Two-loop NMHV octagons from $\bar{Q}$ equations

Based on 1911.01290 with Song He and Zhenjie Li

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Symmetries of planar $\mathcal{N} = 4$ sYM and $\bar{Q}$ equations

planar $\mathcal{N} = 4$ sYM: Harmonic oscillator of QFT

1. Solvable 4-dimensional QFT
2. New mathematical structures
3. Fruitful playground for Feynman loop integrals
4. SUSY cousin of QCD

Field Content and Superamplitude

Simplicity of field content:

- 2 gauge bosons with $h = \pm 1$: $|a\rangle^{+1}, |a\rangle^{-1}_{ABCD}$,
- 8 fermions with $h = \pm 1/2$: $|a\rangle^{+1/2}_{A}, |a\rangle^{-1/2}_{BCD}$,
- 6 scalars: $|a\rangle^{0}_{AB}$.

Related by SUSY generators $\mathcal{Q}^{\alpha}_{A}$ and $\widetilde{\mathcal{Q}}^{\dot{\alpha}}_{A}$,

grouped into a single supermultiplet:

$$
\begin{aligned}
|a\rangle &:= \exp(\widetilde{\mathcal{Q}}_A \cdot \widetilde{\lambda} \cdot \eta^A)|a\rangle^+ \\
&= |a\rangle^+ + \eta^A |a\rangle^{1/2}_A + \cdots + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D|a\rangle^-_{ABCD}
\end{aligned}
$$

We are considering the scattering of n supermultiplets:

$$
A_n(\{p_i, \eta_i\}) = \frac{\delta^4(P)\delta^8(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\left(\mathcal{A}_{n,0}(\{p_i\}) + \mathcal{A}_{n,1}(\{p_i,\eta_i\}) + \cdots\right)
$$

Dual super conformal symmetries and notations

Amplitudes in planar $\mathcal{N}=4$ sYM enjoy not only superconformal symmetries, but also dual superconformal symmetries, [Drummond,Henn,Smirnov,Sokatchev] which is manifest in a chiral superspace cooidnates (x, θ)

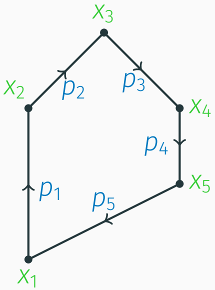
$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\widetilde{\lambda}_i^{\dot{\alpha}} = p_i^{\mu}\sigma_{\mu}^{\alpha\dot{\alpha}}, \qquad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^{\alpha}\eta_i^A$$

$$\text{planar poles:} \quad (p_i + p_{i+1} + \cdots + p_{j-1})^2 = x_{ij}^2$$

In dual space, an amplitude become a light-like polygonal Wilson loop which is invariant under conformal transformation:

$$I(x_i^{\alpha\dot{\alpha}}) = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2}$$

$$D(x_i^{\alpha\dot{\alpha}}) = t x_i^{\alpha\dot{\alpha}}$$



Symmetry generator

Dual superconformal symmetry $\mathrm{SL}(4|4)$ is linearly realized in terms of (super-)momentum twistor

$$\mathcal{Z}_i = (Z_i^a|\chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha\dot{\alpha}}\lambda_{i\alpha}|\theta_i^{\alpha A}\lambda_{i\alpha}). \qquad \text{[Hodges]}$$

For funture convenience, we introduce two basic invariants:

Plücker coordinate : $\langle ijkl\rangle := \varepsilon_{abcd}Z_i^a Z_j^b Z_k^c Z_l^d, \left(x_{ij}^2 = \frac{\langle i{-}1\, i\, j{-}1\, j\rangle}{\langle i{-}1\, i\rangle\langle j{-}1\, j\rangle}\right)$

R invariant : $[i\, j\, k\, l\, m] := \frac{\delta^{0|4}(\chi_i^A\langle jklm\rangle + \text{cyclic})}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}$

In terms of moment twistors, the generators of dual superconformal symmetries can be written as

$$G_I{}^J = \sum_i \mathcal{Z}_i^J \frac{\partial}{\partial \mathcal{Z}_i^I}\ .$$

Tree amplitudes satisfy

$$G R_n^{\text{tree}} = 0 \qquad A_n^{\text{tree}} = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\cdots\langle n1\rangle} R_n^{\text{tree}}$$

General structure of amplitude in planar $\mathcal{N} = 4$ sYM

- The dual conformal invariance of amplitudes is broken at the loop-level due to the infrared divergence.
- This symmtery can be restored by subtracting the infrared part A_n^{BDS} [Bern, Dixon, Smirnov].

$$A_n = \underbrace{A_n^{\text{BDS}}}_{IR} \times \overbrace{\underbrace{\exp(R_n)}_{\substack{\text{Remainder}\\ \text{function}}} \times \underbrace{\left(1 + \mathcal{P}_n^{\text{NMHV}} + \cdots + \mathcal{P}_n^{\overline{\text{MHV}}}\right)}_{\text{helicity structure}}}^{\text{finite functions of dual conformal invariants}}$$

For example, R_6 will be a function of $u = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}$, $v = \frac{\langle 3456\rangle\langle 6123\rangle}{\langle 3461\rangle\langle 5623\rangle}$, $w = \frac{\langle 5612\rangle\langle 2345\rangle}{\langle 5623\rangle\langle 1245\rangle}$.

We are interested in the function $R_{8,1}^{(2)} = \left(\exp(R_8)\mathcal{P}_8^{\text{NMHV}}\right)^{(2)}$

In the following, we will denote $\exp(R_n)\mathcal{P}_n^{\text{N}^k\text{MHV}}$ by $R_{n,k}$

Why octagons?

The cut structures of L-loop hexagon and heptagon amplitudes are described by polylogarithms [Goncharov] of weight $2L$ which satisfy

$$\mathrm{d}F^{(2L)} = \sum_{\beta} F_{\beta}^{(2L-1)} \mathrm{d}\log s_{\beta}$$

This define the symbol of F:

$$\mathcal{S}(F^{2L}) := \sum_{\beta} \mathcal{S}(F_{\beta}^{(2L-1)}) \otimes s_{\beta}$$

where s_{β} are called symbol letters.

Some example:

$$\mathcal{S}(\log x \log y) = x \otimes y + y \otimes x, \quad \mathcal{S}(\mathrm{Li}_2(1-z)) = -\big(z \otimes 1 - z\big)$$

The first entries of symbol indicate the locus of cuts of F

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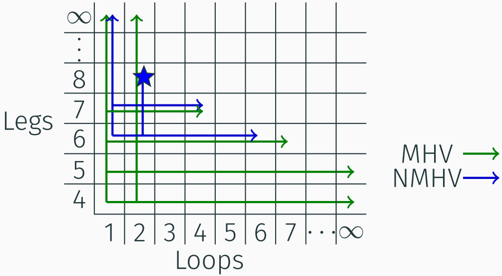
The alphabets (collection of all possible letters) for hexagon and heptagon are highly constrained by the corresponding cluster algebras $G_{4,6}$ and $G_{4,7}$ which consist of 9 and 42 variables, respectively.

Why octagons



[Bern,Caron-Huot, Dixon, Drummond,...]

For more than seven particles, symbol alphabets are not well understood

- $G_{4,n\geq 8}$ are infinite-type cluster algebras.
- Square roots appear in symbol letters even at one-loop in N^2MHV amplitudes

Dual superconformal anomaly and $\bar{Q}$ equations

BDS-normalized amplitudes $R_{n,k}$ are dual conformal invariants, but $R_{n,k}$ are not dual superconformal invariants, they have anomalies under the symmetries generated by

$$\bar{Q}_a^A = \sum_i \chi_i^A \frac{\partial}{\partial Z_i^a}$$

An OPE analysis tell us the action of $\bar{Q}$ on $R_{n,k}$ will yield an integral over higher-point amplitudes [Caron-Huot, He]

$$\bar{Q}_a^A R_{n,k} = \frac{\Gamma_{\text{cusp}}}{4} \int_{\tau=0}^{\tau=\infty} \oint_{\epsilon=0} \left(\mathrm{d}^{2|3} \mathcal{Z}_{n+1}\right)_a^A [R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\text{tree}}] + \text{cyclic}$$

where the particle $n+1$ is added in a collinear limit

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + \frac{\langle n{-}1\, n\, 2\, 3\rangle}{\langle n\, 1\, 2\, 3\rangle} \epsilon \tau \mathcal{Z}_1 + \frac{\langle n{-}2\, n{-}1\, n\, 1\rangle}{\langle n{-}2\, n{-}1\, 2\, 1\rangle} \epsilon^2 \mathcal{Z}_2$$

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Perturbatively, this equation becomes

$$\bar{Q}_a^A R_{n,k}^{(L)} = \int_{\tau=0}^{\tau=\infty} \oint_{\epsilon=0} \left(\mathrm{d}^{2|3}\mathcal{Z}_{n+1}\right)_a^A [R_{n+1,k+1}^{(L-1)} - R_{n,k}^{(L-1)} R_{n+1,1}^{\text{tree}}] + \text{cyclic}$$

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The integral measure

The basic operation $\int (\mathrm{d}^{2|3}\mathcal{Z}_{n+1})^A_a$ consist of bosonic part and fermionic part:

$$(\mathrm{d}^{2|3}\mathcal{Z}_{n+1})^A_a \begin{cases} \varepsilon_{abcd} Z^b_{n+1} \mathrm{d}Z^c_{n+1} \mathrm{d}Z^d_{n+1} = C(\bar{n})_a \epsilon \mathrm{d}\epsilon \mathrm{d}\tau & \text{(Bosonic Part)} \\ \\ (\mathrm{d}^3\chi_{n+1})^A & \text{(Fermionic Part)} \end{cases}$$

where $(\bar{n})_a := \varepsilon_{abcd} Z^b_{n-1} Z^c_n Z^d_1$

The order of performing integral:

- Fermionic integral $(\mathrm{d}^3\chi_{n+1})^A$
- The substitution $\mathcal{Z}_{n+1} \to \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + C\epsilon\tau \mathcal{Z}_1 + C'\epsilon^2 \mathcal{Z}_2$
- Take the residue $\oint_{\epsilon=0} \mathrm{d}\epsilon$ (Collinear limit)
- 1-D integral $\int_0^\infty \mathrm{d}\tau$ (Real integral)

Structures of Loop amplitudes and action of $\bar{Q}$

Poles and Cuts

In general, the BDS-normalized amplitudes $R_{n,k}$ can be written as

$$R_{n,k}^{(L)} = \sum_{\alpha} Y_{n,k}^{\alpha} F_{\alpha}^{(2L)}$$

where $Y_{n,k}$ are Yangian invariants[1] (which means $\bar{Q} Y_{n,k} = 0$)

- $Y_{n,k}$ bear the pole structure of amplitudes

[1] Objects are invariant under both superconformal and dual superconformal symmetries.

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$Y_{n,0} = 1, Y_{n,1} := \{[i, j, k, l, m]\}$

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F are transcendental functions which bear the cut structure of amplitudes and are expected to be generalized polylogarithms of weight $2L$ at L-loop level for MHV and NMHV cases.

$\bar{Q}$ as Differenial

Due to dual conformal invariance(DCI), F are functions of cross ratios of Plücker coordinates. Since F are expected to be generalized polylogarithms, they satisfy

$$\mathrm{d}F^{(2L)} = \sum_{\beta} F_{\beta}^{(2L-1)} \mathrm{d} \log s_{\beta} \qquad \left(\mathrm{d} := \sum_{i} \mathrm{d}Z_i \frac{\partial}{\partial Z_i} \right)$$

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Thus, the action of $\bar{Q}$ on $R_{n,k}^{(L)}$ gives

$$\bar{Q}R_{n,k}^{(L)} = \sum_{\alpha,\beta} Y_{n,k}^{\alpha} F_{\alpha,\beta}^{(2L-1)} \bar{Q}\log s_{\alpha,\beta} \qquad \left(\bar{Q} := \sum_i \chi_i \frac{\partial}{\partial Z_i}\right)$$

where $s_{\alpha,\beta}$ are some DCI of Plücker coordinates and referred to the last entries of amplitudes

Kernel of $\bar{Q}$

$\bar{Q}$-equation can not determine N^2MHV amplitudes on its own due to the non-trivial dependence of its kernel on k:

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- For $k = 0$, the kernel of $\bar{Q}$ is trivial
- For $k = 1$, it's non-trivial, since

$$\begin{aligned}\bar{Q}[1,2,3,4,5] \log \frac{\langle 1234\rangle}{\langle 2345\rangle} &= [1,2,3,4,5]\bar{Q} \log \frac{\langle 1234\rangle}{\langle 2345\rangle} \\ &= (\bar{3})_a [1,2,3,4,5] \frac{\langle 1234\rangle \chi_5^A + \text{cyclic}}{\langle 2345\rangle\langle 2341\rangle}\end{aligned}$$

Kernel of $\bar{Q}$

$\bar{Q}$-equation can not determine N^{2}MHV amplitudes on its own due to the non-trivial dependence of its kernel on k:

- For $k = 0$, the kernel of $\bar{Q}$ is trivial
- For $k = 1$, it's non-trivial, but has no space of DCI functions
- For $k \geq 2$, it's non-trivial, and it indeed contain DCI functions.

RHS of $\bar{Q}$ equations

Now let us consider the RHS of $\bar{Q}$ equation:

$$\int_{\tau=0}^{\tau=\infty} \oint_{\epsilon=0} \left(\mathrm{d}^{2|3} \mathcal{Z}_{n+1}\right)_a^A [R_{n+1,k+1}^{(L-1)} - \underbrace{R_{n,k}^{(L-1)} R_{n+1,1}^{\text{tree}}}] + \text{cyclic}$$

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$$R_{n+1,k+1}^{(L-1)} \begin{cases} Y_{n+1,k+1} \\ F^{(2L-2)} \end{cases} \xrightarrow{C(\bar{n})_a \oint_{\epsilon=0} \epsilon \mathrm{d}\epsilon \mathrm{d}^3 \chi_{n+1}} \sum_{I,J} Y_{n,k}^{I,J} \bar{Q} \log \frac{\langle \bar{n} I \rangle}{\langle \bar{n} J \rangle} \mathrm{d} \log f_{I,J}(\tau)$$

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The second step:

$$\int_0^\infty \mathrm{d} \log f_{I,J}(\tau) F^{(2L-2)}(\tau, \epsilon \to 0) = F^{(2L-1)}$$

where $f_{I,J}(\tau)$ are rational functions of τ (with some exceptions discussed later)

RHS of $\bar{Q}$ equations

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$$\int_{\tau=0}^{\tau=\infty} \oint_{\epsilon=0} \left(\mathrm{d}^{2|3}\mathcal{Z}_{n+1}\right)_a^A [R_{n+1,k+1}^{(L-1)} - \underbrace{R_{n,k}^{(L-1)} R_{n+1,1}^{\text{tree}}]}_{\text{trivial}} + \text{cyclic}$$

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The operation $C(\bar{n})_a \oint_{\epsilon=0} \epsilon \mathrm{d}\epsilon \mathrm{d}^3\chi_{n+1}$ is independent of loop order which gives last entry conditions on amplitudes

Last entry conditions

For NMHV yangian invariants $[i\,j\,k\,l\,m]$, the operation gives either 0 or

$$\bar{Q} \log \frac{\langle \bar{n} i \rangle}{\langle \bar{n} j \rangle}$$

Then cyclic permutations gives the well-known MHV last entries $\langle \bar{\imath} j \rangle$ [Caron-Huot]

Last entry conditions

The N^{2}MHV Yangian invariants have already classified. [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]

They are

$Y_1^{(2)} = [1,2,(23)\cap(456),(234)\cap(56),6][2,3,4,5,6]$

$Y_2^{(2)} = [1,2,(34)\cap(567),(345)\cap(67),7][3,4,5,6,7]$

$Y_3^{(2)} = [1,2,3,(345)\cap(67),7][3,4,5,6,7]$

$Y_4^{(2)} = [1,2,3,(456)\cap(78),8][4,5,6,7,8]$

$Y_5^{(2)} = [1,2,3,4,8][4,5,6,7,8]$

$Y_6^{(2)} = [1,2,3,(45)\cap(678),8][4,5,6,7,8]$

$Y_7^{(2)} = [1,2,3,(45)\cap(678),(456)\cap(78)][4,5,6,7,8]$

$Y_8^{(2)} = [1,2,3,4,(456)\cap(78)][4,5,6,7,8]$

$Y_9^{(2)} = [1,2,3,4,9][5,6,7,8,9]$

$Y_{10}^{(2)} = [1,2,3,4,(567)\cap(89)][5,6,7,8,9]$

$Y_{11}^{(2)} = [1,2,3,4,(56)\cap(789)][5,6,7,8,9]$

$Y_{12}^{(2)} = \varphi[1,2,3,(45)\cap(789),(46)\cap(789)][(45)\cap(123),(46)\cap(123),7,8,9]$

$Y_{13}^{(2)} = [1,2,3,4,5][6,7,8,9,10]$

$Y_{14}^{(2)} = \psi[A,1,2,3,4][B,5,6,7,8]$

where

$$(ij)\cap(klm) = \mathcal{Z}_i\langle j\,k\,l\,m\rangle - \mathcal{Z}_j\langle i\,k\,l\,m\rangle$$

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For N^{2}MHV yangian invariants, this operation gives

$$[i\,j\,k\,l\,m]\bar{Q}\log\frac{\langle\bar{n}I\rangle}{\langle\bar{n}J\rangle}$$

where I, J can generally be intersections of momentum twistors of the form $(ij)\cap(klm)$

Two-loop NMHV octagons

Input

To compute the 2-loop NMHV octagon, we need the input of the one-loop N^2MHV BDS-normalized amplitude $R^{(1)}_{9,2}$, which can be obtained from the chiral box expansion [Bourjaily, Caron-Huot, Trnka]:

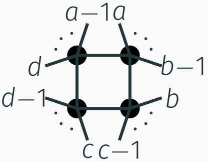
$$R^{(1)}_{n+1,2} = \sum_{a<b<c<d} (f_{a,b,c,d} - R^{\text{tree}}_{n+1,2} f^{\text{MHV}}_{a,b,c,d}) \mathcal{I}^{\text{fin}}_{a,b,c,d}$$

where

- $f_{a,b,c,d}$ are linear combinations of N^2MHV yangian invariants
- $f^{\text{MHV}}_{a,b,c,d}$ are either 1 or 0
- $\mathcal{I}^{fin}_{a,b,c,d}$ denote the finite part of DCI-regulated box integrals

Four-mass box

The most generic term in chiral box expansion arise form four-mass

box: 

$$\begin{cases} x_{ab}^2 := \frac{\langle a{-}1\,a\,b{-}1\,b\rangle}{\langle a{-}1\,a\rangle\langle b{-}1\,b\rangle} = (p_a + \cdots + p_{b-1})^2, \\ u = \frac{x_{ad}^2 x_{bc}^2}{x_{ac}^2 x_{bd}^2} = z\bar{z}, \quad v = \frac{x_{ab}^2 x_{cd}^2}{x_{ac}^2 x_{bd}^2} = (1-z)(1-\bar{z}), \\ \Delta_{abcd} = \sqrt{(1-u-v)^2 - 4uv} \end{cases}$$

For such a box,

$$f_{a,b,c,d} = \frac{1-u-v\pm\Delta}{2\Delta}[\alpha_\pm, b{-}1, b, c-1, c][\delta_\pm, d{-}1, d, a{-}1, a]$$

$$\mathcal{I}^{\text{fin}}_{a,b,c,d} = \text{Li}_2(z) - \text{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}$$

where $\alpha_\pm$ and $\delta_\pm$ are two solutions of Schubert problem
$\alpha = (a{-}1\,a) \cap (d\,d{-}1\,\gamma)$, $\gamma = (c{-}1\,c) \cap (b\,b{-}1\,\alpha)$

Four-mass box

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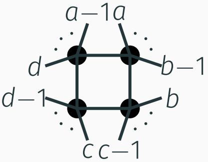
box: 

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The square root will disappear when one mass corner become massless, *e.g.* $b = a{+}1$

Rationalize the square root Δ

Now it is difficult to perform τ-integral for four-mass box coefficients $f_{a,b,c,d}$ due to the appearance of square root Δ whose collinear limit $\Delta(\tau)$ may be not a rational function.

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Example: For the quadratic $y^2 = x^2 + 1$, the rational parameterization

$$(x, y) = \left(\frac{2t}{1-t^2}, \frac{1+t^2}{1-t^2} \right)$$

is obtained by inserting $y - 1 = (x - 0)t$.

Example: $f_{2,4,6,9}$

Let's focus on the octagon (n=8), for which nine 9-point four-mass boxes is needed. It is easy to see only

$$f_{2,4,7,9}, \qquad f_{2,5,7,9}, \qquad f_{2,4,6,9}, \qquad f_{3,5,7,9}$$

can potentially contribute square root. Where $f_{2,4,6,9}$ and $f_{3,5,7,9}$ indeed contribute two square roots $\Delta_{1,3,5,7}$ and $\Delta_{2,4,6,8}$ respectively.

$\Delta_{2,4,6,9}(\tau)$ owns $\tau = \infty$ as its rational point. Thus, we can find the following substitution

$$\tau = \frac{\rho(t - z_{2,4,6,8})(t - \bar{z}_{2,4,6,8})}{t - \sigma}$$

where ρ and σ are some cross ratios of Plücker coordinates.

The square root in $z_{2,4,6,8}$ enters the final result via the limit of integration $\int_0^\infty \mathrm{d}\tau \to \int_z^\infty \mathrm{d}t\, \mathrm{d}\tau/\mathrm{d}t$

Symbol alphabet for 2-loop NMHV octagons

At the end, we obtain the symbol of 2-loop NMHV octagons as

$$\mathcal{S}(R_{8,1}^{(2)}) = \sum_{1\leq i<j<k<l<8} [i,j,k,l,8]\mathcal{S}_{i,j,k,l}$$

where $\mathcal{S}_{i,j,k,l}$ are symbols of weight 4. We find 18 algebraic letters which are generated by cyclic shifts the following 7 seeds

$$\frac{x_* - z}{x_* - \bar{z}} \begin{cases} x_a = \frac{\langle 1(52)(34)(78)\rangle\langle 3456\rangle}{\langle 1345\rangle\langle 1256\rangle\langle 3478\rangle}\,, \quad x_b = x_a|_{5\leftrightarrow 6}\,, \\ x_c = \frac{\langle 1378\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 3478\rangle}\,, \; x_d = x_c|_{3\leftrightarrow 4}\,, \; x_e = \frac{\langle 187(34)\cap(256)\rangle}{\langle 1256\rangle\langle 3478\rangle} \\ x_f = 1, \quad x_g = 0, \quad z = z_{2,4,6,8} \end{cases}$$

Further more, all algebraic letters always enter the symbol in the following combinations

$$\underbrace{\left(u \otimes \frac{1-z}{1-\bar{z}} + v \otimes \frac{\bar{z}}{z}\right)}_{\text{symbol of 4-mass box}} \otimes \frac{x_* - z}{x_* - \bar{z}}$$

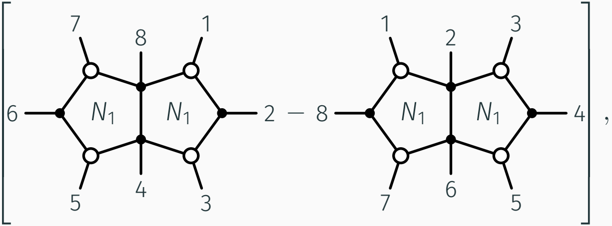
New kind of cuts

Besides 18 algebraic ones, we find 180 rational letters which are contained in the prediction from Laudau equations [Prlina, Spradlin, Stankowicz, Stanojevic].

The algebraic letters can be rewritten as $(a \pm \sqrt{a^2 - 4b})$, where (a, b) are polynomials of Plücker coordinates. Such letters indicate two kinds of cuts. One arise from the discriminant $a^2 - 4b$, which are square root branch points from Landau equations. The other arise from $b \to 0$ which is the same as the cut of $\log b$. Thus the branch points $b = 0$ correspond to zero locus of some rational letters.

Comparison with Feynman integral computation

The component $\chi_1\chi_3\chi_5\chi_7$ of two-loop NMHV octagon is completely free of algebraic letters, which is given by the coefficient of [1,3,5,7,8] in our basis. This component correspond to the difference of two Feynman integrals [Bourjaily, *et al*]



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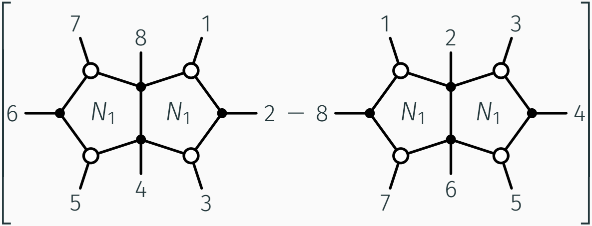

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The symbol size $\sim 2.8 \times 10^4$ terms

Outlook

Summary

- Three-loop MHV octagon.
- The connection to cluster algebra, tropical Grassmannian
- $\bar{Q}$ equations for individual integral and other theories.

Thank You

Questions?