Off-diagonal Bethe ansatz method

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The Hamiltonian of the closed XXZ chain is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}, \quad \alpha = x,\, y,\, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar\frac{\partial}{\partial t}h_i = [H, h_i] = 0, \qquad i = 1, \ldots.$$

and

$$[h_i, h_j] = 0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i.$$

Then

$$[t(u),\ t(v)] = 0, \qquad H \propto \frac{\partial}{\partial u} \ln t(u)|_{u=0} + const,$$

or

$$H \propto h_0^{-1} h_1 + const,$$
$$h_0\, \sigma_i^{\alpha}\, h_0^{-1} = \sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u - \theta_N) \ldots R_{01}(u - \theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \frac{1}{\sinh \eta} \begin{pmatrix} \sinh(u+\eta) & & & \\ & \sinh u & \sin \eta & \\ & \sinh \eta & \sin u & \\ & & & \sinh(u+\eta) \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$.

In the case of N=1,

$$A(u) = \frac{1}{\sinh\eta}\begin{pmatrix} \sinh(u-\theta_1+\eta) & \\ & \sinh(u-\theta_1) \end{pmatrix}, \quad B(u) = \begin{pmatrix} & \\ 1 & \end{pmatrix},$$

$$C(u) = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad D(u) = \frac{1}{\sinh\eta}\begin{pmatrix} \sinh(u-\theta_1) & \\ & \sinh(u-\theta_1+\eta) \end{pmatrix}.$$

In the case of N=2

$$A(u) = A_2(u)A_1(u) + B_2(u)C_1(u), \quad B(u) = A_2(u)B_1(u) + B_2(u)D_1(u),$$

$$C(u) = C_2(u)A_1(u) + D_2(u)C_1(u), \quad D(u) = C_2(u)B_1(u) + D_2(u)D_1(u).$$

$$\vdots$$

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v)=R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{1}$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{0\,0'}(u-v)\,T_0(u)\,T_{0'}(v)=T_{0'}(v)\,T_0(u)\,R_{0\,0'}(u-v). \tag{2}$$

This leads to $[t(u),t(v)]=0$.

In terms of the matrix elements of the monodromy matrix, the RLL relation read

$$B(u)B(v) = B(v)B(u), \quad (3)$$

$$A(u)B(v) = \frac{\sinh(u-v-\eta)}{\sinh(u-v)} B(v)A(u) + \frac{\eta}{\sinh(u-v)} B(u)A(v), \quad (4)$$

$$D(u)B(v) = \frac{\sinh(u-v+\eta)}{\sinh(u-v)} B(v)D(u) - \frac{\eta}{\sinh(u-v)} B(u)D(v), \quad (5)$$

$$\vdots$$

There exists a quasi-vacuum state (or reference state) $|\Omega\rangle$ such that

$$|\Omega\rangle = |\uparrow, \ldots, \uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (6)$$

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle = \prod_{j=1}^{N} (\sinh u - \theta_j + \eta)|\Omega\rangle, \quad (7)$$

$$D(u)|\Omega\rangle = d(u)|\Omega\rangle = \prod_{j=1}^{N} \sinh(u - \theta_j)|\Omega\rangle, \quad (8)$$

$$C(u)|\Omega\rangle = 0, \quad B(u)|\Omega\rangle \neq 0. \quad (9)$$

Let us introduce the Bethe state

$$|\lambda_1, \ldots, \lambda_M\rangle = B(\lambda_1)\ldots B(\lambda_M)\,|\Omega\rangle. \tag{10}$$

The action of the transfer matrix reads

$$\begin{aligned} t(u)|\lambda_1, \ldots, \lambda_M\rangle \quad &= \quad \prod_{i=1}^{M} \frac{\sinh(u-\lambda_i-\eta)}{\sinh(u-\lambda_i)} a(u)|\lambda_1, \ldots, \lambda_M\rangle \\ &\quad + \prod_{i=1}^{M} \frac{\sinh(u-\lambda_i+\eta)}{\sinh(u-\lambda_i)} d(u)|\lambda_1, \ldots, \lambda_M\rangle \\ &\quad + \text{unwanted terms.} \end{aligned}$$

Quantum Spin Chains with $U(1)$-symmetry

Periodic boundary condition

If the parameters $\{\lambda_i\}$ needs satisfy Bethe ansatz equations,

$$\prod_{k\neq j}\frac{\sinh(\lambda_j-\lambda_k+\eta)}{\sinh(\lambda_j-\lambda_k-\eta)}=\prod_{j=1}^{N}\frac{\sinh(\lambda_j-\theta_k+\eta)}{\sinh(\lambda_j-\theta_k)},\qquad j=1,\ldots,M. \tag{11}$$

Then the Bethe states become the common eigenstates of $t(u)$ with eigenvalue $\Lambda(u)$

$$t(u)|\lambda_1,\ldots,\lambda_M\rangle=\Lambda(u)|\lambda_1,\ldots,\lambda_M\rangle,$$

where $\Lambda(u)=\Lambda(u;\lambda_1,\ldots,\lambda_M)$ is given by

$$\begin{aligned}\Lambda(u) &= a(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i-\eta)}{\sinh(u-\lambda_i)}+d(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i+\eta)}{\sinh(u-\lambda_i)},\\ &= a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(\eta)},\end{aligned} \tag{12}$$

where

$$Q(u)=\prod_{i=1}^{M}\sinh(u-\lambda_i). \tag{13}$$

The Hamiltonian of the XXZ chain with twisted boundary condition is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha} = e^{i\phi\sigma_1^z}\,\sigma_1^{\alpha}\,e^{-i\phi\sigma_1^z}, \quad \alpha = x,\, y,\, z.$$

The phase factor ϕ can be arbitrary complex number. The system is **integrable**, i.e., the corresponding transfer matrix can be constructed as

$$t(u) = tr(e^{i\phi\sigma^z}\,T(u)) = e^{i\phi}A(u) + e^{-i\phi}D(u).$$

The transfer matrix can diagonalized by algebraic Bethe ansatz similar as that of periodic case. The Bethe state is the same as (10), namely,

$$|\lambda_1,\ldots,\lambda_M\rangle = B(\lambda_1)\ldots B(\lambda_M)\,|\Omega\rangle.$$

If the parameters $\{\lambda_i\}$ satisfies Bethe ansatz equations,

$$\prod_{k\neq j}\frac{\sinh(\lambda_j-\lambda_k+\eta)}{\sinh(\lambda_j-\lambda_k-\eta)}=e^{2i\phi}\prod_{j=1}^{N}\frac{\sinh(\lambda_j-\theta_k+\eta)}{\sinh(\lambda_j-\theta_k)},\qquad j=1,\ldots,M. \tag{14}$$

Then the Bethe states become the common eigenstates of $t(u)$ with eigenvalue $\Lambda(u)$

$$\begin{aligned}\Lambda(u) &= e^{i\phi}a(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i-\eta)}{\sinh(u-\lambda_i)}+e^{-i\phi}d(u)\prod_{i=1}^{M}\frac{\sinh(u-\lambda_i+\eta)}{\sinh(u-\lambda_i)},\\ &= e^{i\phi}a(u)\frac{Q(u-\eta)}{Q(u)}+e^{-i\phi}d(u)\frac{Q(u+\eta)}{Q(\eta)}.\end{aligned} \tag{15}$$

The Hamiltonian of the XXZ chain with antiperiodic boundary condition is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha} = \sigma_1^x\,\sigma_1^{\alpha}\,\sigma_1^x, \quad \alpha = x,\, y,\, z.$$

The system is **integrable**, i.e., the corresponding transfer matrix can be constructed as

$$t(u) = tr(\sigma^x T(u)) = B(u) + C(u).$$

The model is a typical integrable without $U(1)$ symmetry. Most of conventional Bethe ansatz method fails to give the solution because of the lack of a proper vacuum (or reference) state.

Recently, we give a solution to the spectrum problem of the corresponding transfer matrix in

- *Phys. Rev. Lett.* **111** (2013), 137201 [arXiv:1305.7328].

Let $|\Psi\rangle$ be an eigenstate of the transfer matrix with an eigenvalue

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle.$$

Due to the fact that $|\Psi\rangle$ does not depend on u, we can derive the following properties which can determine $\Lambda(u)$ completely

$$\Lambda(u), \text{ as a function of } u, \text{ is a trigonometric polynomial of degree } N-1, \quad (16)$$

$$\Lambda(u+i\pi) = (-1)^{N-1}\Lambda(u), \quad (17)$$

$$\Lambda(\theta_j)\,\Lambda(\theta_j-\eta) = -f_1(\theta_j)f_2(\theta_j-\eta), \quad j=1,\ldots,N. \quad (18)$$

The functions $f_1(u)$, $f_2(u)$ are

$$f_1(u) = \prod_{i=1}^{N}\sinh(u-\theta_i+\eta), \quad f_2(u) = f_1(u-\eta). \quad (19)$$

The solution to the above equations is given by

$$\Lambda(u) = e^{u} f_1(u)\frac{Q_1(u-\eta)}{Q_2(u)} - e^{-u-\eta} f_2(u)\frac{Q_2(u+\eta)}{Q_1(u)} - b(u)\frac{f_1(u)f_2(u)}{Q_1(u)Q_2(u)}, \quad (20)$$

$$Q_1(u) = \prod_{j=1}^{M}\sinh(u-\mu_j), \quad Q_2(u) = \prod_{j=1}^{M}\sinh(u-\nu_j),$$

$$b^{(e)}(u) = e^{u+\phi_1} - e^{-u-\eta+\phi_2}, \quad b^{(o)}(u) = e^{2u+\phi_1} + e^{-2u-2\eta+\phi_2}.$$

$$\phi_1 = \sum_{j=1}^{N}\theta_j - M\eta - 2\sum_{j=1}^{M}\mu_j, \quad \phi_2 = -\sum_{j=1}^{N}\theta_j + M\eta + 2\sum_{j=1}^{M}\nu_j,$$

The parameters $\{\mu_j\}$ and $\{\nu_j\}$ satisfy the associated Bethe ansatz equations

$$f_2(\nu_j) = \frac{e^{\nu_j}}{b(\nu_j)} Q_1(\nu_j-\eta)Q_1(\nu_j), \quad j = 1, \ldots, M, \quad (21)$$

$$f_1(\mu_j) = -\frac{e^{-\mu_j-\eta}}{b(\mu_j)} Q_2(\mu_j+\eta)Q_2(\mu_j), \quad j = 1, \ldots, M. \quad (22)$$

Open XXZ chain Hamiltonian

$$H = -\frac{1}{2}\sum_{k=1}^{N-1}\left(\sigma_k^x\sigma_{k+1}^x+\sigma_k^y\sigma_{k+1}^y+\cosh\eta\sigma_k^z\sigma_{k+1}^z\right) + f_1^x\sigma_1^x+f_1^y\sigma_1^y+f_1^z\sigma_1^z + f_N^x\sigma_N^x+f_N^y\sigma_N^y+f_N^z\sigma_N^z$$

The model is **integrable**. If the components of boundary fields are parameterized by

$$F_1=(f_1^x,f_1^y,f_1^z)=\frac{\sinh\eta}{\sinh\alpha_-\cosh\beta_-}(\coth\alpha_-\sinh\beta_-,\cosh\theta_-,i\sinh\theta_-)$$

$$F_N=(f_N^x,f_N^y,f_N^z)=\frac{\sinh\eta}{\sinh\alpha_+\cosh\beta_+}(-\coth\alpha_+\sinh\beta_+,\cosh\theta_+,i\sinh\theta_+).$$

The corresponding transfer matrix $t(u)$ can be constructed by the six-vertex R-matrix and the associated K-matrices, i.e.,

$$t(u)=tr(\mathbb{T}(u))=tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the most general solutions of the reflection equation and its dual.

The K-matrix $K^-(u)$ is given by

$$\begin{aligned} K^-(u) &= \begin{pmatrix} K^-_{11}(u) & K^-_{12}(u) \\ K^-_{21}(u) & K^-_{22}(u) \end{pmatrix}, \\ K^-_{11}(u) &= 2\left(\sinh(\alpha_-)\cosh(\beta_-)\cosh(u) + \cosh(\alpha_-)\sinh(\beta_-)\sinh(u)\right), \\ K^-_{22}(u) &= 2\left(\sinh(\alpha_-)\cosh(\beta_-)\cosh(u) - \cosh(\alpha_-)\sinh(\beta_-)\sinh(u)\right), \\ K^-_{12}(u) &= e^{\theta_-}\sinh(2u), \quad K^-_{21}(u) = e^{-\theta_-}\sinh(2u), \end{aligned} \tag{23}$$

and it satisfies the reflection equation (RE)

$$\begin{aligned} & R_{12}(u_1-u_2)K^-_1(u_1)R_{21}(u_1+u_2)K^-_2(u_2) \\ & \quad = K^-_2(u_2)R_{12}(u_1+u_2)K^-_1(u_1)R_{21}(u_1-u_2). \end{aligned} \tag{24}$$

The dual K-matrix $K^+(u)$ satisfies the following dual RE

$$
\begin{aligned}
&R_{12}(u_2-u_1)K_1^+(u_1)R_{21}(-u_1-u_2-2)K_2^+(u_2)\\
&\quad = K_2^+(u_2)R_{12}(-u_1-u_2-2)K_1^+(u_1)R_{21}(u_2-u_1).
\end{aligned} \tag{25}
$$

The most general solution to the DRE is

$$
K^+(u) = K^-(-u-\eta)\big|_{(\alpha_-,\beta_-,\theta_-)\to(-\alpha_+,-\beta_+,\theta_+)}\,. \tag{26}
$$

The Hamiltonian can be expressed in terms of the transfer matrix

$$
H = \sinh\eta\frac{\partial \ln t(u)}{\partial u}|_{u=0,\theta_j=0} - N\cosh\eta - \tanh\eta\sinh\eta.
$$

For the very special case of $F_1 = (f_1^x, f_1^y, f_1^z) = (0, 0, f_1^z)$ and $F_N = (0, 0, f_N^z)$, namely,

$$H = -\frac{1}{2}\sum_{k=1}^{N-1}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\sigma_{k+1}^z\right) + f_1^z\sigma_1^z + f_N^z\sigma_N^z$$

the model was solved by Sklyanin (J. Phys. A 21 (1988), 2375). The boundary QISM has failed to solve the spectral problem of the general case for many years. However, it can be solved by a generalized boundary QISM developed (Fan et al Nucl. Phys. B 478 (1996), 723, Cao et al Nucl. Phys. B 663 (2003), 487) for some case. In these cases, a local vacuum state does exist and the corresponding Bethe sates have similar structure as that of closed but with a different quasi-particle creation operator $\mathcal{B}(u)$ and reference state $\tilde{\Omega}$. The corresponding Bethe states are

$$|\lambda_1, \ldots, \lambda_M\rangle = \mathcal{B}(\lambda_1)\ldots\mathcal{B}(\lambda_M)\,|\tilde{\Omega}\rangle,$$

where the parameters $\{\lambda_i\}$ needs satisfy the associate Bethe ansatz equations.

For the generic $F_1 = (f_1^x, f_1^y, f_1^z)$, $F_N = (f_N^x, f_N^y, f_N^z)$ and generic anisotropic parameter Δ , the model has not been solved since Sklyanin's work in 1988 until our recent works:

- *Nucl. Phys.* **B 875** (2013), 152-165 [`arXiv:1306.1742`];
- *Nucl. Phys.* **B 877** (2013), 152-175 [`arXiv:1307.2023`].

In the solutions, we can give the eigenvalues in terms of some parameters which satisfy associated Bethe ansatz equations, without the explicit expressions of the corresponding eigenstates (such as Bethe states). However, in principle we can reconstruct the corresponding eigenstates with some recursion relations.

Besides the quantum Yang-Baxter equation, the R-matrix satisfies

$$\text{Initial condition}: \ R_{12}(0) = P_{12}, \tag{27}$$

$$\text{Unitarity relation}: \ R_{12}(u)R_{21}(-u) = -\frac{\sinh(u+\eta)\sinh(u-\eta)}{\sinh\eta\sinh\eta} \times \mathrm{id}, \tag{28}$$

$$\text{Crossing relation}: \ R_{12}(u) = V_1 R_{12}^{t_2}(-u-\eta)V_1, \quad V = -i\sigma^y, \tag{29}$$

$$\text{PT-symmetry}: \ R_{12}(u) = R_{21}(u) = R_{12}^{t_1\, t_2}(u), \tag{30}$$

$$Z_2\text{-symmetry}: \quad \sigma_1^i\sigma_2^i R_{1,2}(u) = R_{1,2}(u)\sigma_1^i\sigma_2^i, \quad \text{for } i = x, y, z, \tag{31}$$

$$\text{Antisymmetry}: \ R_{12}(-\eta) = -\eta(1-P) = -2\eta P^{(-)}. \tag{32}$$

In addition to reflection equations, the K-matric satisfies

$$K^-(0) = \frac{1}{2}tr(K^-(0)) \times \mathrm{id}, \quad K^-(\frac{i\pi}{2}) = \frac{1}{2}tr(K^-(\frac{i\pi}{2})) \times \sigma^z.$$

These properties and the quasi-periodic properties of R-matrix and K-matrices imply

$$\begin{aligned} t(-u-\eta) &= t(u), \quad t(u+i\pi)=t(u), \\ t(0) &= -2^3 \sinh\alpha_- \cosh\beta_- \sinh\alpha_+ \cosh\beta_+ \cosh\eta \\ &\quad \times \prod_{l=1}^{N} \frac{\sinh(\eta-\theta_l)\sinh(\eta+\theta_l)}{\sinh\eta\,\sinh\eta} \times \mathrm{id}, \\ t(\frac{i\pi}{2}) &= -2^3 \cosh\alpha_- \sinh\beta_- \cosh\alpha_+ \sinh\beta_+ \cosh\eta \\ &\quad \times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh\eta\,\sinh\eta} \times \mathrm{id}, \\ \lim_{u\to\pm\infty} t(u) &= -\frac{\cosh(\theta_- - \theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta} \times \mathrm{id} + \ldots, \end{aligned}$$

and the very operator identity

$$t(\theta_j)t(\theta_j-\eta) = -\frac{\sinh^2\eta\,\Delta_q^{(o)}(\theta_j)}{\sinh(2\theta_j+\eta)\sinh(2\theta_j-\eta)}, \quad \Delta_q^{(o)}(u)=\delta(u)\times\mathrm{id}.$$

Let $|\Psi\rangle$ be a common eigenstate of the transfer matrix with an eigenvalue $\Lambda(u)$, then

$$\Lambda(-u-\eta) = \Lambda(u), \quad \Lambda(u+i\pi)=\Lambda(u), \tag{33}$$

$$\Lambda(0) = -2^3 \sinh\alpha_- \cosh\beta_- \sinh\alpha_+ \cosh\beta_+ \cosh\eta \tag{34}$$

$$\times \prod_{l=1}^{N} \frac{\sinh(\eta-\theta_l)\,\sinh(\eta+\theta_l)}{\sinh\eta\,\sinh\eta}, \tag{35}$$

$$\Lambda(\frac{i\pi}{2}) = -2^3 \cosh\alpha_- \sinh\beta_- \cosh\alpha_+ \sinh\beta_+ \cosh\eta \tag{36}$$

$$\times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh\eta\,\sinh\eta}, \tag{37}$$

$$\lim_{u\to\pm\infty}\Lambda(u) = -\frac{\cosh(\theta_- - \theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta} + \ldots, \tag{38}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -\frac{\sinh^2\eta\,\delta(\theta_j)}{\sinh(2\theta_j+\eta)\sinh(2\theta_j-\eta)}. \tag{39}$$

The function $\delta(u)$ is given by

$$\begin{aligned}\delta(u) &= -2^4\frac{\sinh(2u-2\eta)\sinh(2u+2\eta)}{\sinh\eta\,\sinh\eta}\sinh(u+\alpha_-)\sinh(u-\alpha_-)\cosh(u+\beta_-)\\&\quad\times\cosh(u-\beta_-)\sinh(u+\alpha_+)\sinh(u-\alpha_+)\cosh(u+\beta_+)\cosh(u-\beta_+)\\&\quad\times\prod_{l=1}^{N}\frac{\sinh(u+\theta_l+\eta)\sinh(u-\theta_l+\eta)\sinh(u+\theta_l-\eta)\sinh(u-\theta_l-\eta)}{\sinh(\eta)\sinh(\eta)\sinh(\eta)\sinh(\eta)}.\end{aligned}$$

Moreover, it follows that $\Lambda(u)$, as an entire function of u, is a trigonometric polynomial of degree $2N+4$. Hence (33)-(39) completely determine the function $\Lambda(u)$. For this purpose, let us introduce the following functions:

$$\begin{aligned}A(u) &= \prod_{l=1}^{N}\frac{\sinh(u-\theta_l+\eta)\,\sinh(u+\theta_l+\eta)}{\sinh\eta\,\sinh\eta},\\a(u) &= -2^2\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\sinh(u-\alpha_-)\cosh(u-\beta_-)\\&\quad\times\sinh(u-\alpha_+)\cosh(u-\beta_+)A(u),\\d(u) &= a(-u-\eta).\end{aligned}$$

$$\Lambda(u) = a(u)\frac{Q_1(u-\eta)}{Q_2(u)} + d(u)\frac{Q_2(u+\eta)}{Q_1(u)} + \frac{2c\sinh(2u)\sinh(2u+2\eta)}{Q_1(u)Q_2(u)}A(u)A(-u-\eta), \quad (40)$$

where the functions $Q_1(u)$ and $Q_2(u)$ are some trigonometric polynomials parameterized by N Bethe roots $\{\mu_j|j=1,\ldots,N\}$ as follows,

$$Q_1(u) = \prod_{j=1}^{N}\frac{\sinh(u-\mu_j)}{\sinh(\eta)}, \quad Q_2(u) = Q_1(-u-\eta). \quad (41)$$

the parameters $\bar{c}$ is determined by the boundary parameters and μ_j

$$c = \cosh((N+1)\eta + \alpha_- + \beta_- + \alpha_+ + \beta_+ + 2\sum_{j=1}^{N}\mu_j) - \cosh(\theta_- - \theta_+). \quad (42)$$

The N parameters $\{\mu_j\}$ satisfy the associated Bethe ansatz equations

$$\frac{2c\sinh(2\mu_j)\sinh(2\mu_j+2\eta)\,A(\mu_j)A(-\mu_j-\eta)}{d(\mu_j)Q_2(\mu_j)Q_2(\mu_j+\eta)} = -1,\ j=1,\ldots,N, \quad (43)$$

and with the following selection rule for the roots of the above equations

$$\mu_j \neq \mu_l \quad \text{and} \quad \mu_j \neq -\mu_l - \eta. \quad (44)$$

$$\Lambda(u) = a(u)\frac{Q_1(u-\eta)}{Q_2(u)} + d(u)\frac{Q_2(u+\eta)}{Q_1(u)} + \frac{2^3 c \sinh(2u)\sinh(2u+2\eta)}{Q_1(u)Q_2(u)}\frac{\sinh u \sinh(u+\eta)}{\sinh\eta\sinh\eta}A(u)A(-u-\eta), \quad (45)$$

$$Q_1(u) = \prod_{j=1}^{N+1}\frac{\sinh(u-\mu_j)}{\sinh(\eta)}, \quad Q_2(u) = Q_1(-u-\eta). \quad (46)$$

$$c = \cosh((N+3)\eta + \alpha_- + \beta_- + \alpha_+ + \beta_+ + 2\sum_{j=1}^{N+1}\mu_j) - \cosh(\theta_- - \theta_+). \quad (47)$$

The $N+1$ parameters $\{\mu_j\}$ satisfy the associated Bethe ansatz equations

$$\frac{2^3 c \sinh(2\mu_j)\sinh(2\mu_j+2\eta)\,A(\mu_j)A(-\mu_j-\eta)}{d(\mu_j)Q_2(\mu_j)Q_2(\mu_j+\eta)} = -\frac{\sinh\eta\sinh\eta}{\sinh\mu_j\sinh(\mu_j+\eta)},$$
$$j = 1, \ldots, N+1, \quad (48)$$

with the very selection rule (44) for the roots of the above equations.

- QYBE& REs $\Rightarrow$ **Integrability** $\Leftrightarrow$ **Transfer matrix**

- *Intrinsic Prop.* of R(or K)-matrix $\Rightarrow$ **Operator Id.** $\Leftrightarrow$ **Solvability**

Moreover, almost all of standard R-matrices and K-matrices have such intrinsic properties.

The R-matrix is given by

$$R_{12}(u) = u + \eta P_{12}, \quad P|i,j\rangle = |j,i\rangle, \quad i,j = 1,\ldots,n, \tag{49}$$

and the associated the most general K-matrices are given by

$$K^-(u) \quad = \quad \xi + uM, \quad M^2 = \mathrm{id}, \tag{50}$$

$$K^+(u) \quad = \quad \bar{\xi} - (u + \frac{n}{2}\eta)\bar{M}, \quad \bar{M}^2 = \mathrm{id},. \tag{51}$$

The R-matrix satisfies QYBE and the K-matrices satisfy REs. The transfer matrix is given by

$$t(u) = tr_0\left\{K_0^+(u)T(u)K_0^-(u)\hat{T}_0(u)\right\}, \quad [t(u),t(v)] = 0,$$

$$T_0(u) = R_{0N}(u-\theta_N)R_{0\,N-1}(u-\theta_{N-1})\ldots R_{01}(u-\theta_1),$$

$$\hat{T}_0(u) = R_{10}(u+\theta_1)\ldots R_{N-1\,0}(u+\theta_{N-1})\ldots R_{N0}(u+\theta_N).$$

Intrinsic properties of the R-matrix

$$R_{12}(0) = \eta P_{12}, \quad R_{12}(\pm\eta) = \pm 2\eta P_{12}^{(\pm)}, \tag{52}$$

$$R_{12}(u)\, R_{21}(-u) = \rho_1(u)\,\mathrm{id}, \quad R_{12}^{t_1}(u)\, R_{21}^{t_1}(-u-n\eta) = \rho_2(u)\,\mathrm{id}. \tag{53}$$

and the corresponding properties of the K-matrices:

$$K^-(0) = \xi, \quad K^+(-\frac{n}{2}\eta) = \bar{\xi}, \tag{54}$$

$$K^-(u)K^-(-u) \propto \mathrm{id}, \quad K^+(u)\, K^+(-u-n\eta) \propto \mathrm{id}. \tag{55}$$

These intrinsic properties of the R-matrix and K-matrices lead to the operator identities:

$$t(\pm\theta_j)t_m(\pm\theta_j-\eta) = t_{m+1}(\pm\theta_j)\prod_{k=1}^{m}\rho_2^{-1}(\pm 2\theta_j-k\eta)\rho_0(\pm\theta_j), \quad (56)$$

$$m=1,\ldots,n-1,\quad j=1,\ldots,N,$$

$$\rho_0(u) = \prod_{l=1}^{N}(u-\theta_l-\eta)(u+\theta_l-\eta)\prod_{k=2}^{m}(2u-k\eta)(-2u-k\eta+(n-2)\eta),$$

$$t_n(u)=\mathrm{Det}_q(u)\mathrm{id},$$

and others $n(n-1)$ relations among $\{t_m(u)\}$. The above relations completely determine the eigenvalues of all fused transfer matrices.

- *JHEP* **04** (2014), 143 [arXiv:1312.4770].

The R-matrix reads

$$R_{12}(u)=\left(\begin{array}{ccc|ccc|ccc}
c(u) & & & & & & & & \\
 & b(u) & & e(u) & & & & & \\
 & & d(u) & & g(u) & & f(u) & & \\
\hline
 & \bar{e}(u) & & b(u) & & & & & \\
 & & \bar{g}(u) & & a(u) & & g(u) & & \\
 & & & & & b(u) & & e(u) & \\
\hline
 & & \bar{f}(u) & & \bar{g}(u) & & d(u) & & \\
 & & & & & \bar{e}(u) & & b(u) & \\
 & & & & & & & & c(u)
\end{array}\right).$$

It is the first simplest model beyond A-type.

The matrix elements are

$$a(u) = \sinh(u-3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh\eta, \;\; b(u) = \sinh(u-3\eta) + \sin 3\eta,$$

$$c(u) = \sinh(u-5\eta) + \sinh\eta, \quad d(u) = \sinh(u-\eta) + \sinh\eta,$$

$$e(u) = -2e^{-\frac{u}{2}} \sinh 2\eta \cosh(\frac{u}{2} - 3\eta), \quad \bar{e}(u) = -2e^{\frac{u}{2}} \sinh 2\eta \cosh(\frac{u}{2} - 3\eta),$$

$$f(u) = -2e^{-u+2\eta} \sinh\eta \sinh 2\eta - e^{-\eta} \sinh 4\eta,$$

$$\bar{f}(u) = 2e^{u-2\eta} \sinh\eta \sinh 2\eta - e^{\eta} \sinh 4\eta,$$

$$g(u) = 2e^{-\frac{u}{2}+2\eta} \sinh\frac{u}{2} \sinh 2\eta, \quad \bar{g}(u) = -2e^{\frac{u}{2}-2\eta} \sinh\frac{u}{2} \sinh 2\eta.$$

The associated non-diagonal K-matrices $K^-(u)$ is

$$K^-(u) = \begin{pmatrix} 1+2e^{-u-\epsilon}\sinh\eta & 0 & 2e^{-\epsilon+\sigma}\sinh u \\ 0 & 1-2e^{-\epsilon}\sinh(u-\eta) & 0 \\ 2e^{-\epsilon-\sigma}\sinh u & 0 & 1+2e^{u-\epsilon}\sinh\eta \end{pmatrix},$$

$$K^+(u) = \mathcal{M}K^-(-u+6\eta+i\pi)\big|_{(\epsilon,\sigma)\to(\epsilon',\sigma')},$$

$$\mathcal{M} = \mathrm{Diag}(e^{2\eta},1,e^{-2\eta}).$$

There four boundary parameters ϵ, σ, ϵ', σ'.

Intrinsic properties of the R-matrix

$$R_{12}(0) \propto \eta P_{12}, \quad R_{12}(u)\,R_{21}(-u) = \rho_1(u), \tag{57}$$
$$R_{12}^{t_1}(u)\,\mathcal{M}_1\,R_{21}^{t_1}(-u+12\eta)\,\mathcal{M}_1^{-1} = \rho_2(u)\ \times \mathrm{id}, \tag{58}$$
$$R_{12}(6\eta+i\pi) = P_{12}^{(1)} \times S_{12}^{(1)}, \quad R_{12}(4\eta) = P_{12}^{(3)} \times S_{12}^{(3)}. \tag{59}$$

and the corresponding properties of the K-matrices:

$$K^-(0) = \xi, \quad K^+(-\frac{n}{2}\eta) = \bar{\xi}, \tag{60}$$
$$K^-(u)K^-(-u) \propto \mathrm{id}, \quad K^+(u)\,K^+(-u+6\eta+i\pi) \propto \mathrm{id}. \tag{61}$$

$P_{12}^{(1)}$ and $P_{12}^{(3)}$ are projectors with rank 1 and 3 respectively.

These properties lead to the following operator identities which complete characterize the spectrum of the transfer matrix

$$t(\theta_j)t(\theta_j+6\eta+i\pi) = \frac{\delta_1(u)\ \times \mathrm{id}}{\rho_1(2u)}\Big|_{u=\theta_j}, \quad j=1,\ldots,N, \tag{62}$$

$$t(\theta_j)t(\theta_j+4\eta) = \frac{\delta_2(u)\ \times t(u+2\eta+i\pi)}{\rho_2(-2u+8\eta)}\Big|_{u=\theta_j}, \quad j=1,\ldots,N, \tag{63}$$

$$t(u) = t(-u+6\eta+i\pi), \quad t(u)=t(u+2i\pi), \tag{64}$$

and the values of the transfer matrix at 0, $i\pi$, ∞.

- *JHEP* 06 (2014), 128 [arXiv:1403.7915].

Conclusion and comments

So far, many typical $U(1)$-symmetry-broken models have been solved by the method:

- The spin torus.
- The XYZ closed spin chain.
- The spin-$\frac{1}{2}$ Heisenberg chain with arbitrary boundary fields and its higher spin generalization.
- The open spin chains with general boundary condition associated with A-type algebras.
- The Hubbard model with unparallel boundary fields.
- The t-J model with unparallel boundary fields.
- The Izergin-Korepin model with non-diagonal boundary terms.

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Integrability $\Leftrightarrow$ Solvability

Thank for your attentions