needs to distinguish quantitatively the RDMs of a subsystem
- Many measures can distinguish various states: correlation functions, entanglement entropy, Rényi entropy, Holevo information

  $\cdots$, He-Lin-JZ 1703.08724, 1708.05090, Guo-Lin-JZ 1808.02873, 1812.11753, $\cdots$
- More importantly, we need *distance* in the Hilbert space of the RDM

- Mathematically, a distance should: Nielsen-Chuang 2000
  - be nonnegative $d(\rho,\sigma) \geq 0$
  - be equal to zero if and only if its two inputs are exactly the same $d(\rho,\sigma) = 0 \Leftrightarrow \rho = \sigma$
  - be symmetric in its inputs $d(\rho,\sigma) = d(\sigma,\rho)$
  - obeys the triangular inequality $d(\rho,\sigma) + d(\sigma,\lambda) \geq d(\rho,\lambda)$
- Well-studied quantity in literature: relative entropy Nielsen-Chuang 2000

$$S(\rho\|\sigma) = \mathrm{tr}(\rho\log\rho) - \mathrm{tr}(\rho\log\sigma)$$

- It is a very important quantity and can be calculated in qauntum field theory Lashkari 1404.3216, 1508.03506
- Drawbacks: it is not a distance! Watrous 2018
  - It is not symmetric in its inputs
  - It does not satisfy the triangle inequality
  - It can be infinity for some (very close) inputs

- There are an infinite number of definitions of distance
- Any matrix norm $\|A\|$ gives a definition of distance $\|\rho-\sigma\|$
  - $\|A\| \geq 0$
  - $\|A\| = 0 \Leftrightarrow A = 0$
  - $\|\lambda A\| \geq |\lambda|\|A\|$
  - $\|A\| + \|B\| \geq \|A+B\|$
- Schatten n-norm gives n-distance with $n \geq 1$ Watrous 2018

$$\boxed{D_n(\rho,\sigma) = \frac{1}{2^{1/n}}\|\rho-\sigma\|_n}$$

- n-norm $\|A\|_n = (\sum_i a_i^n)^{1/n}$ with a_i being the singular values of A
- By definition $0 \leq D_n(\rho,\sigma) \leq 1$
  - The nearest states $D_n(\rho,\sigma) = 0 \Leftrightarrow \rho = \sigma$
  - The farthest states $D_n(\rho,\sigma) = 1 \Leftrightarrow \rho\sigma = 0$

- $D = D_1$ trace distance, D_2 Frobenius distance or Hilbert-Schmidt distance, D_∞ operator distance
- $D_{2,4}$ have been calculated Basu-Das-Datta-Pal 1705.03001, He-Lin-JZ 1708.05090
- Trace distance is special $\boxed{D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}|\rho - \sigma|}$
- It provides an upper bound for the difference between expectation values of observable $\mathcal{O}$ (with $\mathcal{O}^2 = 1$) in different states Fagotti-Essler 1302.6944

$$|\mathrm{tr}[(\rho - \sigma)\mathcal{O}]| \leq 2D(\rho,\sigma)$$

- Trace distance is UV cutoff-independent and scale invariant in CFT, being different from the n-distance with $n > 1$, as we will see later
- In subsystem eigenstate thermalization hypothesis, trace distance is proposed to characterize the difference of the RDMs of the pure and thermal states Lashkari-Dymarsky-Liu 1610.00302, 1611.08764

- Difficult to impose the absolute value in definition of trace distance
- RDMs are Hermitian: $\|\rho-\sigma\|_{n_e}^{n_e} = \mathrm{tr}(\rho-\sigma)^{n_e}$ for an even integer n_e
- Replica trick for trace distance JZ-Ruggiero-Calabrese 1901.10993, ...
  - ◇ We first calculate $\mathrm{tr}(\rho-\sigma)^{n_e}$ for general even integer n_e
  - ◇ And then take $n_e \to 1$ limit

$$\mathrm{tr}|\rho-\sigma| = \lim_{n_e\to 1} \mathrm{tr}(\rho-\sigma)^{n_e}$$

- The matrix $\rho-\sigma$ is not semi-positive definite
- $\mathrm{tr}(\rho-\sigma)^{n_e}$ and $\mathrm{tr}(\rho-\sigma)^{n_o}$ behave very differently

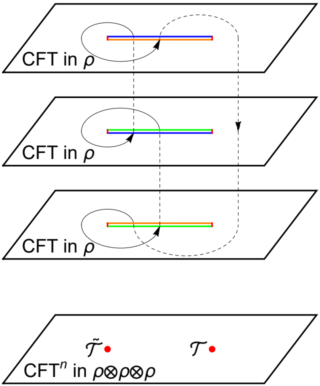
$$\lim_{n_o\to 1} \mathrm{tr}(\rho-\sigma)^{n_o} = 0$$

- No guarantee that the strategy would work for all systems
- It works at least for the examples we calculate
- Similar to replica trick for entanglement entropy and entanglement negativity Calabrese-Cardy 0405152, Calabrese-Cardy-Tonni 1206.3092

Replica trick for entanglement entropy Calabrese-Cardy 0405152

- Replica trick leads to a CFT on an n-fold Riemann surface
- It is equivalent to an n-fold CFT, which we call CFT^n, on a one-fold Riemann surface
- Boundary conditions in CFT^n can be replaced by insertion of twist operators $\mathcal{T}$, $\tilde{\mathcal{T}}$
- For one intervals $A = [0, \ell]$

$$\mathrm{tr}_A \rho_A^n = \langle \mathcal{T}(\ell,\ell)\tilde{\mathcal{T}}(0,0)\rangle_{\rho_n}$$



- We review two methods to evaluate the entanglement entropy
- Both of them will be useful for the calculation of trace distance

Short interval expansion from OPE of twist operators

- A short interval A in a general translationally invariant state ρ
- Twist operators are primary operators Calabrese-Cardy 0405152
- Operator product expansion (OPE) of twist operators Headrick 1006.0047, Calabrese-Cardy-Tonni 0905.2069, Chen-JZ 1309.5453

$$\mathcal{T}(z,\bar{z})\tilde{\mathcal{T}}(0,0) = \frac{c_n}{z^{2h_n}\bar{z}^{2\bar{h}_n}} \sum_K d_K \sum_{r,s\geq 0} \frac{a_K^r}{r!} \frac{\bar{a}_K^s}{s!} z^{h_K+r} \bar{z}^{\bar{h}_K+s} \partial^r \bar{\partial}^s \Phi_K(0,0)$$

- We need to construct quasaprimary operators Φ_K in CFT^n
- The OPE coefficient d_K Calabrese-Cardy-Tonni 0905.2069
- Thanks to the translation symmetry of the state, we only need

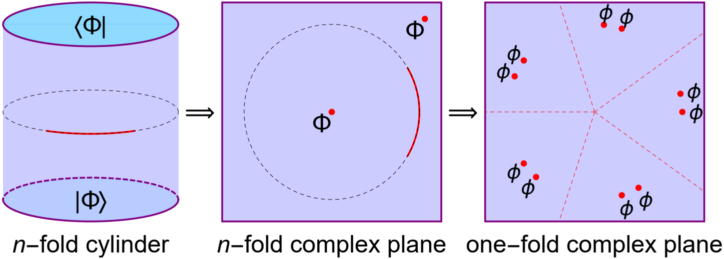
$$\Phi_K^{j_1 j_2 \cdots j_k} = \mathcal{X}_1^{j_1} \mathcal{X}_2^{j_2} \cdots \mathcal{X}_k^{j_k}$$

- We will use the OPE of twist operators to calculate the leading short interval expansion of trace distance

Exact excited state EE from conformal transformations

- State/operator correspondence in 2D CFT $|\phi\rangle = \phi(0)|0\rangle$
- One interval $A = [0, \ell]$ on a cylinder capped with primary operator ϕ
- Replica trick leads to a one-fold CFT on an n-fold cylinder
- Note the n-fold operator $\Phi = \prod_{j=0}^{n-1} \phi_j$
- The n-fold cylinder can be mapped to an n-fold complex plane, which can be further mapped to a one-fold complex plane

  Alcaraz-Berganza-Sierra 1101.2881, 1109.5673



- Partition function on the n-fold excited state cylinder

$$F_\phi^{(n)}(\ell) \equiv \frac{\mathrm{tr}_A \rho_{A,\phi}^n}{\mathrm{tr}_A \rho_{A,0}^n} = \frac{\mathrm{i}^{2ns_\phi}}{\alpha_\phi^n} \Big(\frac{2}{n} \sin \frac{\pi \ell}{L} \Big)^{2n\Delta_\phi}$$
$$\times \Big\langle \prod_{j=0}^{n-1} \big[f_{j,\ell}^{h_\phi} \bar f_{j,\ell}^{\bar h_\phi} f_j^{h_\phi} \bar f_j^{\bar h_\phi} \phi(f_{j,\ell}, \bar f_{j,\ell}) \phi^\dagger(f_j, \bar f_j) \big] \Big\rangle_{\mathcal{C}}$$

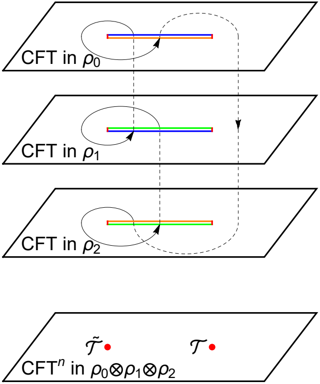
- Universal ground state partition function

$$\mathrm{tr}_A \rho_{A,0}^n = c_n \Big(\frac{L}{\pi \epsilon} \sin \frac{\pi \ell}{L} \Big)^{-4h_n}$$

- Positions of operators on the plane $f_{j,\ell} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}(j + \frac{\ell}{L})}$, $f_j = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} j}$
- The function $F_\phi^{(n)}(\ell)$ would be useful to calculations of the leading order short interval expansion of trace distance
- The method would also be useful to calculate exact n-distances

Replica trick for trace distance JZ-Ruggiero-Calabrese 1901.10993, ...

- The previous techniques for the entanglement entropy can be applied to the trace distance
- $\mathrm{tr}_A(\rho_A - \sigma_A)^n$ is a sum of 2^n terms, each term corresponds to an n-fold Riemann surface
- It is equivalent to CFT^n with each copy of the CFT in different states
- Twist operators still apply, at least after we sum them together He-Lin-JZ 1708.05090



$$\mathrm{tr}_A(\rho_{A,0}\rho_{A,1}\cdots\rho_{A,n-1}) + \mathrm{perm.} = \langle \mathcal{T}(\ell,\ell)\tilde{\mathcal{T}}(0,0)\rangle_{\otimes_{j=0}^{n-1}\rho_j} + \mathrm{perm.}$$

- From OPE of twist operators He-Lin-JZ 1708.05090

$$\mathrm{tr}_A(\rho_A-\sigma_A)^n = c_n\Big(\frac{\epsilon}{\ell}\Big)^{4h_n} \sum_{\{\mathcal{X}_1,\cdots,\mathcal{X}_n\}} \ell^{\Delta_{\mathcal{X}_1}+\cdots+\Delta_{\mathcal{X}_n}} b_{\mathcal{X}_1\cdots\mathcal{X}_n} \times \big(\langle\mathcal{X}_1\rangle_\rho-\langle\mathcal{X}_1\rangle_\sigma\big)\cdots\big(\langle\mathcal{X}_n\rangle_\rho-\langle\mathcal{X}_n\rangle_\sigma\big)$$

- Many terms have been canceled because of permutations
- Leading term in short interval expansion

$$\mathrm{tr}_A(\rho_A-\sigma_A)^n = c_n\Big(\frac{\epsilon}{\ell}\Big)^{4h_n}\Big[\ell^{n\Delta_\phi} b_{\phi^n}\big(\langle\phi\rangle_\rho-\langle\phi\rangle_\sigma\big)^n + o(\ell^{n\Delta_\phi})\Big]$$

- ϕ is the Hermitian quasiprimary operator with the lowest scaling dimension that satisfies $\langle\phi\rangle_\rho \neq \langle\phi\rangle_\sigma$
- It only applies to the cases without degeneracy, i.e. that at scaling dimension Δ_ϕ there is only one such operator
- The power $\ell^{n\Delta_\phi}$ still applies to degenerate cases, but the overall coefficients are more complex

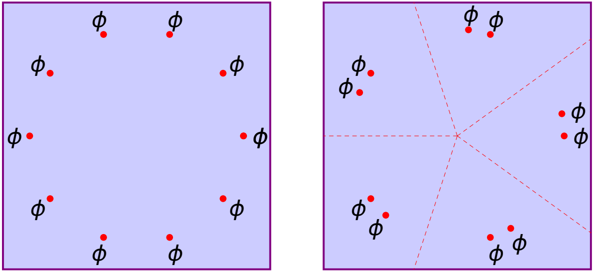
- We take $n_e \to 1$ limit for even integer $n_e = 2p$, $p = 1, 2, \cdots$
- Leading short interval expansion of trace distance

$$D(\rho_A, \sigma_A) = \frac{x_\phi \ell^{\Delta_\phi}}{2} \left| \frac{\langle\phi\rangle_\rho - \langle\phi\rangle_\sigma}{\mathrm{i}^{s_\phi}\sqrt{\alpha_\phi}} \right| + o(\ell^{\Delta_\phi})$$

- The coefficient x_ϕ from the OPE coefficient of twist operators

$$x_\phi = \lim_{p \to 1/2} \frac{\mathrm{i}^{2ps_\phi}}{\alpha_\phi^p} \Big\langle \prod_{j=0}^{2p-1} \big[f_j^{h_\phi} \bar{f}_j^{\bar{h}_\phi} \phi(f_j, \bar{f}_j) \big] \Big\rangle_{\mathcal{C}}, \quad f_j = \mathrm{e}^{\frac{\pi \mathrm{i} j}{p}}$$

- It is independent of the UV cutoff ϵ, being different from $D_n(\rho_A, \sigma_A)$, with $n > 1$
- It is scaling invariant, i.e. that it is dependent on ℓ and L only through ℓ/L, noting $\langle\phi\rangle_\rho \sim 1/L^{\Delta_\phi}$
- It is universal in the sense that it applies to general translationally invariant states in general CFTs (with no degeneracy)



- Remember the function $F_\phi^{(p)}(\ell)$ in calculation of Rényi entropy

$$F_\phi^{(p)}(\ell) = \frac{\mathrm{i}^{2ps_\phi}}{\alpha_\phi^p}\Big(\frac{2}{p}\sin\frac{\pi\ell}{L}\Big)^{2p\Delta_\phi}\Big\langle \prod_{j=0}^{p-1}\big[f_{j,\ell}^{h_\phi}\bar f_{j,\ell}^{\bar h_\phi} f_j^{h_\phi}\bar f_j^{\bar h_\phi}\phi(f_{j,\ell},\bar f_{j,\ell})\phi(f_j,\bar f_j)\big]\Big\rangle_{\mathcal{C}}$$

- The way to calculate the coefficient $x_\phi = \dfrac{F_\phi^{(1/2)}(L/2)}{2^{2\Delta_\phi}}$
- It is related to the Rényi entropy $S_{A,\phi}^{(1/2)}$ with index $1/2$

Exact n_e-distance from conformal transformations

- From conformal transformations ($\alpha_{\mathcal{X}} = \alpha_{\mathcal{Y}} = 1$)

$$
\begin{aligned}
\mathrm{tr}_A(\rho_{A,\mathcal{X}} - \rho_{A,\mathcal{Y}})^n = c_n \Big(\frac{L}{\pi\epsilon} \sin\frac{\pi\ell}{L} \Big)^{-4h_n} \\
\times \sum_{\mathcal{S}\subseteq\mathcal{S}_0} \Big\{ (-)^{|\mathcal{S}|} \mathrm{i}^{2(|\bar{\mathcal{S}}|s_{\mathcal{X}} + |\mathcal{S}|s_{\mathcal{Y}})} \Big(\frac{2}{n} \sin\frac{\pi\ell}{L} \Big)^{2(|\bar{\mathcal{S}}|\Delta_{\mathcal{X}} + |\mathcal{S}|\Delta_{\mathcal{Y}})} \\
\times \Big\langle \Big[\prod_{j\in\bar{\mathcal{S}}} \Big(f_{j,\ell}^{h_{\mathcal{X}}} \bar{f}_{j,\ell}^{\bar{h}_{\mathcal{X}}} f_j^{h_{\mathcal{X}}} \bar{f}_j^{\bar{h}_{\mathcal{X}}} \mathcal{X}(f_{j,\ell}, \bar{f}_{j,\ell}) \mathcal{X}^\dagger(f_j, \bar{f}_j) \Big) \Big] \\
\times \Big[\prod_{j\in\mathcal{S}} \Big(f_{j,\ell}^{h_{\mathcal{Y}}} \bar{f}_{j,\ell}^{\bar{h}_{\mathcal{Y}}} f_j^{h_{\mathcal{Y}}} \bar{f}_j^{\bar{h}_{\mathcal{Y}}} \mathcal{Y}(f_{j,\ell}, \bar{f}_{j,\ell}) \mathcal{Y}^\dagger(f_j, \bar{f}_j) \Big) \Big] \Big\rangle_{\mathbb{C}} \Big\}
\end{aligned}
$$

- The n_e-distance $D_{n_e}(\rho_{A,\mathcal{X}}, \rho_{A,\mathcal{Y}})$ with $n_e \geq 2$ is UV cutoff-dependent and is not scale invariant
- We use another UV cutoff-independent and scale invariant quantity

$$
\mathcal{D}_{n_e}(\rho_{A,\mathcal{X}}, \rho_{A,\mathcal{Y}}) = \frac{\mathrm{tr}_A(\rho_{A,\mathcal{X}} - \rho_{A,\mathcal{Y}})^{n_e}}{2\mathrm{tr}_A \rho_{A,0}^{n_e}}
$$

2D free compact massless boson theory

- 2D free massless real boson theory compactified on a unit circle

$$\mathcal{L} \propto \partial\varphi\bar{\partial}\varphi, \quad \varphi \sim \varphi + 2\pi$$

- A 2D CFT with $c = 1$
- Equation of motion $\partial\bar{\partial}\varphi = 0$, leads to $\varphi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$
- Spectrum: primary operators with conformal weights $(h, \bar{h})$
- Identity with $(0, 0)$
- Currents $J = \mathrm{i}\partial\phi$, $\bar{J} = \mathrm{i}\bar{\partial}\bar{\phi}$ and $J\bar{J}$ with $(1, 0)$, $(0, 1)$ and $(1, 1)$
- Vertex operators $V_{\alpha,\bar{\alpha}} = \exp(\mathrm{i}\alpha\phi + \mathrm{i}\bar{\alpha}\bar{\phi})$ with $(\alpha^2/2, \bar{\alpha}^2/2)$, $\alpha, \bar{\alpha} = 0, \pm 1, \cdots$
- RDMs $\rho_{\mathcal{X}} = \mathrm{tr}_{\bar{A}}|\mathcal{X}\rangle\langle\mathcal{X}|$: ρ_0, $\rho_{\alpha,\bar{\alpha}}$, ρ_J, $\rho_{\bar{J}}$, $\rho_{J\bar{J}}$
- Excited state entanglement entropy Alcaraz-Berganza-Sierra 1101.2881, 1109.5673, Essler-Läuchli-Calabrese 1211.2474, 1406.7477

Leading short interval trace distance JZ-Ruggiero-Calabrese 1901.10993, ...

Cases with no degeneracy

- We take the two RDMs $\rho_{\alpha,\bar{\alpha}}$, $\rho_{\alpha',\bar{\alpha}}$ as an example
- The expectation values

$$\langle J\rangle_{\alpha,\bar{\alpha}} = \frac{2\pi \mathrm{i}\alpha}{L}, \quad \langle \bar{J}\rangle_{\alpha,\bar{\alpha}} = -\frac{2\pi \mathrm{i}\bar{\alpha}}{L}$$

- The coefficient $x_J = 1/\pi$ from the function Essler-Läuchli-Calabrese 1211.2474, 1406.7477

$$F_J^{(p)}(\ell) = \left(\frac{2}{p}\sin\frac{\pi\ell}{L}\right)^{2p} \frac{\Gamma^2\left(\frac{1+p+p\csc\frac{\pi\ell}{L}}{2}\right)}{\Gamma^2\left(\frac{1-p+p\csc\frac{\pi\ell}{L}}{2}\right)}$$

- The leading short interval trace distance from the universal formula

$$D(\rho_{\alpha,\bar{\alpha}}, \rho_{\alpha',\bar{\alpha}}) = \frac{|\alpha-\alpha'|\ell}{L} + o\left(\frac{\ell}{L}\right)$$

- The other examples are similar

Cases with degeneracy

- Cases with degeneracy, i.e. $\langle J\rangle_\rho \neq \langle J\rangle_\sigma$, $\langle \bar{J}\rangle_\rho \neq \langle \bar{J}\rangle_\sigma$, are more complicated
- The universal formula for nondegenerate cases does not apply
- We need to redo the calculation from the beginning
- One example of leading trace distances for degenerate cases

$$D(\rho_{\alpha,\bar\alpha},\rho_{\alpha',\bar\alpha'}) = \sqrt{(\alpha-\alpha')^2+(\bar\alpha-\bar\alpha')^2}\,\frac{\ell}{L} + o\Big(\frac{\ell}{L}\Big)$$

Other cases

- There are other cases that we only get the leading powers of ℓ/L but cannot get the overall coefficients
- One example of such cases

$$D(\rho_0,\rho_J) \propto \frac{\ell^2}{L^2} + o\Big(\frac{\ell^2}{L^2}\Big)$$

Exact n-distance JZ-Ruggiero-Calabrese 1901.10993, ...

- Exact n_e-distances for a small even integer $n_e = 2, 4, \cdots$ are easy
- Exact trace distances and other n_o-distances are hard
- For vertex operators and from conformal transformations

$$\mathcal{D}_{n_e}(\rho_{\alpha,\bar\alpha}, \rho_{\alpha',\bar\alpha'}) = \frac{1}{2} \sum_{\mathcal{S} \subseteq \mathcal{S}_0} (-)^{|\mathcal{S}|} [H_{n_e}(\mathcal{S})]^{(\alpha-\alpha')^2 + (\bar\alpha - \bar\alpha')^2}$$

$$H_{n_e}(\mathcal{S}) = \Big(\frac{\sin \frac{\pi\ell}{L}}{n_e \sin \frac{\pi\ell}{n_e L}} \Big)^{|\mathcal{S}|} \prod_{j_1, j_2 \in \mathcal{S}}^{j_1 < j_2} \frac{\sin^2 \frac{\pi(j_1 - j_2)}{n_e}}{\sin \frac{\pi(j_1 - j_2 + \ell/L)}{n_e} \sin \frac{\pi(j_1 - j_2 - \ell/L)}{n_e}}$$

For the special case $(\alpha - \alpha')^2 + (\bar\alpha - \bar\alpha')^2 = 1$

- Exact n_e-distance that can be analytically continued to arbitrary real n-distance with $n > 1$

$$\log \mathcal{D}_{n_e}[1] = n_e \log(2\pi) - \log 2 - 2 \int_0^{+\infty} dt \frac{\mathrm{e}^{-t}}{t} \Big\{ \frac{1}{1 - \mathrm{e}^{-t}} \times \Big[\frac{(\mathrm{e}^{\frac{t\ell}{2L}} - 1)\big[\mathrm{e}^{\frac{t\ell}{2n_e L}} + \mathrm{e}^{\big(1 - \frac{(n_e - 1)\ell}{2 n_e L}\big) t}\big]}{\mathrm{e}^{\frac{t\ell}{n_e L}} - 1} - n_e \Big] - \frac{n_e}{2} \Big\}$$

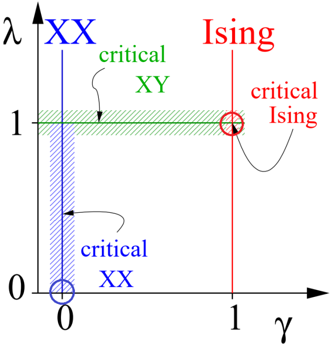
- Exact trace distance $D = \dfrac{\ell}{L}$

XY spin chain

- XY spin chain with transverse field

$$H = -\sum_{l=1}^{L}\left(\frac{1+\gamma}{4}\sigma_l^x\sigma_{l+1}^x + \frac{1-\gamma}{4}\sigma_l^y\sigma_{l+1}^y + \frac{\lambda}{2}\sigma_l^z\right)$$

- There are totally L sites of Pauli matrices $\sigma_j^{x,y,z}$
- Periodic (P) boundary conditions $\sigma_{L+1}^{x,y,z} = \sigma_1^{x,y,z}$
- Anti-periodic (AP) boundary conditions $\sigma_{L+1}^{x,y} = -\sigma_1^{x,y}$, $\sigma_{L+1}^z = \sigma_1^z$
- Critical behaviors Barouch-McCoy 1971, Latorre-Rico-Vidal 0304098
- At $\gamma = \lambda = 0$, critical XX model without transverse field $\Rightarrow$ free massless boson theory compactified on a unit circle $c = 1$
- At $\gamma = \lambda = 1$, critical Ising model $\Rightarrow$ free massless fermion theory $c = 1/2$



$$\sigma_l^\mu \overset{\text{JW}}{\Longrightarrow} a_l, a_l^\dagger \overset{\text{F}}{\Longrightarrow} b_k, b_k^\dagger \overset{\text{B}}{\Longrightarrow} c_k, c_k^\dagger$$

$$a_l, a_l^\dagger \Downarrow d_m$$

- It can be exactly and analytically diagonalized Lieb-Schultz-Mattis 1961
- Jordan-Wigner transformation with $\sigma_l^\pm = \frac{1}{2}(\sigma_l^x \pm \mathrm{i}\sigma_l^y)$

$$a_l = \Big(\prod_{j=1}^{l-1}\sigma_j^z\Big)\sigma_l^+, \quad a_l^\dagger = \Big(\prod_{j=1}^{l-1}\sigma_j^z\Big)\sigma_l^-$$

- Fourier transform with $\varphi_k = \frac{2\pi k}{L}$

$$b_k = \frac{1}{\sqrt{L}}\sum_{l=1}^{L}\mathrm{e}^{\mathrm{i}l\varphi_k}a_l, \quad b_k^\dagger = \frac{1}{\sqrt{L}}\sum_{l=1}^{L}\mathrm{e}^{-\mathrm{i}l\varphi_k}a_l^\dagger$$

- Bogoliubov transformation

$$c_k = b_k\cos\frac{\theta_k}{2} + \mathrm{i}b_{-k}^\dagger\sin\frac{\theta_k}{2}, \quad c_k^\dagger = b_k^\dagger\cos\frac{\theta_k}{2} - \mathrm{i}b_{-k}\sin\frac{\theta_k}{2}$$

- The parameter $\theta_k \in (-\pi, \pi]$ is determined by

$$\mathrm{e}^{\mathrm{i}\theta_k} = \frac{\lambda - \cos\varphi_k + \mathrm{i}\gamma\sin\varphi_k}{\varepsilon_k}, \quad \varepsilon_k = \sqrt{(\lambda - \cos\varphi_k)^2 + \gamma^2 \sin^2\varphi_k}$$

- Special cases $\theta_0 = \theta_{L/2} = 0$
- Different boundary conditions for fermionic modes $a_l, a_l^\dagger$
- Antiperiodic boundary conditions $a_{L+1} = -a_1$, $a_{L+1}^\dagger = -a_1^\dagger$, Neveu-Schwarz (NS) sector, half-integer modes k
- Periodic boundary conditions $a_{L+1} = a_1$, $a_{L+1}^\dagger = a_1^\dagger$, Ramond (R) sector, integer modes k
- Parity operator $P = \exp\left(\pi\mathrm{i}\sum_{l=1}^L a_l^\dagger a_l\right) = \exp\left(\pi\mathrm{i}\sum_k b_k^\dagger b_k\right)$
- Totally four sectors: PNS sector $P = 1$, PR sector $P = -1$, APNS sector $P = -1$, APR sector $P = 1$
- The Hamiltonian in each sector

$$H = \sum_k \varepsilon_k \left(c_k^\dagger c_k - \frac{1}{2}\right)$$

Entanglement entropy in XY model Vidal-Latorre-Rico-Kitaev 0211074, Latorre-Rico-Vidal 0304098, Alba-Fagotti-Calabrese 0909.1999, Alcaraz-Berganza-Sierra 1101.2881, 1109.5673

- Empty states in NS and R sectors

$$c_k|\emptyset,\mathrm{NS}\rangle = 0, \quad k \in \text{half integers},$$
$$c_k|\emptyset,\mathrm{R}\rangle = 0, \quad k \in \text{integers}$$

- Ground and excited states by imposing $c_k^\dagger$ on the empty states
- Each state is represented by the excited modes, e.g. $K = \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$
- Majorana modes d_m, $m = 1, 2, \cdots, 2\ell$ for an interval with ℓ sites

$$d_{2l-1} = a_l + a_l^\dagger, \quad d_{2l} = \mathrm{i}(a_l - a_l^\dagger)$$

- Correlation matrix Γ_K in state K

$$\langle d_m d_{m'} \rangle_K = \delta_{mm'} + \Gamma^K_{mm'}$$

- It is Hermitian and antisymmetric, and contains all the information of the RDM!

- The correlation matrix can be transformed to the standard form

$$\tilde{\Gamma}^K = \bigoplus_{j=1}^{\ell} \left(\begin{array}{cc} 0 & \mathrm{i}\gamma_j^K \\ -\mathrm{i}\gamma_j^K & 0 \end{array} \right)$$

- RDM in the particular basis

$$\tilde{\rho}_K = \bigotimes_{j=1}^{\ell} \left(\begin{array}{cc} (1+\gamma_j^K)/2 & 0 \\ 0 & (1-\gamma_j^K)/2 \end{array} \right)$$

- Entanglement entropy does not change under change of basis

$$S_K = -\sum_{j=1}^{\ell} \Big(\frac{1+\gamma_j^K}{2} \log \frac{1+\gamma_j^K}{2} + \frac{1-\gamma_j^K}{2} \log \frac{1-\gamma_j^K}{2} \Big)$$

- In terms of the 2ℓ eigenvalues γ_m^K of Γ_K

$$S_K = -\sum_{m=1}^{2\ell} \frac{1+\gamma_m^K}{2} \log \frac{1+\gamma_m^K}{2}$$

Subsystem trace distance in XY model JZ-Ruggiero-Calabrese 1901.10993, ...

- $2^\ell \times 2^\ell$ RDM in state K in the complete basis Latorre-Rico-Vidal 0304098

$$\rho_K = \frac{1}{2^\ell} \sum_{s_1,\cdots,s_{2\ell}\in\{0,1\}} \langle d_{2\ell}^{s_{2\ell}} \cdots d_1^{s_1} \rangle_K d_1^{s_1} \cdots d_{2\ell}^{s_{2\ell}}$$

- As it is a free theory, all the multi-point correlations are calculated from correlation matrix by Wick contractions
- Subsystem trace distance

$$D(\rho_K, \rho_{K'}) = \frac{1}{2} \sum_{i=1}^{2^\ell} |\lambda_i|$$

- λ_i are the eigenvalues of $\rho_K - \rho_{K'}$
- The size of RDM ρ_K grows exponentially with ℓ
- Numerically we choose $\ell = 4, 5, 6, 7$, $L = 8, 16, \cdots, 2048$
- We can probe a large scope of ℓ/L, especially when it is small
- With explicit RDMs we can also calculate relative entropy, fidelity, ...

n_e-distance in XY model JZ-Ruggiero-Calabrese 1901.10993

- Correlation matrix $\Gamma \leftrightarrow$ RDM ρ_Γ
- Algebra of the RDM Fagotti-Calabrese 1003.1110

$$\rho_\Gamma \rho_{\Gamma'} = \mathrm{tr}(\rho_\Gamma \rho_{\Gamma'}) \rho_{\Gamma \times \Gamma'}$$

- The trace of two RDMs

$$\mathrm{tr}(\rho_\Gamma \rho_{\Gamma'}) = \prod_{\lambda \in [\mathrm{spectrum}(\Gamma\Gamma')]/2} \frac{1+\lambda}{2}$$

- The new correlation matrix

$$\Gamma \times \Gamma' = 1 - (1-\Gamma')(1+\Gamma\Gamma')^{-1}(1-\Gamma)$$

- Can be used successively for $\mathrm{tr}(\rho_\Gamma \rho_{\Gamma'} \rho_{\Gamma''} \cdots)$ and for n_e-distance
- The size of the correlation matrix Γ grows linearly with ℓ
- We can probe the full range of $\ell/L \in (0,1)$
- It cannot be used for trace distance, relative entropy, or fidelity

XX spin chain with no transverse field

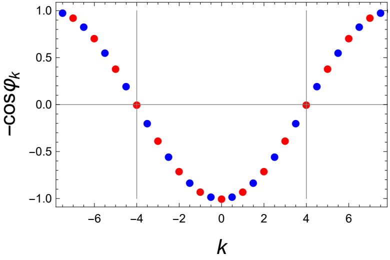
- It is a special case of XY spin chain

$$H = -\frac{1}{2}\sum_{l=1}^{L}(\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y)$$

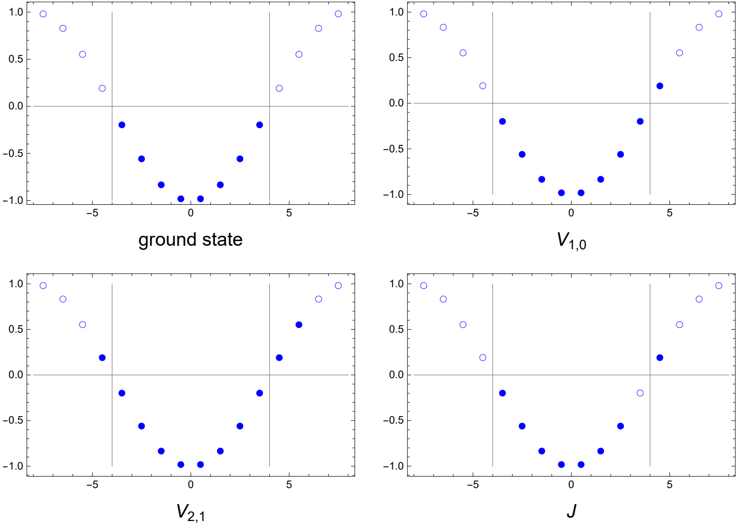
- It is critical and gapless
- Exactly digonoalized with a trivial Bogoliubov transformation $\theta_k = 0$

$$H = -\sum_k \cos\varphi_k \left(c_k^\dagger c_k - \frac{1}{2}\right)$$

- Energy of modes in the NS (blue) and R (red) sectors

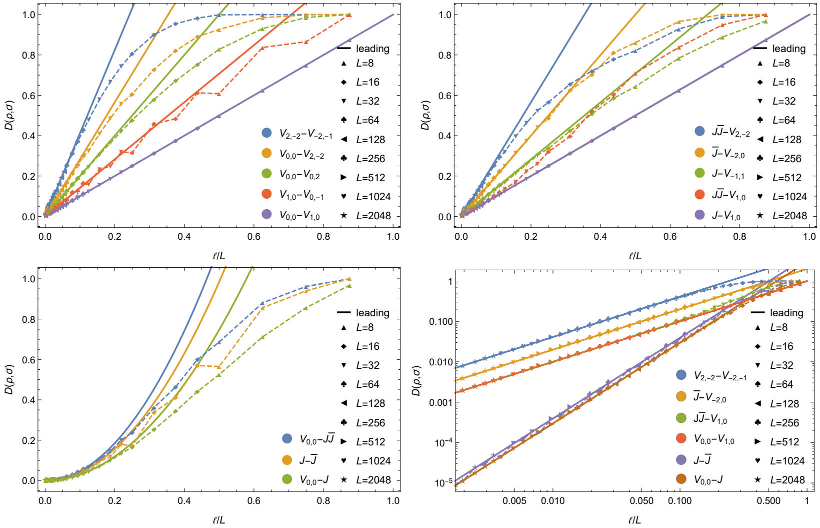


- Its continuous limit is the free massless real boson theory compactified on a unit circle
- One-to-one correspondence of the low energy excited states



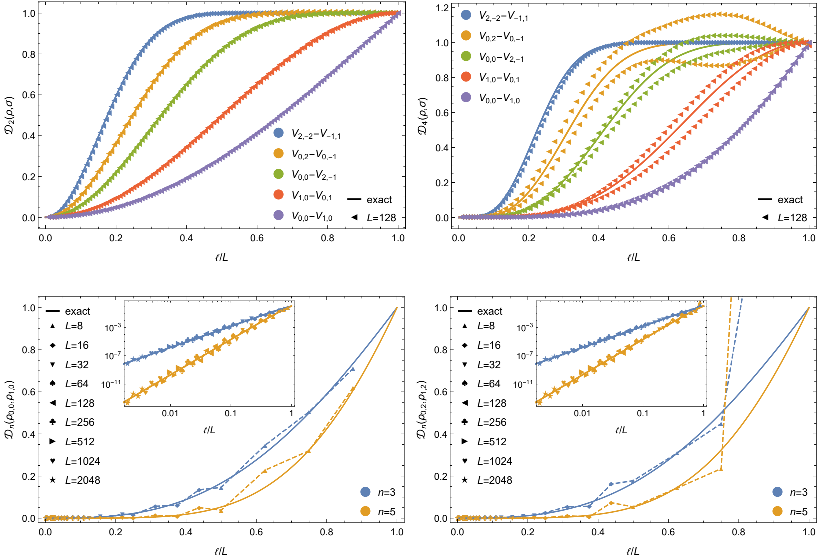
Trace distance in XX spin chain JZ-Ruggiero-Calabrese 1901.10993, ...

- We compare the leading short interval trace distances in the boson theory and XX spin chain



n-distance in XX spin chain JZ-Ruggiero-Calabrese 1901.10993, ...

- We compare the *n*-distances in the boson theory and XX spin chain



2D free massless fermion theory and Ising spin chain

- 2D free massless Majorana fermion theory

$$\mathcal{L} \propto \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}$$

- It is a 2D CFT with $c = 1/2$
- Related to boson theory by bosonization: two-fold fermion theory $\Leftrightarrow$ boson theory Di Francesco-Mathieu-Sénéchal 1997
- Results in fermion theory can be got from results in boson theory
- The fermion theory is the continuous limit of critical Ising spin chain
- Ising spin chain is another special case of the XY spin chain
- The calculations and results are similar to those in the boson theory and XX model case

Summary

- We developed replica trick to subsystem trace distance in 2D CFT
- Universal leading order short interval trace distances from OPE of twist operators
- The method was applied to several low lying excited states in 2D free massless boson the fermion theories
- For a special case we got the exact trace distance as well as the n-distance with arbitrary $n > 1$ for an interval with arbitrary length
- We calculate trace distance numerically in XX and Ising spin chains
- Short interval trace distance and exact n_e-distance in both CFTs and spin chains
- Perfect agreement of analytic CFT results with numerical spin chain calculations

Discussion

- It is the first time that subsystem trace distance is calculated in QFT
- There are several interesting possible generalizations
- Other states in 2D CFT: open systems, disjoint intervals, finite temperature, generalzied Gibbs ensemble, inhomogeneous systems, states after quantum quench, etc?
- 2D large c CFT?
- 2D massive theories?
- Higher dimensional boson and fermion theories?
- Holographic theories and gravity?

Thanks for your attention!