Braneworld in f(R) gravity and critical gravity

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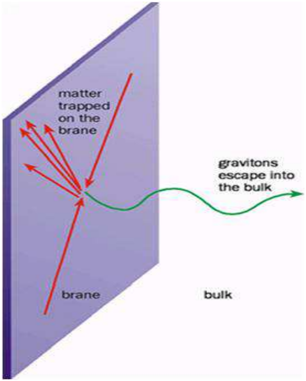
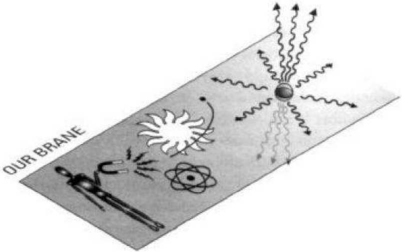
Brief history of extra dimensions and braneworlds

- 1920's, Kaluza and Klein, KK Theory
- 1980's, Akama, Rubakov, Domain Wall Braneworld
- 1998, Arkani-Hamed, Dimopoulos, and Dvali, Large Extra Dimension (ADD Braneworld Scenario)
- 1999, Randall and Sundrum(RS), Warped Extra Dimension (RS thin Braneworld Scenario)
- 1999, DeWolfe, Freedman, Gubser, and Karch, Thick Braneworld Scenario

Picture of braneworlds

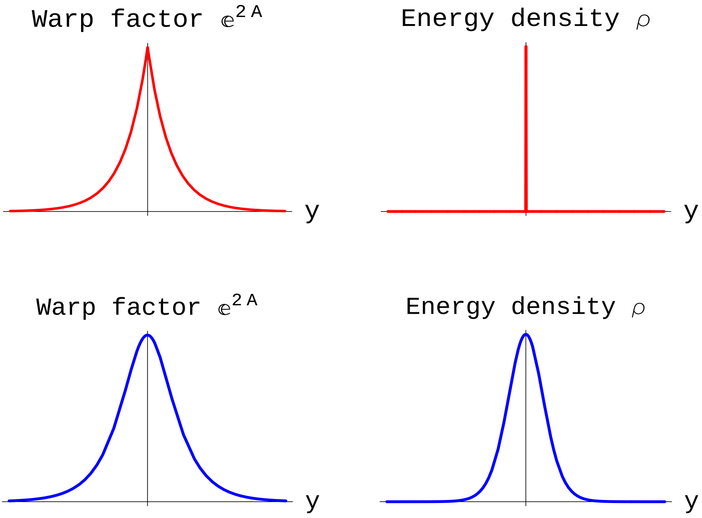




- 4D space-time is seen as a brane (hypersurface) embedded in higher-dimensional space-time
- Matter and gauge fields are confined on the brane, only the gravity can propagate in the bulk

Thin brane & thick brane

$ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu - dy^2$, $e^{2A(y)}$ is the warp factor.



Why braneworlds in $f(R)$ and critical gravities?

- General relativity is not a renormalizable theory.
- By adding some higher order curvature terms (such as $R^2, R^2_{\mu\nu}, R^2_{\mu\nu\lambda\sigma}$)) to the original Einstein-Hilbert action, we can get a renormalizable theory of gravity.
- Higher-order curvature terms also appear in low-energy effective action of string theory.
- It is necessary and important to reconsider braneworld scenarios in higher-order curvature modified theories.

Why braneworlds in $f(R)$ and critical gravities? (cont.)

- $f(R)$ gravity and critical gravity are such simple toy models.
- $f(R)$ gravity has been applied in discussing dark energy, inflation, thermodynamics, entropic force, the holographic principle, etc..
- Critical gravity [Lu and Pope, PRL 106 (2011) 181302] possess such an AdS vacuum, for which there is only massless tensor, and the linearized excitations have vanishing energy.

Related works about thin $f(R)$-brane

About thin $f(R)$-brane

1. In 1988, Barrow and Cotsakis found that $f(R)$ gravity conformally equivalents to the second order Einstein gravity plus a scalar degree of freedom.
2. BC's discovery was applied to construct thin $f(R)$-brane by M. Parry, S. Pichler, and D. Deeg in 2005 [hep-ph/0502048]
3. For thin branes the Israel junction conditions were investigated by two groups: Balcerzak etc. [arXiv:0710.3670] and Dabrowski etc. [arXiv:0711.1150] in 2007.
4. Bouhmadi-Lopez etc. discussed $f(R)$ brane cosmology [arXiv:1001.3028] recently.

Related works about thick $f(R)$-brane

For thick $f(R)$-branes, the related works are

1. [1] V.I. Afonso, D. Bazeia, R. Menezes, and A.Yu. Petrov,
"$f(R)$-Brane",
PLB 658(2007)71, arXiv:0710.3790.
2. [2] V. Dzhunushaliev, V. Folomeev, B. Kleihaus, and J. Kunz, "Some thick brane solutions in $f(R)$-gravity",
JHEP 1004(2010)130, arXiv:0912.2812.
3. [3] Y. Zhong, Y.-X. Liu and K. Yang,
"Tensor perturbations of $f(R)$-branes,
Phys. Lett. B699 (2011) 398, arXiv:1010.3478.
4. [4] Y.-X. Liu, Y. Zhong, Z.-H. Zhao and H.-T. Li,
"Domain wall brane in squared curvature gravity",
JHEP 1106 (2011) 135,arXiv:1104.3188.

5. [5] H. Liu, H. Lu and Z.-L. Wang, "f(R) Gravities, Killing Spinor Equations, "BPS" Domain Walls and Cosmology", arXiv:1111.6602.

In Refs. [1,2] $f(R) \propto R + R^n$ are assumed.

[1] PLB 658(2007)71:

- Thick $f(R)$-brane solutions exist either for constant and variant bulk curvatures with a background scalar field.
- The curvature equals to a constant is only a special case.
- For the non-constant curvature case, the solution has a singular point, which we don't want to see in thick braneworld models.

Related works about thick $f(R)$-brane (cont.)

[2] JHEP 1004(2010)130:

- Some numerical thick $f(R)$-brane solutions exist without the introducing of background scalar field.
- The numerical solution can not be applied to discuss more complex problems.
- There is a problem in localizing fermions on the brane.

[5] H. Liu, H. Lu and Z.-L. Wang, arXiv:1111.6602:

- The Killing spinor equations are used to reduce the fourth-order differential equations of motion to the first order for both the domain wall and FLRW cosmological solutions.
- "BPS" domain walls that describe the smooth Randall-Sundrum II were obtained.

The context of the report

1. **The thick $f(R)$-brane solution in $f(R)$ gravity with $f(R) = R + \gamma R^2$ [4].**
2. **The analysis of the tensor perturbations of brane metric in $f(R)$ gravity [3,4].**
3. **The thin and thick brane solutions in critical gravity.**

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Recently, we have found an analytical thick brane solution of the action

$$S = \int d^4xdy\sqrt{-g}\left[f(R) - \frac{1}{2}g^{AB}(\nabla_A\phi)(\nabla_B\phi) - V(\phi)\right], \quad (1)$$

where $f(R)$ and $V(\phi)$ are given by

$$f(R) = \frac{1}{2\kappa_5^2}(R + \gamma R^2). \quad (2)$$

$$V(\phi) = \lambda(\phi^2 - v^2)^2 + \Lambda_5, \quad (3)$$

and the metric has the following from

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \quad (4)$$

$f(R)$ thick brane model (cont.)

The (four-order) field equations are

$$
\begin{aligned}
f(R) + 2f_R\left(4A'^2 + A''\right) - 6f'_R A' - 2f''_R &= \kappa_5^2(\phi'^2 + 2V), && (5)\\
-8f_R\left(A'' + A'^2\right) + 8f'_R A' - f(R) &= \kappa_5^2(\phi'^2 - 2V), && (6)\\
4A'\phi' + \phi'' &= \frac{\partial V}{\partial \phi}. && (7)
\end{aligned}
$$

For the case $f(R) = R + \gamma R^2$,

$$
\begin{aligned}
f_R &= 1 + 2\gamma R, && (8)\\
f'_R &= 2\gamma R', && (9)\\
f''_R &= 2\gamma R'', && (10)\\
R' &= -(8A''' + 40A'A''), && (11)\\
R'' &= -(8A'''' + 40A''^2 + 40A'A'''). && (12)
\end{aligned}
$$

The solution

1. The solution is [1]

$$e^{2A(y)} = \cosh^{-2}(ky), \qquad (13)$$
$$\phi(y) = v\tanh(ky). \qquad (14)$$

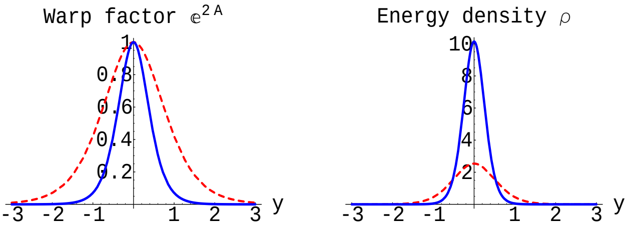
where $v = 7\sqrt{3/(29\kappa^2)}$.

2. At the boundary of the extra dimension, i.e., $y = \pm\infty$, the bulk curvature $R = -20k^2$, which means that the spacetime is asymptotically anti-de Sitter. The corresponding cosmological constant $\Lambda = -\frac{159k^2}{29\kappa_5^2}$.

3. As shown in the following figure, this solution describes a typical thick braneworld model.

[1] Y.-X. Liu, Y. Zhong, Z.-H. Zhao and H.-T. Li, JHEP 1106 (2011) 135

The solution



$\kappa_5 = 1$, $k = 1$(red dotted) and $k = 2$(blue solid).

The stability of the solution

The analysis of a full set of gravitational fluctuations of the metric is a hard work.

However, the problem can be simplified when one only considers the transverse and traceless (TT) part of the metric fluctuation.

- Consider the tensor perturbations of the background metric:

$$ds^2 = e^{2A(y)}(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2, \tag{15}$$

  where $h_{\mu\nu} = h_{\mu\nu}(x^\mu,\ y)$.
- The perturbation of the scalar field $\delta\phi(x^\mu,\ y)$ decouples from the metric perturbations under the transverse and traceless gauge $\partial_\mu h^\mu_\nu = 0 = \eta^{\mu\nu} h_{\mu\nu}$.

The stability of the solution (cont.)

- **Thus, the perturbed Einstein equation is [Y. Zhong, Y.-X. Liu and K. Yang, PLB 699 (2011) 398]**

$$\Box^{(5)} h_{\mu\nu} = \frac{f_R'}{f_R} \partial_y h_{\mu\nu}, \tag{16}$$

For $f(R) = R$, this equation reduces to the KG equation for massless spin-2 particles.

- **By doing the coordinate transformation $dz = e^{-A(y)} dy$, we can rewrite the metric (4) into a conformally flat one:**

$$ds^2 = e^{2A(y(z))}(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2). \tag{17}$$

Then the perturbed equation (16) reads

$$\left[\partial_z{}^2 + \left(6\partial_z A + \frac{\partial_z f_R}{f_R}\right)\partial_z + \Box^{(4)}\right] h_{\mu\nu} = 0. \tag{18}$$

The stability of the solution (cont.)

With the KK decomposition $h_{\mu\nu} = e^{ipx} e^{-\frac{3}{2}A} f_R^{-1/2} \varepsilon_{\mu\nu} h(z)$, the above equation is reduced to the following Schrodinger equation for $h(z)$:

$$\left[-\partial_z^2 + V_G(z)\right] h(z) = m^2 h(z), \tag{19}$$

where the potential is given by

$$V_G(z) = \left(\frac{3}{2}A'' + \frac{9}{4}(A')^2\right) + \left(3A'\frac{f_R'}{f_R} - \frac{1}{4}\frac{f_R'^2}{f_R^2} + \frac{1}{2}\frac{f_R''}{f_R}\right). \tag{20}$$

One can also factorize the Schrodinger equation (19) as

$$\left[\partial_z + \left(3A' + \frac{1}{2}\frac{f_R'}{f_R}\right)\right]\left[-\partial_z + \left(3A' + \frac{1}{2}\frac{f_R'}{f_R}\right)\right] h(z) = m^2 h(z), \tag{21}$$

The stability of the solution (cont.)

which indicates that

- there is no gravitational mode with $m^2 < 0$ and therefore the solution for our model is stable.
- The zero mode ($m^2 = 0$)

$$h_0(z) = N_0 e^{\frac{3}{2}A(y(z))} f_R^{1/2}(R(z)) \qquad (22)$$

  is the ground state of the Schrodinger equation (19).
- In $f(R)$ gravity, the zero mode depends not only on the form of the warp factor, but also on the form of $f(R)$.

The localization of gravity

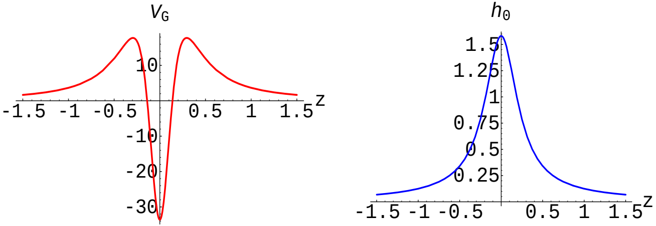
For our model, the warp factor $e^{2A(y)} = \cosh^{-2}(ky)$ becomes $e^{2A(y(z))} = (1 + k^2z^2)^{-1}$ in z coordinate, and the potential takes the form

$$V_G(z) = \frac{15k^2\left(-14 + 37k^2z^2 + 28k^4z^4 + 4k^6z^6\right)}{4\left(5 + 7k^2z^2 + 2k^4z^4\right)^2}. \quad (23)$$

When $|z| \to \infty$, $V_G(z) \to \frac{15}{4z^2}$. The corresponding zero mode can be normalized as

$$h_0(z) = \sqrt{\frac{k}{8}}\frac{\sqrt{5 + 2k^2z^2}}{(1 + k^2z^2)^{5/4}}. \quad (24)$$

The localization of gravity (cont.)



- **The existence of the zero mode indicates that the gravity can be localized on the brane.**
- **The zero mode gives the Newtonian potential $U \propto \frac{1}{r}$ felt by the massive objects on the brane.**
- **In addition to the zero mode, there is a series of continues massive KK states, which would give a correction to the Newtonian potential.**

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Thin brane solutions in critical gravity

First, we consider the thin braneworld with co-dimension one generated in $n = (p+1)$-dimensional critical gravity, where $n \geq 3$. The action is

$$S = S_{\mathbf{g}} + S_{\mathbf{b}}, \tag{25}$$

where the gravity part $S_{\mathbf{g}}$ and the brane part $S_{\mathbf{b}}$ are given by

$$S_{\mathbf{g}} = \frac{1}{2\kappa^2}\int d^n x\sqrt{-g}\Big[R-(n-2)\Lambda_0+\alpha R^2+\beta R_{MN}R^{MN}\Big], \tag{26}$$

$$S_{\mathbf{b}} = \int d^{n-1}x\sqrt{-g^{(\mathbf{b})}}\; V_0. \tag{27}$$

Here $g^{(\mathbf{b})}_{\mu\nu}$ is the induced metric on the brane, and V_0 is the brane tension.

Thin brane solutions in critical gravity (cont.)

The line-element is assumed as

$$ds^2 = g_{MN}dx^M dx^N = \mathrm{e}^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \qquad (28)$$

where e^{2A} **is the warp factor with the normalized condition** $\mathrm{e}^{2A(0)} = 1$ **on the brane located at** $y = 0$**. We introduce the** Z_2 **symmetry by setting** $A(y) = A(-y)$**.**

The equations of motion are given by

$$G_{MN} + \alpha E^{(1)}_{MN} + \beta E^{(2)}_{MN} = \kappa^2 T_{MN} - \frac{1}{2}(n-2)\Lambda_0 g_{MN}, \qquad (29)$$

where $T_{\mu\nu} = -V_0\, g_{\mu\nu}\delta(y)$**,** $T_{nn} = 0$**,**

$$E^{(1)}_{MN} = 2R\Big(R_{MN} - \frac{1}{4}R\, g_{MN}\Big) + 2g_{MN}\Box R - 2\nabla_M\nabla_N R,$$

$$E^{(2)}_{MN} = 2R^{PQ}(R_{MPNQ} - \frac{1}{4}R_{PQ}\, g_{MN}) + \Box(R_{MN} + \frac{1}{2}R\, g_{MN}) - \nabla_M\nabla_N R.$$

Thin brane solutions in critical gravity (cont.)

The junction conditions are determined by

$$\int_{0^-}^{0^+} dy\left[R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} + \alpha E^{(1)}_{\mu\nu} + \beta E^{(2)}_{\mu\nu}\right] = -\kappa^2 V_0\, g_{\mu\nu}(y=0).(30)$$

At $y \neq 0$, the explicit forms of the above EOMs are

$$\begin{aligned} & (p-1)(\Lambda_0 + pA'^2 + 2A'') - p^2(p-3)[(p+1)\alpha + \beta]A'^4 \\ - \ & 2p\left[(6p^2 - 4p - 6)\alpha + (p^2 + 3p - 6)\beta\right] A'^2 A'' \\ - \ & \left[4p\alpha + (p+1)\beta\right]\left(2A^{(4)} + 4pA'A^{(3)} + 3pA''^2\right) = 0, \qquad (31) \end{aligned}$$

$$\begin{aligned} & (p-1)\left(\Lambda_0 + pA'^2\right) - p^2(p-3)((p+1)\alpha + \beta)A'^4 \\ + \ & p\left[4p\alpha + (p+1)\beta\right]\left(A''^2 - 2A'A^{(3)} - 2pA'^2 A''\right) = 0. (32) \end{aligned}$$

It is very difficult to find the solution of thin brane for arbitrary α and β for the above four-order differential equations and the junction conditions (30).

However, at the critical point $4p\alpha + (p+1)\beta = 0$, the EOMs are reduced to:

$$\begin{aligned} 4\Lambda_0 + 4pA'^2 + p(p-1)(p-3)\beta A'^4 &= 0, && (33) \\ \left[2 + (p-1)(p-3)\beta A'^2\right] A'' &= 0, && (34) \end{aligned}$$

and the junction condition is

$$\int_{0^-}^{0^+} dy \left[2 + (p-1)(p-3)\beta A'^2\right] A'' = -\frac{2\kappa^2}{p-1} V_0. \qquad (35)$$

It can be seen from Eqs. (33) and (34) that the curvature-squared modifications in the 4D critical gravity ($p = 3$ or $n = 4$) has no effect on the brane solution.

Thin brane solutions in critical gravity (cont.)

In the following, we give the solutions of the above brane equations for $n = 4$ and $n \neq 4$, respectively.

The case $n = 4$

For this case ($p = 3$), the solution is

$$A(y) = -\sqrt{\frac{-\Lambda_0}{3}}|y|, \qquad (36)$$

$$V_0 = \frac{4}{\kappa^2}\sqrt{\frac{-\Lambda_0}{3}} \quad (> 0)\,. \qquad (37)$$

Thus, we get a brane with positive tension and the warp factor exponentially falling from the brane to infinity. The brane is embedded in a 4D AdS spacetime.

This is nothing but the RS solution in four dimensions. But the perturbation structure of the system is very different from the RS one.

The case $n \neq 4$

For the case $n \neq 4$, we first give the solution corresponding to $\beta = 0$:

$$A^{(\mathbf{RS})}(y) = -\sqrt{-\frac{\Lambda_0}{p}}|\mathbf{y}|, \quad (38)$$

$$V_0^{(\mathbf{RS})} = \frac{2(p-1)}{\kappa^2}\sqrt{-\frac{\Lambda_0}{p}} \quad (>0) \,, \quad (39)$$

which describes the RS positive tension brane embedded in an AdS spacetime.

Note that the tension brane is positive.

In critical gravity ($\beta \neq 0$), we have two solutions:

$$A_\pm(y) = -\sqrt{\frac{2\left[-p \pm \sqrt{p[p-(p-1)(p-3)\beta\Lambda_0]}\,\right]}{p(p-1)(p-3)\beta}}\,|\mathbf{y}|, \quad (40)$$

$$V_{0\pm} = \pm\frac{p-1}{p\kappa^2}\sqrt{\frac{-8\Lambda_0 p[p-(p-1)(p-3)\beta\Lambda_0]}{p \pm \sqrt{p[p-(p-1)(p-3)\beta\Lambda_0]}}}, \quad (41)$$

which correspond to positive and negative tension branes, respectively.

For the positive tension brane solution, the constrain conditions for the parameters are $\Lambda_0 < 0$ and $\beta > 0$, or $\Lambda_0 < 0$ and $\frac{p}{(p-1)(p-3)\Lambda_0} < \beta < 0$.

For the negative tension case, the constrain conditions are $\beta < 0$ and $\Lambda_0 > 0$, or $\beta < 0$ and $\frac{p}{(p-1)(p-3)\beta} < \Lambda_0 < 0$.

The case $n \neq 4$ (cont.)

Next, we study the limits of the solutions (40) under the condition of $\beta\Lambda_0 \to 0$.

For the negative tension brane solution $A_-(y)$, the limit is divergent.

While, for the case of positive tension, $A_+(y)$ and V_{0+} can be expanded as

$$A_+(y) = -\sqrt{-\frac{\Lambda_0}{p}}\left[1 + \frac{(p-1)(p-3)}{8p}\beta\Lambda_0 + \mathcal{O}(\beta\Lambda_0)^2\right]|y|, \qquad (42)$$

$$V_{0+} = \frac{2(p-1)}{\kappa^2}\sqrt{\frac{-\Lambda_0}{p}}\left[1 - \frac{3(p-1)(p-3)}{8p}\beta\Lambda_0 + \mathcal{O}(\beta\Lambda_0)^2\right]. \quad (43)$$

So, when $\beta \to 0$ but keep Λ_0 as a constant, the above positive tension brane solution (40)-(41) can be reduced to the RS one (38)-(39), while the negative one cannot.

It is interesting to note that, when Λ_0 and β satisfy the following relation

$$\Lambda_0 = \frac{p}{(p-1)(p-3)\beta}, \tag{44}$$

the brane tension vanishes and the warp factor is simplified as

$$A(y) = -\sqrt{-\frac{2\Lambda_0}{p}}|y|. \tag{45}$$

Obviously, such solution could not appear in standard Einstein gravity theory. While, in critical gravity theory, although the naked brane tension is zero, but with the identification of $\kappa^2 T^{(\text{eff})}_{\mu\nu} \equiv -\alpha E^{(1)}_{\mu\nu} - \beta E^{(2)}_{\mu\nu}$, there will be a positive effective brane tension coming from the contribution of the curvature-squared terms.

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Next, we consider the thick brane generated by a scalar field in n-dimensional critical gravity. The action reads as

$$S = S_{\mathbf{g}} + S_{\mathbf{m}}, \tag{46}$$

where $S_{\mathbf{g}}$ is given by (26) and the matter part is

$$S_{\mathbf{m}} = \int d^n x \sqrt{-g} \Big[-\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - V(\phi) \Big]. \tag{47}$$

The line-element is also assumed as (28) and the scalar field $\phi = \phi(y)$ for static branes.

Thick brane solution in critical gravity (cont.)

The EOMs for general α and β are four-order, while they are reduced to the following two-order ones in critical case:

$$\left[(p-1)(p-3)\beta A'^2 - 2\right] A'' = \frac{2\kappa^2}{p-1}\phi'^2, \tag{48}$$

$$p(p-1)(p-3)\beta A'^4 + 4pA'^2 + 4\Lambda_0 = \frac{8\kappa^2}{p-1}\left(\frac{1}{2}\phi'^2 - V\right), \tag{49}$$

$$\phi'' + pA'\phi' = V_\phi, \tag{50}$$

where $V_\phi \equiv \frac{dV}{d\phi}$.

Note that eq. (50) can be derived from eqs. (49) and (48). Hence, the above three equations are not independent.

We can solve the above second-order differential equations by the superpotential method.

Thick brane solution in critical gravity (cont.)

Introducing the superpotential function $W(\phi)$, the EOMs (49)-(50) can be solved by the first-order equations:

$$
\begin{aligned}
A' &= -\frac{\kappa^2}{p-1} W, && (51)\\
\phi' &= \left(1 - hW^2\right) W_\phi, && (52)\\
V &= \frac{1}{2}\left(1 - hW^2\right)^2 W_\phi^2 - \frac{p-1}{2\kappa^2}\Lambda_0 \\
&\quad - \frac{p\kappa^2}{2(p-1)} W^2 \left(1 - \frac{1}{2} hW^2\right), && (53)
\end{aligned}
$$

where $h = -\frac{p-3}{2(p-1)}\beta\kappa^4$.

Again, the parameter β has no effect on the Einstein equations in four-dimensional case.

Thick brane solution in critical gravity (cont.)

The energy density $\rho(y)$ of the system is given by

$$\rho(y) = -T_{MN}\, u^M u^N = -T^0{}_0 = \frac{1}{2}\phi'^2 + V, \tag{54}$$

where $u^M = (E^{-A}, 0, 0, 0, 0)$.

For brane solutions, we require that the energy density on the boundaries of the extra dimension y should vanish:

$$\rho(|y| \to \infty) \to 0, \tag{55}$$

with which the naked cosmological constant Λ_0 will be determined.

For $n = 4$, the EOMs read

$$A' = -\frac{\kappa^2}{2} W, \tag{56}$$
$$\phi' = W_\phi, \tag{57}$$
$$V = -\frac{3}{4}\kappa^2 W^2 + \frac{1}{2} W_\phi^2 - \frac{\Lambda_0}{\kappa^2}. \tag{58}$$

In order for the scalar to get a kink solution, the potential $V(\phi)$ should at least has two finite vacua. And the usual ϕ^4 potential is a natural choice.

However, with the superpotential method, the ϕ^4 potential derived from the superpotential $W(\phi) = a\phi + b\phi^2$ can not support kink solution for the scalar.

Hence, we turn to use another superpotential $W(\phi) = a\left(\phi - \frac{\phi^3}{3v_0^2}\right)$, which corresponds to the ϕ^6 model:

$$V(\phi) = -\frac{a^2\kappa^2}{12v_0^4}\left(\phi^2 - v_0^2\right)^2\left[\phi^2 - 2(3\kappa^{-2} + 2v_0^2)\right] \qquad (59)$$

with the two vacua are at $\phi_\pm = \pm v_0$ (the extreme points of the superpotential $W(\phi)$).

The solution is found to be

$$\begin{aligned} \phi(y) &= v_0 \tanh(ky), & (60) \\ e^{2A(y)} &= \left[\cosh(ky)\right]^{-\frac{2}{3}\kappa^2 v_0^2} \exp\left(-\frac{1}{6}\kappa^2 v_0^2 \tanh^2(ky)\right), & (61) \\ \Lambda_0 &= -\frac{1}{3}a^2 v_0^2 \kappa^4 \quad < 0\,, & (62) \end{aligned}$$

where $k = a/v_0$.

The scalar curvature and the energy density are

$$R(y) = \frac{1}{3}a^2\kappa^2\left[\kappa^2 v_0^2 \mathbf{sech}^6(ky) + 3(\kappa^2 v_0^2 + 3)\mathbf{sech}^4(ky) - 4\kappa^2 v_0^2\right], \quad (63)$$

$$\rho(y) = a^2\left[\frac{\kappa^2 v_0^2}{12}\left(3 + \mathbf{sech}^2(ky)\right) + 1\right]\mathbf{sech}^4(ky). \quad (64)$$

The energy density vanishes at the boundaries:

$$\rho(|y| \to \infty) \propto e^{-4k|y|} \to 0. \quad (65)$$

And the bulk curvature $R(|y| \to \infty) \to -\frac{4}{3}a^2 v_0^2 \kappa^4 = 4\Lambda_0$**, which means that the spacetime is asymptotically AdS. The corresponding cosmological constant is just the naked one:** $\Lambda = \Lambda_0$**.**

As shown in the following figure, the above solution describes a typical thick braneworld embedded in an AdS spacetime.

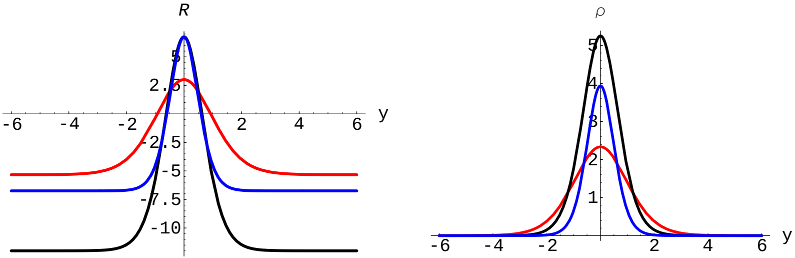

Figure: The shapes of the scalar curvature $R(y)$ and the energy density $\rho(y)$ for the case $n = 4$.

For general $n \neq 4$ ($n \geq 5$ and $n = 3$) and $\beta \neq 0$, $h \neq 0$ and so we have chance to get the usual ϕ^4 potential by setting $W = a\phi$. The potential is

$$V(\phi) = b(\phi^2 - v_0^2)^2, \tag{66}$$

where

$$b = \frac{p-3}{8(p-1)^2}\left[(p-3)a^2\kappa^2\beta^2 - p\beta\right]a^4\kappa^6, \tag{67}$$

$$v_0^2 = -\frac{2(p-1)}{(p-3)a^2\kappa^4\beta}. \tag{68}$$

When $(p-3)\beta > 0$, the above $V(\phi)$ (66) is not a usual ϕ^4 potential with two degenerate vacua since $v_0^2 < 0$. Such potential does not support a thick brane solution.

The case $n \neq 4$ (cont.)

So we consider the case of $(p-3)\beta < 0$, namely,

- $p > 3$ **and** $\beta < 0$, **or**
- $p = 2$ **and** $\beta > 0$,

for which $v_0^2 > 0$, $b > 0$, **so the potential** $V(\phi) = b(\phi^2 - v_0^2)^2$ **has two vacua at** $\phi_\pm = \pm v_0$. **The solution is**

$$\phi(y) = v_0 \tanh(ky), \quad (69)$$

$$e^{2A(y)} = \left[\cosh(ky)\right]^{-\frac{2}{p-1}\kappa^2 v_0^2}, \quad (70)$$

$$\Lambda_0 = \frac{p}{(p-1)(p-3)\beta} < 0\,, \quad (71)$$

where $k = a/v_0 = \sqrt{-\frac{(p-3)}{2(p-1)}\kappa^4\beta}$. **This solution stands for a thick flat brane with the energy density given by**

$$\rho(y) = \frac{1}{2} v_0^2 \left(k^2 + 2bv_0^2\right) \mathbf{sech}^4(ky). \quad (72)$$

The case $n \neq 4$ (cont.)

On the boundaries $|y| \to \infty$,

$$A(|y| \to \infty) \to -\sqrt{-\frac{2\Lambda_0}{p}}|y|. \qquad (73)$$

So the asymptotic solution (73) with the relation (71) ($\Lambda_0 = \frac{p}{(p-1)(p-3)\beta}$) is in accord with the thin brane solution (44)-(45) given in previous section.

From the asymptotic solution (73), we have $R_{MN}(|y| \to \infty) \to 2\Lambda_0 g_{MN} = \Lambda g_{MN}$.

Therefore, the thick flat brane is embedded in an AdS spacetime with the effective cosmological constant $\Lambda = 2\Lambda_0$.

The case $n \neq 4$ (cont.)

In $n(\neq 4)$-dimensional critical gravity without matter fields, there are two disconnect AdS vacua with the cosmological constants determined by $\Lambda_0 = \Lambda - \frac{(p-1)(p-3)\beta}{4p}\Lambda^2$ [2].

In the case here with matter field, because of the relation $\Lambda_0 = \frac{p}{(p-1)(p-3)\beta}$, which is caused by the condition (55) $\rho(|y| \to \infty) \to 0$, the two asymptotic AdS vacua become the same one with the cosmological constant $\Lambda = 2\Lambda_0$.

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[2] H. Lu and C. N. Pope, *Critical Gravity in Four Dimensions*, Phys. Rev. Lett. **106** (2011) 181302 [arXiv:1101.1971[hep-th]].

The tensor fluctuation of the metric

The tensor perturbation of the background metric is

$$ds^2 = e^{2A(y)}(\eta_{\mu\nu} + \bar{h}_{\mu\nu})dx^\mu dx^\nu + dy^2. \quad (74)$$

The fluctuation equations in TT gauge are given by

$$G^{(L)}_{MN} + \frac{1}{2}(n-2)\Lambda_0 e^{2A}\bar{h}_{\mu\nu} + \alpha E^{(1)(L)}_{MN} + \beta E^{(2)(L)}_{MN} = 0, \quad (75)$$

where the fluctuation of $G^{(1)}_{\mu\nu}$ **is**

$$G^{(L)}_{\mu\nu} = -\frac{1}{2}\left[\Box^{(4)}\bar{h}_{\mu\nu} + e^{2A}\bar{h}''_{\mu\nu} + 4A'e^{2A}\bar{h}'_{\mu\nu} + e^{2A}(4A'^2 + A'')\bar{h}_{\mu\nu}\right], \quad (76)$$

and the fluctuations of $E^{(1)}_{\mu\nu}$ and $E^{(2)}_{\mu\nu}$ read

$$
\begin{aligned}
E^{(1)(L)}_{\mu\nu} = & \; 4(2A'' + 5A'^2)\Box^{(4)}\bar{h}_{\mu\nu} + 4(2A'' + 5A'^2)e^{2A}\bar{h}''_{\mu\nu} \\
& + 8e^{2A}(A''' + 9A'A'' + 10A'^3)\bar{h}'_{\mu\nu} \\
& - 8e^{2A}(2A'''' + 16A'A''' + 12A''^2 + 37A'^2A'' + 5A'^4)\bar{h}_{\mu\nu}, \\
E^{(2)(L)}_{\mu\nu} = & -\frac{1}{2}e^{-2A}\Box^{(4)}\Box^{(4)}\bar{h}_{\mu\nu} - \Box^{(4)}\bar{h}''_{\mu\nu} \\
& - 2A'\Box^{(4)}\bar{h}'_{\mu\nu} + 2(A'' + 3A'^2)\Box^{(4)}\bar{h}_{\mu\nu} \\
& - \frac{1}{2}e^{2A}\bar{h}''''_{\mu\nu} - 4e^{2A}A'\bar{h}'''_{\mu\nu} - 4e^{2A}A'^2\bar{h}''_{\mu\nu} \\
& + e^{2A}(40A'^3 + 16A'A'' + 2A''')\bar{h}'_{\mu\nu} \\
& - e^{2A}(5A'''' + 40A'A''' + 30A''^2 - 72A'^4 + 20A''A'^2)\bar{h}_{\mu\nu}.
\end{aligned}
$$

It is unclear whether the tensor perturbation of the brane metric is stable.

Summary

For $f(R)$ gravity theory:

- In the frame of fourth-order $f(R)$ gravity theory, we have found a smooth thick brane solution.
- The brane is generated by a background scalar field with the usual ϕ^4 potential.
- The tensor perturbation of the metric solution is stable.
- The 4D massless graviton can be localized on such brane.
- By introducing the Yukawa coupling between the fermion and the background scalar field, the fermions can be trapped on the brane.

For critical gravity theory:

- The curvature-squared modifications in the 4D critical gravity has no effect on the brane solution. So the brane solutions are same with the RS ones in four dimensions
- In other dimensions, the thin brane solutions are also similar to the RS ones. But the brane tension can be positive, negative or vanishing in some conditions.
- The thick branes are also similar to the RS ones. They are embedded in AdS space-times with effective cosmological constant $\Lambda = \Lambda_0$ and $\Lambda = 2\Lambda_0$ for $n = 4$ and $n \neq 4$, respectively.
- The fluctuation equations of the brane solution are very different for the critical gravity (four-order) and the standard Einstein gravity (two-order).

Open issues

There are some questions should be addressed:

- It is unclear whether the scalar perturbations of the brane metric are stable in $F(R)$ gravity with matter fields.
- It is unclear whether the tensor and scalar perturbations of the brane metric are stable in critical gravity.
- It is not known whether the four-dimensional Newton's Law can be recovered on the brane in critical gravity.
- The reduced Einstein equations on the branes may be applied to cosmology.

Thanks!