The universal structure of large logarithms in scattering amplitudes and cross sections

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Introduction

Scattering amplitudes:

- The central objects in theories of fundamental interactions.
- A bridge between theories and experiments.
- Hidden simple structures, e.g., MHV, BCFW, color-kinematics duality, double copy.
- Connection with mathematics, e.g. algebraic geometry, combinatorics.

"Scattering amplitudes are the most perfect microscopic structures in the universe." —by Lance Dixon

However, it is still in general difficult to calculate scattering amplitudes at higher orders (loops) and of many external particles.

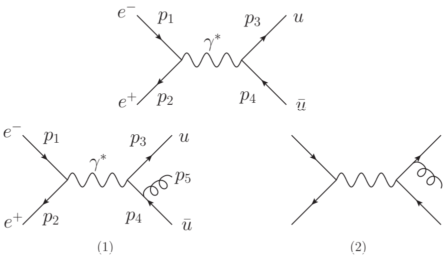
Introduction

Cross sections (decay rates): constructed from amplitudes squared

$$\frac{d\sigma}{dO} \sim \sum_{n=m} \int d\Phi_n |\mathcal{M}_n|^2 O(\{p_i\}) \tag{1}$$

- O is an observable, e.g., transverse momentum, rapidity, event shape, spin correlation.
- Φ_n is the n-body phase space.
- O can depend on $m, m+1, m+2, \ldots$
- Optical theorem can be applied for a few observables.
- Most of the observables are difficult to calculate precisely.
- Simplicity appears for large scale hierarchy.

An example



$$\frac{d\sigma}{\sigma_B dT} = \frac{\alpha_s \mathrm{C_F}}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)}\right]$$

with

$$T \equiv \max_{\vec{n}} T_{\vec{n}} = \max_{\vec{n}} \frac{\sum_i |\vec{n}\cdot\vec{p}_i|}{\sum_i |\vec{p}_i|}$$

No analytical NLO results though only one parameter appears. Numerical NNLO results have been obtained [Gehrmann-De Ridder, Gehrmann, Glover, Heinrich, '07]

An example

In the limit $\tau \equiv 1 - T \to 0$,

$$\frac{d\sigma}{\sigma_B dT} = \frac{\alpha_s C_F}{2\pi}\left[\frac{4}{\tau}\ln\left(\frac{1}{\tau}\right) + O(\tau^0)\right]$$

Can we obtain this large logarithm without performing the complicated phase space integral? (Is there a simple way to calculate this logarithm?)

Actually, since $\alpha_s \ln\tau \sim 1$ or even larger than 1, it is not valid any more to expand the cross section in α_s. Infinite higher orders of such kind of logarithms matter.

An example: soft limit

In the soft limit of $p_5 \to 0$ with $p_5 \sim \mathcal{O}(\lambda)$.

$$|M_1^{(1)}|_s^2 = \mathcal{O}(\lambda^0), \quad (2)$$

$$|M_2^{(1)}|_s^2 = \mathcal{O}(\lambda^0), \quad (3)$$

$$2\mathrm{Re}[M_1^{(1)} M_2^{(1)*}]_s = |M_\mathrm{B}|^2 g_s^2 C_F \frac{4 s_{34}}{s_{35} s_{45}} + \mathcal{O}(\lambda^{-1}) \quad (4)$$

After phase space integration (factorized),

$$\begin{aligned} \frac{1}{\sigma_B}\frac{d\sigma_s^{(1)}}{d\tau} =& \frac{g_s^2 C_F}{2(2\pi)^3} \int dn_+ p_5 \, dn_- p_5 \, d^{d-2} p_{5\perp} \delta(p_5^2) \frac{4}{n_+ p_5 \, n_- p_5} \\ & \times \left[\delta(\tau - \frac{n_+ p_5}{E_\mathrm{cm}}) \theta(n_- p_5 - n_+ p_5) + (n_- \leftrightarrow n_+) \right] \\ =& \frac{2\alpha_s C_F}{\pi} \frac{1}{\epsilon} \tau^{-1-2\epsilon} E_\mathrm{cm}^{-2\epsilon} + \mathcal{O}(\epsilon^0) \end{aligned} \quad (5)$$

An example: collinear limit

In the collinear limit of $p_5 \parallel p_3$ with $p_5 \cdot p_3 \sim \mathcal{O}(\lambda)$ and $n_+ p_5 = z\ n_+(p_3 + p_5)$.

$$|M_1^{(1)}|_c^2 = \mathcal{O}(\lambda^0), \tag{6}$$

$$|M_2^{(2)}|_c^2 = |M_{\rm B}|^2 g_s^2 C_F \frac{2}{s_{35}} z, \tag{7}$$

$$2{\rm Re}[M_1^{(1)} M_2^{(1)*}]_c = |M_{\rm B}|^2 g_s^2 C_F \frac{2}{s_{35}} \frac{2(1-z)}{z} \tag{8}$$

After phase space integration (factorized),

$$\begin{aligned} \frac{1}{\sigma_B}\frac{d\sigma_c^{(1)}}{d\tau} =& \frac{g_s^2 C_F}{16\pi^2} \int ds_{35} \int_0^1 dz [z(1-z)]^{-\epsilon} s_{35}^{-\epsilon} \frac{2}{s_{35}} \frac{1+(1-z)^2}{z} \delta(\tau - \frac{s_{35}}{E_{\rm cm}^2}) \\ =& -\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon}\tau^{-1-\epsilon} E_{\rm cm}^{-2\epsilon} + \mathcal{O}(\epsilon^0) \end{aligned} \tag{9}$$

An example

The sum of the soft and collinear contribution is

$$\frac{1}{\sigma_{\mathrm{B}}}\frac{d(\sigma_s^{(1)}+\sigma_c^{(1)}+\sigma_{\bar{c}}^{(1)})}{d\tau}$$

$$=\frac{2\alpha_s C_F}{\pi}E_{\mathrm{cm}}^{-2\epsilon}\left[\frac{1}{\epsilon}\tau^{-1-2\epsilon}-\frac{1}{\epsilon}\tau^{-1-\epsilon}\right] \tag{10}$$

$$=\frac{2\alpha_s C_F}{\pi}E_{\mathrm{cm}}^{-2\epsilon}\left[\frac{1}{\epsilon^2}\delta(\tau)-\left(\frac{\ln\tau}{\tau}\right)_+ + \mathcal{O}(\epsilon)\right] \tag{11}$$

- The poles are cancelled with virtual corrections $\sim\delta(\tau)$, as the requirement of infra-red safety.
- The large logarithm arises from the mismatch of the scales in the soft and collinear regions; τE_{cm} vs. $\sqrt{\tau}E_{\mathrm{cm}}$.

An example: Renormalization group view

Scale hierarchy: $E_{\rm cm} \gg \sqrt{\tau} E_{\rm cm} \gg \tau E_{\rm cm}$, or equivalently $t_{hard} \ll t_{coll} \ll t_{soft}$, or $\lambda_{hard} \ll \lambda_{coll} \ll \lambda_{soft}$. The physics at different scales decouples from each other; no interference between waves of different length happens; the process factorizes into hard, jet, and soft functions.

$$\frac{d\sigma}{\sigma_B d\tau} = |C_H(\mu)|^2 \int d\tau_s d\tau_c d\tau_{\bar{c}} \delta(\tau - \tau_s - \tau_c - \tau_{\bar{c}}) \\ J(\tau_c, \mu) J(\tau_{\bar{c}}, \mu) S(\tau_s, \mu)$$

Laplace transform $\tilde{f}(N) = \int_0^\infty dx e^{-xN} f(x)$:

$$\frac{d\tilde{\sigma}(N)}{\sigma_B d\tau} = |C_H(\mu)|^2 \tilde{J}(N, \mu) \tilde{J}(N, \mu) \tilde{S}(N, \mu)$$

The RG equation:

$$\frac{d}{d \ln \mu^2} \tilde{J}(N, \mu) = \left[\Gamma_J \ln \frac{\mu^2}{E_{\rm cm}^2 / N} + \gamma_J \right] \tilde{J}(N, \mu) \tag{12}$$

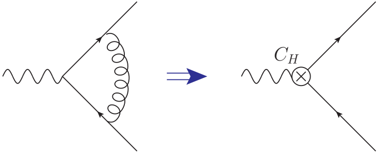
Lessons learned

The logarithms (at leading power) can be derived by

1. Factorization of the cross section
2. Calculation of the anomalous dimension of each ingredient

Factorization

Factorization of hard function (integrating out hard fluctuation in loops)



$$C_H(\mu) = 1 - \frac{\alpha_s C_F}{4\pi}\left(\ln^2\frac{\mu^2}{E_{\rm cm}^2} + 3\ln\frac{\mu^2}{E_{\rm cm}^2} + c_H\right) \qquad (13)$$

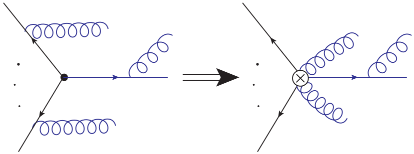
RG equation

$$\frac{d}{d\ln\mu^2} C_H(\mu) = \left(\Gamma_H \ln\frac{\mu^2}{E_{\rm cm}^2} + \gamma_H\right) C_H(\mu) \qquad (14)$$

Solution

$$C_H(\mu) = C_H(\mu_h)\exp\left[\frac{\Gamma_H}{2}\ln^2\frac{\mu^2}{\mu_h^2} + \gamma_H \ln\frac{\mu^2}{\mu_h^2}\right]\left(\frac{E_{\rm cm}^2}{\mu_h^2}\right)^{-\Gamma_H \ln\frac{\mu^2}{\mu_h^2}} \qquad (15)$$

The interaction between collinear modes in different directions.



All attachments can be summed to a collinear Wilson line

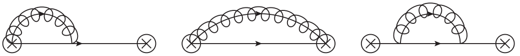
$$W_c(x) = P \exp \left[i g_s \int_{-\infty}^{0} ds \bar{n} \cdot A_c(x + s\bar{n}) \right]$$

Necessary to form a gauge invariant building block, $W_c^\dagger \xi_c$, which is independent of the other directions.

Factorization

Factorization of jet function (integrating out collinear modes in loops and final states)

$$J(p^2,\mu) = \mathrm{Disc}\left\{ \quad \right\}$$



$$J(p^2,\mu) = \delta(p^2)\,[1+c_J] + \left[\frac{\Gamma_J \log\frac{p^2}{\mu^2} + \gamma_J}{p^2}\right]_\star^{[p^2,\mu^2]} ,$$

$$\frac{dJ(p^2,\mu)}{d\log\mu} = \left[-2\Gamma_J \log\frac{p^2}{\mu^2} - 2\gamma_J\right] J(p^2,\mu) + 2\Gamma_J \int_0^{p^2} dq^2\,\frac{J(p^2,\mu)-J(q^2,\mu)}{p^2-q^2}.$$

$$J(p^2,\mu) = \exp\left[\frac{\Gamma_J}{2}\log^2\frac{\mu^2}{\mu_j^2} - \gamma_J \log\frac{\mu^2}{\mu_j^2}\right] \tilde{j}\left(\partial_{\eta_j}\right) \left[\frac{1}{p^2}\left(\frac{p^2}{\mu_j^2}\right)^{\eta_j}\right]_\star^{[p^2,\mu_j^2]} \frac{e^{-\gamma_E \eta_j}}{\Gamma[\eta_j]},$$

Factorization

The factorization of jet function is closely related to the property

$$|\mathcal{M}(1+2\to 3+4+5)|^2_{\rm coll} = |\mathcal{M}(1+2\to 3+4')|^2 \times P_{44'}(z,\epsilon)\frac{2g_s^2}{s_{45}}$$

and

$$d\Phi_3|_{\rm coll} = d\Phi_2 \frac{1}{16\pi^2} dz ds_{45} [s_{45} z(1-z)]^{-\epsilon}$$

A similar factorization happens for the initial-state collinear splitting. The parton distribution function satisfies

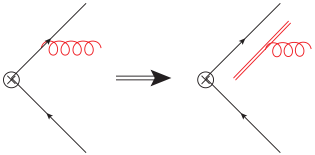
$$\frac{d}{d\ln\mu} f_{i/N}(x,\mu) = \int_x^1 \frac{dz}{z} P_{ij}(z) f_{j/N}(x/z,\mu)$$

with the DGLAP evolution kernel

$$P_{qq}(z) = \frac{\alpha_s \mathrm{C_F}}{2\pi}\left(\frac{1+z^2}{1-z}\right)_+$$

Factorization

The interaction between the soft modes and collinear modes



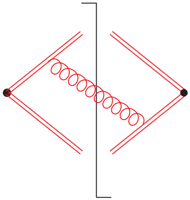
$$M(k,\{p_i\}) = \sum_i (-g_s)\mathbf{T}_i \left(\frac{\varepsilon(k)\cdot p_i}{k\cdot p_i}\right) M_0(\{p_i\}) \qquad (16)$$

The result depends only on the direction and color charge of the collinear mode. The information about the momentum and spin of the collinear particle is irrelevant. This is called Eikonal approximation. All attachments can be summed to a soft Wilson line

$$Y_n(x) = P\exp\left[ig_s \int_{-\infty}^{0} ds n\cdot A_s(x+sn)\right]$$

Factorization

Factorization of soft function (decouple of the soft interaction with collinear mode)

$S(\tau_s, \mu) = F(\tau_s, k) \times$ 

with the measurement function

$$F(\tau_s, k) = \delta(\tau_s - n \cdot k)\theta(\bar{n} \cdot k - n \cdot k) + \delta(\tau_s - \bar{n} \cdot k)\theta(n \cdot k - \bar{n} \cdot k)$$

Only UV poles in S. IR poles cancel between real and virtual corrections. The RG equation is similar to that of jet function.

Large logarithms

Summarize the results based on factorization of the cross section,

$$\sigma(\tau) = \sum_n \alpha_s^n \left[c_n \delta(\tau) + \sum_{m=0}^{2n-1} \left(c_{nm} \frac{\ln^m \tau}{\tau} + \underbrace{d_{nm} \ln^m \tau}_{NLP} \right) + \cdots \right]$$

c_{nm} are fully determined by the anomalous dimensions of (the hard function), jet function and soft function. In this sense, they are universal.

There are another kind of logarithms, whose coefficients are d_{nm}. Though they are suppressed, they are numerically important as well. The question is how to develop a factorization formula for this power suppressed contribution.

Recent development

Actually, τ can be the N-jettiness variable, the threshold variable $1 - M^2/s$, the transverse momentum of a lepton pair q_T, the mass ratio m_h^2/m_b^2, $\cdots$

1. Phenomenology: useful for NN(N)LO differential calculations in q_T/N-jettiness slicing methods [Moult, Rothen, Stewart, Tackmann, Zhu '16, Boughezal, Liu, Petriello,'16]
2. Theory: NLP factorization and resummation [Bonocore, Laenen, Magnea, Melville, Vernazza, White, '15, '16, Liu, Penin, '17, Moult, Stewart, Vita, Zhu, '18, Beneke, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, JW, '18, Laenen, Damste, Vernazza, Waalewijn, Zoppi, '20, Liu, Mecaj, Neubert, Wang '20]
3. Amplitude: soft theorem, soft bootstrap [Strominger '13,Rodina '18]

Improvement for subtraction

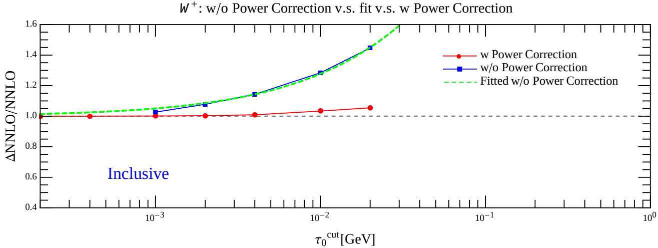


Figure: $O(\alpha_s^2)$ correction for DY production with N-jettiness subtraction from 1612.02911

Without the power corrections, $\tau_{\rm cut}$ should be set to below 10^{-3}GeV to reproduce the exact NNLO coefficient. The cut can be relaxed by a factor of 10 when the power corrections are included.

Recent development

- Beyond leading logarithms (at $O(\alpha_s)$) [Boughezal, Isgro, Petriello, '18, Ebert, Moult, Stewart, Tackmann, Vita, Zhu, '18]
- Beyond $2 \to 1$ or $1 \to 2$ [Beekveld, Beenakker, Laenen, White '19,Boughezal, Isgro, Petriello, '19]
- Threshold/Thrust resummation at NLP [Moult, Stewart, Vita, Zhu, '18, Beneke, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, JW, '18, Bahjat-Abbas, Bonocore, Damste, Laenen, Magnea, Vernazza, White '19, Ajjath, Mukherjee, Ravindran '20]
- Rapidity divergences in q_T spectrum or energy-energy correlators [Ebert, Moult, Stewart, Tackmann, Vita, Zhu, '18, Moult, Vita, Yan, '19]
- Soft quark Sudakov [Liu, Penin, '17, Moult, Stewart, Vita, Zhu, '19, Liu, Mecaj, Neubert, Wang, Fleming, '20, JW, '20]
- Subleading power effects in B physics and heavy quarkonium production [Ma, Qiu, Sterman, Zhang '13, Lee, Sterman '20, Li, Lü, Sheng Wang, Wang, Wei, '17,'20]

The soft limit at NLP

In the soft limit $k^\mu \to 0$, (LBK/soft theorem [Low, '58, Burnett, Kroll, '68])

$$M(k,\{p_i\}) = \sum_i (-g_s)\mathbf{T}_i \left(\frac{\varepsilon(k)\cdot p_i}{k\cdot p_i} + \frac{\varepsilon_\mu k_\nu J_i^{\mu\nu}}{k\cdot p_i} \right) M_0(\{p_i\}) \quad (17)$$

with

$$J_i^{\mu\nu} = p_i^\mu \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\mu}} + \Sigma_i^{\mu\nu}, \quad \Sigma_i^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu] \qquad (18)$$

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Integrating over the constrained phase space,

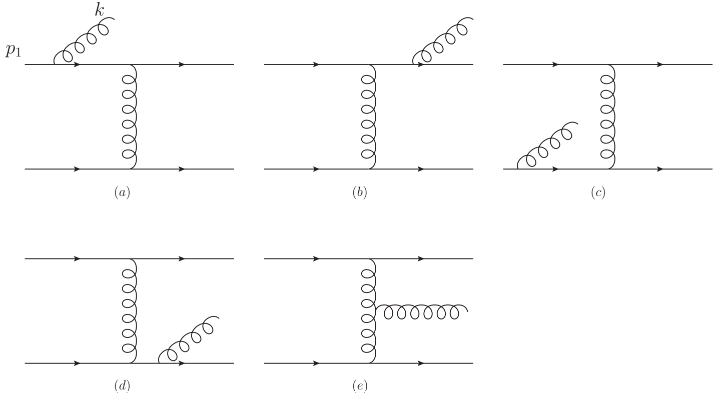
$$\int d^d k \delta(k^2)\theta(k^0) \frac{1}{k\cdot p_i}\frac{1}{k\cdot p_j} f(k) \quad (19)$$

$$\frac{1}{\epsilon}\tau^\epsilon = \frac{1}{\epsilon} + \ln\tau \quad (20)$$

$$\frac{1}{\epsilon^2}\tau^\epsilon = \frac{1}{\epsilon^2} + \frac{\ln\tau}{\epsilon} + \frac{1}{2}\ln^2\tau \quad (21)$$

LBK Theorem

Consider a process $ud \to ud + g$.



$$A(k, \{p_i\}) = \sum_i (-g_s)\mathbf{T}_i \left(\frac{\varepsilon(k) \cdot p_i}{k \cdot p_i} + \frac{\varepsilon_\mu k_\nu J_i^{\mu\nu}}{k \cdot p_i} \right) A_0(\{p_i\}) \quad (22)$$

LBK Theorem

$$J_i^{\mu\nu} = p_i^\mu \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\mu}} + \Sigma_i^{\mu\nu} \tag{23}$$

We expand the propagators in diagram (a)

$$\frac{(\not{p}_1 - \not{k})\not{\epsilon}}{(p_1 - k)^2} = \frac{p_1 \cdot \epsilon}{-p_1 \cdot k} + \frac{i\Sigma^{\mu\nu}\epsilon_\mu k_\nu}{-p_1 \cdot k} \tag{24}$$

$$\frac{1}{(p_1 - p_3 - k)^2} = \frac{1}{(p_1 - p_3)^2} - k \cdot \frac{\partial}{\partial p_1} \frac{1}{(p_1 - p_3)^2} \tag{25}$$

LBK Theorem

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Where is the blue part?

LBK Theorem

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Where is the blue part? It comes from diagram (e),

LBK Theorem

$$J_i^{\mu\nu} = p_i^{\mu}\frac{\partial}{\partial p_{i\nu}} - p_i^{\nu}\frac{\partial}{\partial p_{i\mu}} + \Sigma_i^{\mu\nu} \tag{23}$$

We expand the propagators in diagram (a)

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Where is the blue part? It comes from diagram (e), or from gauge invariance.

Subleading power operators

Understanding from the effective field theory [Beneke, Garny, Szafron, JW, '17,'18]

$$\mathcal{L}_{\mathrm{SCET}} = \sum_{i=1}^{N} \mathcal{L}_i(\psi_i, \psi_s) + \mathcal{L}_s(\psi_s) \tag{26}$$

The general structure of subleading operators

$$J = \int dt \; C(\{t_{i_k}\}) \, J_s(0) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, \dots) \tag{27}$$

where

$$J_i(t_{i_1}, t_{i_2}, \dots) = \prod_{k=1}^{n_i} \psi_{i_k}(t_{i_k} n_{i+}) \tag{28}$$

with gauge-invariant collinear "building blocks"

$$\psi_i(t_i n_{i+}) \in \begin{cases} \chi_i(t_i n_{i+}) \equiv W_i^{\dagger} \xi_i & \text{collinear quark} \\ \mathcal{A}^{\mu}_{\perp i}(t_i n_{i+}) \equiv W_i^{\dagger} [i D^{\mu}_{\perp i} W_i] & \text{collinear gluon} \end{cases}$$

LP:

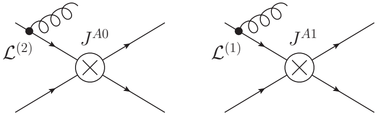
$$J_i^{A0}(t_i) = \psi_i(t_i n_{i+})\,. \tag{29}$$

NLP $[O(\lambda), O(\lambda^2)]$:

- $i\partial_\perp$ $\rightarrow J^{A1} = i\partial_\perp J^{A0}$
- $in_- D_s \equiv in_-\partial + g_s n_- A_s$ $\rightarrow$ eliminated by E.o.M
- more building blocks $\rightarrow J^{B1} = \psi_{i_1}(t_{i_1} n_{i+})\psi_{i_2}(t_{i_2} n_{i+})$
- new building blocks, e.g., $n_-\mathcal{A}$ $\rightarrow$ eliminated by E.o.M
- pure soft sector J_s, e.g., $q \sim O(\lambda^3), F_s^{\mu\nu} \sim O(\lambda^4)$, not needed at NLP
- time-ordered product operators

$$J_i^{T1}(t_i) = i\int d^4x\, \mathbf{T}\left\{J_i^{A0}(t_i), \mathcal{L}_i^{(1)}(x)\right\} \tag{30}$$

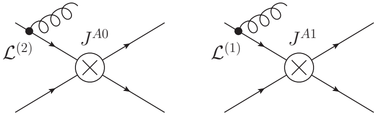
LBK Theorem



We reproduce LBK theorem with two time-ordered products

$$\int d^4x \mathbf{T}\{J^{A0}, \mathcal{L}^{(2)}(x)\}, \quad \int d^4x \mathbf{T}\{J^{A1}, \mathcal{L}^{(1)}(x)\}$$

LBK Theorem



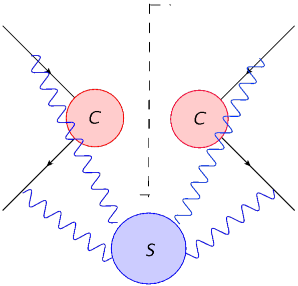
We reproduce LBK theorem with two time-ordered products

$$\int d^4x \mathbf{T}\{J^{A0}, \mathcal{L}^{(2)}(x)\}, \quad \int d^4x \mathbf{T}\{J^{A1}, \mathcal{L}^{(1)}(x)\}$$

No operators with soft fields needed!
No Ward identity needed!
J^{A1} is related to J^{A0}.

Factorization of Drell-Yan process at LP

At LP, the factorization picture is given by [Becher, Neuber, Xu, '08]

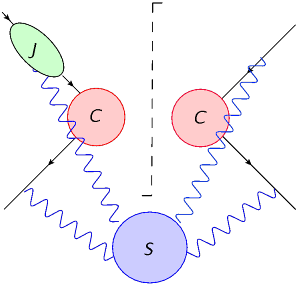


$$\frac{d\sigma_{\rm DY}}{dQ^2} = \frac{4\pi\alpha_{\rm em}^2}{3N_c Q^4}\sum_{a,b}\int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b)\,\hat{\sigma}_{ab}(z)$$

$$\hat{\sigma}(z) = H(Q^2)\, Q S_{\rm DY}(Q(1-z))$$

$$S_{\rm DY}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega/2}\,\frac{1}{N_c}\,{\rm Tr}\,\langle 0|\bar{\mathbf{T}}(Y_+^\dagger(x^0)Y_-(x^0))\,\mathbf{T}(Y_-^\dagger(0)Y_+(0))|0\rangle$$

At NLP, the picture is more complicated [Beneke, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, JW '18]



$$\hat{\sigma}(z) = \sum_{\text{terms}} \int d\omega_i d\bar{\omega}_i d\omega'_i d\bar{\omega}'_i \, D(-\hat{s};\omega_i,\bar{\omega}_i) D^*(-\hat{s};\omega'_i,\bar{\omega}'_i)$$
$$\times Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3 \, 2\sqrt{Q^2+\vec{q}^{\,2}}} \frac{1}{2\pi} \int d^4x \, e^{i(x_a p_A + x_b p_B - q)\cdot x}$$
$$\widetilde{S}(x;\omega_i,\bar{\omega}_i,\omega'_i,\bar{\omega}'_i)\,.$$

The D function combines the hard and jet function (at the amplitude level).

$$
\begin{aligned}
D(-\hat{s};\omega_i,\bar{\omega}_i) &= \int d(n_+p_i)d(n_-\bar{p}_i)\, C(n_+p_i,n_-\bar{p}_i) \\
&\quad \times J(n_+p_i, x_a n_+p_A;\omega_i)\, \bar{J}(n_-\bar{p}_i,-x_b n_-p_B;\bar{\omega}_i)\,.
\end{aligned}
$$

The complexity comes from the fact that the soft modes do not decouple from the collinear modes beyond LP, as seen from the LBK theorem. We have to keep more indices (quantum information) in both the jet and soft function.

$$
\mathcal{L}^{(2)}_{2\xi} = \frac{1}{2}\bar{\chi}_c x_\perp^\mu x_\perp^\nu \left[i\partial_\nu i n_-\partial \mathcal{B}^+_\mu\right] \frac{\not{n}_+}{2}\chi_c, \quad \mathcal{B}^\mu_\pm = Y^\dagger_\pm \left[iD^\mu_s Y_\pm\right]
$$

Factorization of the collinear mode:

$$i\int d^4z\,\mathbf{T}\left[\chi_{c,\alpha a}(tn_+)\mathcal{L}^{(2)}_{2\xi}(z)\right] = 2\pi\int du\int\frac{d(n_+z)}{2}$$
$$\widetilde{J}_{2\xi;\alpha\beta,abde}\left(t,u;\frac{n_+z}{2}\right)\,\chi^{\rm PDF}_{c,\beta b}(un_+)\,\frac{\partial^\mu_\perp}{in_-\partial}\mathcal{B}^+_{\perp\mu;de}(z_-)\,.$$

LO result:

$$\begin{aligned} J_{2\xi;\alpha\beta,abde}(n_+p,n_+p';\,\omega) &\equiv J_{2\xi;\alpha\beta,abde}(n_+p;\,\omega)\delta(n_+p-n_+p') \\ &= -\frac{1}{n_+p}\delta(n_+p-n_+p')\delta_{\alpha\beta}\delta_{ad}\delta_{eb}\,. \end{aligned}$$

We evolve other scales to the collinear scale. So we do not calculate the NLO result.

Factorization of the soft mode: We introduce the soft operator

$$\widetilde{\mathcal{S}}_{2\xi}(x, z_-) = \bar{\mathbf{T}}\left[Y_+^\dagger(x)Y_-(x)\right]\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}_{\perp\nu}^+(z_-)\right],$$

and the Fourier transform of its (colour-traced) vacuum matrix element

$$S_{2\xi}(\Omega,\omega) = \int\frac{dx^0}{4\pi}\int\frac{d(n_+z)}{4\pi}\,e^{ix^0\Omega/2-i\omega(n_+z)/2}\,\frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\widetilde{\mathcal{S}}_{2\xi}(x^0,z_-)|0\rangle\,.$$

Divergences in LO result:

$$S_{2\xi}(\Omega,\omega) = \frac{\alpha_s C_F}{2\pi}\left\{\theta(\Omega)\delta(\omega)\left(-\frac{1}{\epsilon}+\ln\frac{\Omega^2}{\mu^2}\right)+\left[\frac{1}{\omega}\right]_+\theta(\omega)\theta(\Omega-\omega)\right\},$$

Do we need additive renormalization?

Introduce auxiliary soft function

$$S_{x_0}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega/2}\, \frac{-2i}{x^0 - i\varepsilon}\, \frac{1}{N_c} \mathrm{Tr}\, \langle 0|\bar{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right]\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\right]|0\rangle.$$

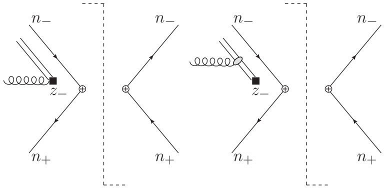
$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

Renormalization of soft operator at NLP

Consider $\langle g|\mathcal{S}_{2\xi}|0\rangle$. The renormalization factor of the soft function is obtained by projecting on the colour singlet part.

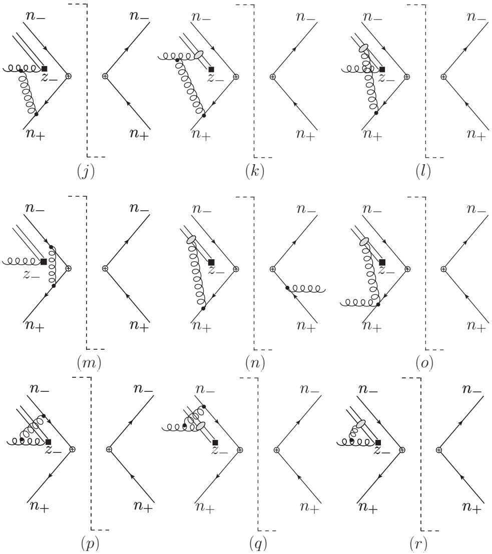


The filled square and the two solid lines connected to it stand for the soft covariant derivative and the Wilson lines contained in $\frac{i\partial_{\perp\mu}}{in_-\partial}\mathcal{B}_+^\mu = \frac{i\partial_{\perp\mu}}{in_-\partial}\, Y_+^\dagger[iD_s^\mu Y_+]$, respectively.

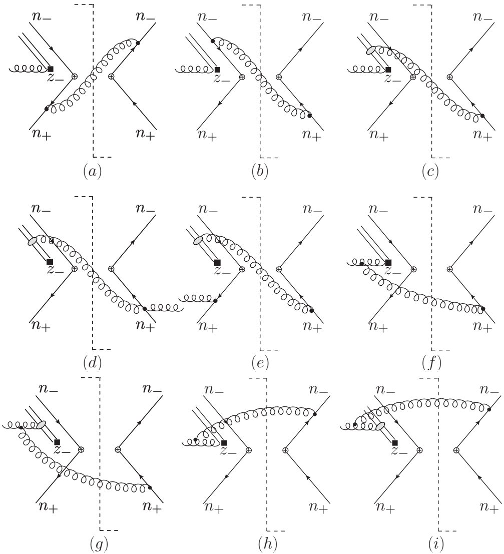
$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}} = g_s T^A\left(\frac{p_\perp\cdot\epsilon_\perp^*}{n_-p} - \frac{p_\perp^2\, n_-\epsilon^*}{(n_-p)^2}\right)\delta(\Omega)\delta(\omega - n_-p).$$

Choose $n_-\epsilon = 0$ for simplicity.

Renormalization of soft operator at NLP



Renormalization of soft operator at NLP



Renormalization of soft function at NLP

RG equation:

$$\frac{d}{d\ln\mu}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega)\end{pmatrix} = \frac{\alpha_s}{\pi}\begin{pmatrix} 4C_F\ln\frac{\mu}{\mu_s} & -C_F\delta(\omega) \\ 0 & 4C_F\ln\frac{\mu}{\mu_s}\end{pmatrix}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x^0}(\Omega)\end{pmatrix}$$

Solution:

$$S_{2\xi}^{\mathrm{LL}}(\Omega,\omega,\mu) = \frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\,\theta(\Omega)\delta(\omega)\,.$$

with

$$\begin{aligned} S^{\mathrm{LL}}(\nu,\mu) &= \frac{C_F}{\beta_0^2}\frac{4\pi}{\alpha_s(\nu)}\left(1-\frac{\alpha_s(\nu)}{\alpha_s(\mu)}+\ln\frac{\alpha_s(\nu)}{\alpha_s(\mu)}\right)\,. \\ &\to -\frac{\alpha_s C_F}{2\pi}\ln^2\frac{\mu}{\nu} \end{aligned}$$

Resummed cross section at NLP

$$\hat{\sigma}^{\rm LL}_{\rm NLP}(z,\mu) = \exp\left[4S^{\rm LL}(\mu_h,\mu) - 4S^{\rm LL}(\mu_s,\mu)\right] \times \frac{-8C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z)\,,$$

Expansion to fixed orders: First N^3LO agrees with [Kramer, Laenen, Spiar, '96]

$$\begin{aligned}
\hat{\sigma}^{\rm LL}_{\rm NLP}(z,\mu) \;=\;& -\theta(1-z)\Bigg\{4C_F\frac{\alpha_s}{\pi}\Big[\ln(1-z) - L_\mu\Big] \\
&+8C_F^2\left(\frac{\alpha_s}{\pi}\right)^2\Big[\ln^3(1-z) - 3L_\mu\ln^2(1-z) + 2L_\mu^2\ln(1-z)\Big] \\
&+8C_F^3\left(\frac{\alpha_s}{\pi}\right)^3\Big[\ln^5(1-z) - 5L_\mu\ln^4(1-z) + 8L_\mu^2\ln^3(1-z) - 4L_\mu^3\ln^2(1-z)\Big] \\
&+\frac{16}{3}C_F^4\left(\frac{\alpha_s}{\pi}\right)^4\Big[\ln^7(1-z) - 7L_\mu\ln^6(1-z) + 18L_\mu^2\ln^5(1-z) - 20L_\mu^3\ln^4(1-z) \\
&\qquad + 8L_\mu^4\ln^3(1-z)\Big] \\
&+\frac{8}{3}C_F^5\left(\frac{\alpha_s}{\pi}\right)^5\Big[\ln^9(1-z) - 9L_\mu\ln^8(1-z) + 32L_\mu^2\ln^7(1-z) - 56L_\mu^3\ln^6(1-z) \\
&\qquad + 48L_\mu^4\ln^5(1-z) - 16L_\mu^5\ln^4(1-z)\Big]\Bigg\} + \mathcal{O}(\alpha_s^6\times(\log)^{11})\,,
\end{aligned}$$

Double logarithms in off-diagonal splitting kernel

The above result is shown for $q\bar{q} \to Z$ ($gg \to H$). If we consider $qg \to Z + X$ ($qg \to H + X$), we need the evolution of parton $g \to q$ ($q \to g$).

The DGLAP splitting kernel [Vogt '10]

$$P_{gq}^{\rm LL}(N) = \frac{1}{N}\frac{\alpha_s C_F}{\pi}\mathcal{B}_0(a), \qquad a = \frac{\alpha_s}{\pi}(C_F - C_A)\ln^2 N\,, \tag{31}$$

where

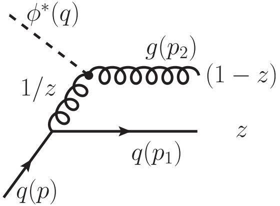
$$\mathcal{B}_0(x) = \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}x^n \quad , B_n = 1, \frac{-1}{2}, 0, \frac{1}{6}, 0, \frac{-1}{30}, 0, \frac{1}{42}\cdots \tag{32}$$

Compared to

$$P_{qq}^{\rm LL}(N) = -2\Gamma_{\rm cusp}(\alpha_s)\ln N \tag{33}$$

Off-diagonal DIS cross section

To calculate the splitting kernel, we consider the off-diagonal DIS process. The partonic process contains IR divergences which must be absorbed into the PDF. [Beneke, Garny, Jaskiewicz, Szafron, Vernazza, JW '20]



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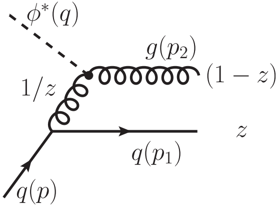


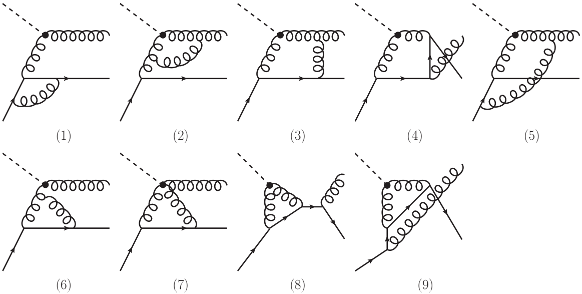
$$W_{\phi,q}\Big|_{q\phi^*\to qg} = \int_0^1 dz \left(\frac{\mu^2}{s_{qg} z\bar{z}}\right)^{\epsilon} \mathcal{P}_{qg}(s_{qg}, z)\Big|_{s_{qg}=Q^2\frac{1-x}{x}}$$

$$\mathcal{P}_{qg}(s_{qg}, z) \equiv \frac{e^{\gamma_E \epsilon} Q^2}{16\pi^2\Gamma(1-\epsilon)} \frac{|\mathcal{M}_{q\phi^*\to qg}|^2}{|\mathcal{M}_0|^2} = \frac{\alpha_s C_F}{2\pi}\frac{\bar{z}^2}{z} + \mathcal{O}(\epsilon, \lambda^2)$$

The $z \to 0$ limit generats a pole. This is an IR pole caused by Soft quark. No simple soft Wilson line.

Off-diagonal DIS cross section

One loop virtual corrections.



Off-diagonal DIS cross section

$$
\begin{aligned}
\mathcal{P}_{qg}(s_{qg}, z)|_{\mathrm{1-loop}} &= \mathcal{P}_{qg}(s_{qg}, z)|_{\mathrm{tree}} \frac{\alpha_s}{\pi}\frac{1}{\epsilon^2} \\
&\left(\mathbf{T}_1 \cdot \mathbf{T}_0 \left(\frac{\mu^2}{zQ^2}\right)^{\epsilon} + \mathbf{T}_2 \cdot \mathbf{T}_0 \left(\frac{\mu^2}{\bar{z}Q^2}\right)^{\epsilon} \right. \\
&\left. + \mathbf{T}_1 \cdot \mathbf{T}_2 \left[\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} - \left(\frac{\mu^2}{zQ^2}\right)^{\epsilon} + \left(\frac{\mu^2}{z s_{qg}}\right)^{\epsilon} \right] \right) \quad (34)
\end{aligned}
$$

Off-diagonal DIS cross section

$$\begin{aligned}
\mathcal{P}_{qg}(s_{qg}, z)|_{\rm 1-loop} \quad = \quad & \mathcal{P}_{qg}(s_{qg}, z)|_{\rm tree} \frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \\
& \left(\mathbf{T}_1 \cdot \mathbf{T}_0 \left(\frac{\mu^2}{z Q^2} \right)^{\epsilon} + \mathbf{T}_2 \cdot \mathbf{T}_0 \left(\frac{\mu^2}{\bar{z} Q^2} \right)^{\epsilon} \right. \\
& \left. + \mathbf{T}_1 \cdot \mathbf{T}_2 \left[\left(\frac{\mu^2}{Q^2} \right)^{\epsilon} - \left(\frac{\mu^2}{z Q^2} \right)^{\epsilon} + \left(\frac{\mu^2}{z s_{qg}} \right)^{\epsilon} \right] \right) \quad (34)
\end{aligned}$$

We get the terms with $\mathbf{T}_1 \cdot \mathbf{T}_0$ and $\mathbf{T}_2 \cdot \mathbf{T}_0$ by standard method.

Off-diagonal DIS cross section

$$\mathcal{P}_{qg}(s_{qg}, z)|_{1-\text{loop}} = \mathcal{P}_{qg}(s_{qg}, z)|_{\text{tree}} \frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left(\mathbf{T}_1 \cdot \mathbf{T}_0 \left(\frac{\mu^2}{zQ^2} \right)^{\epsilon} + \mathbf{T}_2 \cdot \mathbf{T}_0 \left(\frac{\mu^2}{\bar{z}Q^2} \right)^{\epsilon} + \mathbf{T}_1 \cdot \mathbf{T}_2 \left[\left(\frac{\mu^2}{Q^2} \right)^{\epsilon} - \left(\frac{\mu^2}{zQ^2} \right)^{\epsilon} + \left(\frac{\mu^2}{zs_{qg}} \right)^{\epsilon} \right] \right) \quad (34)$$

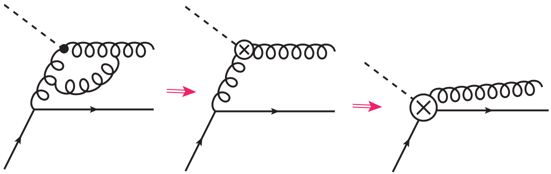
We get the terms with $\mathbf{T}_1 \cdot \mathbf{T}_0$ and $\mathbf{T}_2 \cdot \mathbf{T}_0$ by standard method.
Caution: Keep $z^{-\epsilon}$! End-point singularity

$$\frac{1}{\epsilon^2} \int_0^1 dz \, \frac{1}{z^{1+\epsilon}} \left(1 - z^{-\epsilon}\right) = -\frac{1}{2\epsilon^3} \quad (35)$$

$$\frac{1}{\epsilon^2} \int_0^1 dz \, \frac{1}{z^{1+\epsilon}} \left(\epsilon \ln z - \frac{\epsilon^2}{2!} \ln^2 z + \frac{\epsilon^2}{3!} \ln^3 z + \cdots \right) = -\frac{1}{\epsilon^3} + \frac{1}{\epsilon^3} - \frac{1}{\epsilon^3} + \cdots .$$

Off-diagonal DIS cross section

A new scale $\sqrt{z}Q$ emerges dynamically.



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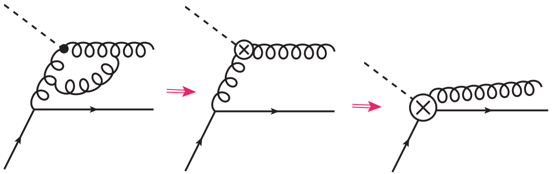

Two step matching:

$$C^{A0}\left(Q^2, Q^2\right) \exp\left[-\frac{\alpha_s C_A}{2\pi}\frac{1}{\epsilon^2}\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\right],$$

$$D^{B1}\left(zQ^2, zQ^2\right) \exp\left[-\frac{\alpha_s}{2\pi}\left(C_F - C_A\right)\frac{1}{\epsilon^2}\left(\frac{zQ^2}{\mu^2}\right)^{-\epsilon}\right]. \quad (36)$$

Off-diagonal DIS cross section

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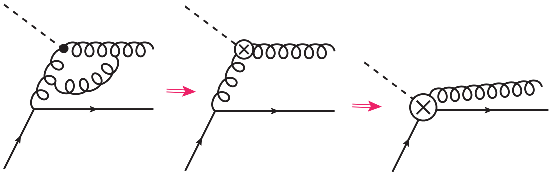

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$$D^{B1}\left(zQ^2, zQ^2\right) \exp\left[-\frac{\alpha_s}{2\pi}\left(C_F - C_A\right)\frac{1}{\epsilon^2}\left(\frac{zQ^2}{\mu^2}\right)^{-\epsilon}\right]. \tag{36}$$

$$\mathcal{P}_{qg,\mathrm{hard}} = \frac{\alpha_s C_F}{2\pi}\frac{1}{z}\exp\left[\frac{\alpha_s}{\pi}\frac{1}{\epsilon^2}\left(-C_A\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} + \left(C_A - C_F\right)\left(\frac{\mu^2}{zQ^2}\right)^{\epsilon}\right)\right],$$

Off-diagonal DIS cross section

$$
\begin{aligned}
&W_{\phi,q}\Big|^{hard}_{q\phi^*\to qg} \\
&= \int_0^1 dz\, \left(\frac{\mu^2}{s_{qg}z}\right)^\epsilon \mathcal{P}_{qg,hard}\left(s_{qg},z\right)\Big|_{s_{qg}=Q^2(1-x)} \\
&= \frac{\alpha_s C_F}{2\pi}\left(-\frac{1}{\epsilon}\right)\left(\frac{\mu^2}{Q^2(1-x)}\right)^\epsilon \exp\left[-\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right] \\
&\times \frac{\exp\left[\frac{\alpha_s(C_A-C_F)}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]-1}{\frac{\alpha_s(C_A-C_F)}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon}
\end{aligned}
$$

The result can be expanded in the strong coupling,

$$
W_{\phi,q}\Big|^{hard}_{q\phi^*\to qg} = \sum_{n=1}\left(\frac{\alpha_s}{4\pi}\right)^n c^{(n)}_{n1}\frac{1}{\epsilon^{2n-1}}\left(\frac{\mu^{2n}}{Q^{2n}(1-x)}\right)^\epsilon
$$

with

$$
c^{(n)}_{n1} = \frac{(-4)^n}{2n!}C_F(C_F^{n-1}+C_F^{n-2}C_A+\cdots+C_A^{n-1})
$$

Consistency relations for DIS

$$W = \sum_i W_{\phi,i}\, f_i \;=\sum_k \tilde{C}_{\phi,k}\tilde{f}_k$$

Multiplicative renormalization factors

$$\tilde{f}_k = Z_{ki} f_i, \qquad W_{\phi,i} = \tilde{C}_{\phi,k} Z_{ki}\,,$$

The splitting kernels are given by

$$P_{ij} = -\gamma_{ij} = \frac{dZ_{ik}}{d\ln\mu}(Z^{-1})_{kj}\,.$$

The four relevant virtualities (scales) are:

- hard, $p^2 = Q^2$
- anti-hardcollinear, $p^2 = Q^2\lambda^2 = Q^2/N$
- collinear, $p^2 = \Lambda^2$
- softcollinear, $p^2 = \Lambda^2\lambda^2 = \Lambda^2/N$

The LP is simple.

$$W_{\phi,g}\, f_g = f_g(\Lambda)\times\sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \frac{1}{\epsilon^{2n}} \sum_{k=0}^{n}\sum_{j=0}^{n} b_{kj}^{(n)}(\epsilon)\left(\frac{\mu^{2n}N^j}{Q^{2k}\Lambda^{2(n-k)}}\right)^{\epsilon} + \mathcal{O}\left(\frac{1}{N}\right)$$

k: hard + anti-hardcollinear, j: anti-hardcollinear and softcollinear.
Boundary condition:

$$W_{\phi,g}^{LP,LL}\Big|_{\text{hard loops}} = \exp\left[-\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\right]$$

Solution:

$$(W_{\phi,g}\, f_g)^{LP,LL} = \exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} - \left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}\right\}(N^{\epsilon}-1)\right] f_g(\Lambda)$$

Consistency relations for DIS

Clearly, the above equation factorizes into

$$W_{\phi,g}^{LP,LL} = \exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(N^{\epsilon}-1)\right]$$

$$f_g^{LP,LL} = \exp\left[-\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}(N^{\epsilon}-1)\right] f_g(\Lambda)$$

$\overline{MS}$ Renormalization factor:

$$Z_{gg}^{LP,LL} = \exp\left[\frac{\alpha_s C_A}{\pi}\frac{\ln N}{\epsilon}\right],$$

$$\tilde{C}_{\phi,g} = \exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(N^{\epsilon}-1)-\epsilon\ln N\right)\right]$$

Anomalous dimension:

$$P_{gg}^{LP,LL}(N) = -\frac{\alpha_s C_A}{\pi}2\ln N$$

Consistency relations for DIS

$$\sum_i (W_{\phi,i} f_i)^{NLP} = W_{\phi,q}^{NLP} f_q^{LP} + W_{\phi,\bar{q}}^{NLP} f_{\bar{q}}^{LP} + W_{\phi,g}^{NLP} f_g^{LP} + W_{\phi,g}^{LP} f_g^{NLP}$$

Using the boundary condition of $W_{\phi,q}\Big|_{q\phi^*\to qg}^{\text{hard}}$, we obtain

$$W_{\phi,q}^{NLP,LP} = \frac{1}{2N\ln N}\frac{C_F}{C_F - C_A}\exp\left[\frac{\alpha_s C_F}{\pi}\frac{\ln N}{\epsilon}\right]\frac{w}{e^w - 1}\left(e^{a/w} e^{\widehat{S}_A} - e^{\widehat{S}_F}\right)$$

with

$$w \equiv -\epsilon \ln N, \qquad a = \frac{\alpha_s}{\pi}(C_F - C_A)\ln^2 N$$

$$\widehat{S}_i = \frac{\alpha_s C_i}{\pi}\frac{1}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(N^\epsilon - 1) - \epsilon \ln N\right\}, \qquad i = A, F$$

Off-diagonal DIS cross section

$$W_{\phi,q}^{NLP} = \tilde{C}_{\phi,q}^{NLP} Z_{qq}^{LP} + \tilde{C}_{\phi,g}^{LP} Z_{gq}^{NLP}$$

Define

$$F(w,a) \equiv \frac{we^{a/w}}{e^{w}-1} = F_{\rm pole}(w,a) + F_{\rm fin}(w,a)$$

$$Z_{gq}^{NLP,LL} = \frac{1}{2N\ln N}\frac{C_F}{C_F - C_A}\exp\left[\frac{\alpha_s C_F}{\pi}\frac{\ln N}{\epsilon}\right] F_{\rm pole}(w,a)$$

The off-diagonal splitting kernel

$$\begin{aligned} P_{gq}^{NLP,LL}(N) &= -\frac{1}{N}\frac{\alpha_s C_F}{\pi}\left[F_{\rm pole}(w,a) - w\frac{d}{da}F_{\rm pole}(w,a)\right] \\ &= \frac{1}{N}\frac{\alpha_s C_F}{\pi}\mathcal{B}_0(a) \end{aligned}$$

Summary

- The universal structure of the large logarithms in cross sections is controlled by the factorization formula and the anomalous dimensions.
- The picture at leading power has been understood up to higher order corrections.
- At subleading power, the factorization becomes complicated.
- For the diagonal channel, the soft function exhibits divergences. One needs to introduce new soft function to perform renormalization.

Summary

- For the off-diagonal channel, the end-point singularity appears. The traditional factorization breaks down. We have to work in d-dimension in order to generate the correct all order result.
- A new scale in the end-point region indicates a two-step matching. Using the consistency relations, we obtain the off-diagonal DGLAP evolution kernel to all orders, which contains double logarithms in itself.

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Thank you !