Some Applications of Computational Algebraic Geometry to String and Particle Phenomenology

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A Trio for an New Age

1. LHC Era: Higgs-like Particle just discovered
   - biggest experimental/theoretical collaboration
   - A plenitude of experimental data, computational challenge: plans to strategically outsource the terabytes per day to global network of computers
2. Polymath Project: Headed by Tim Gowers and Terry Tao:
   - Q: Can mathematics be outsourced by blogging?
   - initial phase a huge success: Hales-Jewett Theorem, new proof in 7 weeks
3. SVP: a global collaboration to probe string vacua

   inter-disciplinary enterprise: field theorist, phenomenologist, string theorist, algebraic/differential geometers, computer scientists . . .

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Triadophilia: A 20-year search

- A 20 Year Old Problem: [Candelas-Horowitz-Strominger-Witten] (1986)
  - $E_8 \supset SU(3) \times SU(2) \times U(1)$ Natural Gauge Unification
  - Mathematically succinct and physically motivated
- CY3 X, tangent bundle $SU(3) \Rightarrow E_6$ GUT: $E_8 \to SU(3) \times E_6$
  - Particle Spectrum:

Particle Spectrum:	Generation	$n_{27} = h^1(X, TX) = h^{2,1}_{\bar{\partial}}(X)$
	Anti-Generation	$n_{\overline{27}} = h^1(X, TX^*) = h^{1,1}_{\bar{\partial}}(X)$

- Net-generation: $\chi = 2(h^{1,1} - h^{2,1})$
- First Challenge to String Pheno:

  Are there Calabi-Yau threefolds with Euler character ± 6?

Heterotic Compactification:

Stable Vector Bundles on CY3

The First Decade

- Complete Intersection Calabi-Yau (CICY) 3-folds ('86-'90)
  - dim(Ambient space) - #(defining Eq.) $= 3$
  - First CY3 data-set (7890) [Candelas-He-Hübsch-Lutken-Schimmrigk]

$$\left[\begin{array}{c|cccc} n_1 & q_1^1 & q_1^2 & \dots & q_1^K \\ n_2 & q_2^1 & q_2^2 & \dots & q_2^K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_m & q_m^1 & q_m^2 & \dots & q_m^K \end{array}\right]_{m \times K}$$

– K equations of multi-degree q_j^i embedded in $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$

– $c_1(X) = 0 \rightsquigarrow \sum_{j=1}^{K} q_r^j = n_r + 1$

- Most Famous Example: $[4|5]^{1,101}_{-200}$ (or simply $[5]$) QUINTIC
- CYCLIC: $m = h^{1,1} = 1$; HYPERSURFACE: $K = 1$
- The Tian-Yau Manifold: $M = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix} / \mathbb{Z}_3$ has $M^{6,9}_{-6}$
- TY: Central to string pheno in the 1st decade [Greene, Ross, et al.]

  E_6 GUTS unfavoured; Many exotics: e.g. 6 entire anti-generations

Heterotic Compactification: Recent Development

- Getting $SU(5)$ and $SO(10)$ GUTS is easy: use general embedding
  - Instead of TX, use (poly-)stable holomorphic vector bundle V
  - Gauge group$(V) = G = SU(n)$, $n = 3, 4, 5$, gives $H =$ Commutant(G, E_8):

$E_8 \to G \times H$	Breaking Pattern		
$SU(3) \times E_6$	248	$\to$	$(1,78) \oplus (3,27) \oplus (\bar{3}, \overline{27}) \oplus (8,1)$
$SU(4) \times SO(10)$	248	$\to$	$(1,45) \oplus (4,16) \oplus (\bar{4}, \overline{16}) \oplus (6,10) \oplus (15,1)$
$SU(5) \times SU(5)$	248	$\to$	$(1,24) \oplus (5,\overline{10}) \oplus (\bar{5}, 10) \oplus (10,5) \oplus (\overline{10}, \bar{5}) \oplus (24,1)$

  - Particle content $\sim H^q(X, \bigwedge^p V)$; Yukawas: Trilinear maps in Cohomology
- MSSM Gauge Group easy: $H \overset{\text{Wilson Line}}{\longrightarrow} SU(3) \times SU(2) \times U(1)$
- Getting even exact spectrum HARD; Yukawas HARDER

  Over a decade of Math/Physics Collaboration initiated by Ovrut w/ Donagi, Pantev at Penn

  Recent collaboration w/ Anderson, Braun, Candelas, Gray, Lukas et al.

Vector Bundles on Elliptic CY3: The Penn Programme

- Another large dataset: **elliptically fibered CY3** over surface B

$$\pi : X \to B, \quad \text{each fibre } \pi^{-1}(b \in B) = T^2_{possibly\ singular}$$

and $\exists$ zero section $\sigma : B \to X, \quad \sigma(b \in B) = \mathbb{I}_{ellipse}$

- Surface B is highly constrained [Morrison-Vafa, Grassi]:
  - del Pezzo surface $d\mathbb{P}_{r=1,\ldots,9}$: $\mathbb{P}^2$ blown up at r points
  - Hirzebruch surface $\mathbb{F}_{r=0,\ldots 12}$: $\mathbb{P}^1$-bundle over $\mathbb{P}^1$
  - Enriques suface $\mathbb{E}$: involution of K3
  - Blowups of $\mathbb{F}_r$
- Isolate manifolds with good discrete symmetries (for Wilson Line) and try to construct equivariant bundles

  Ovrut together with Buchbinder, Donagi, YHH, Pantev, Reinbacher

Heterotic MSSM

- Some initial scans: (Spectral Cover) bundles over elliptically fibered CY3 (Donagi-YHH-Ovrut-Reinbacher, Gabella-YHH-Lukas)
  - $\sim 10^7$ stable bundles so far, $\sim 10^4$ GUTS with $3k$ generations (to give potential 3-generation MSSM)
  - a little difficult to fully computerize
- V stable $SU(n)$ bundle : Generalised Serre Construction
  - $X_0^{19,19}$ a double-fibration over dP_9
  - [Braun-YHH-Ovrut-Pantev] equivariant $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$, $SU(4)$ bundle, $\rightsquigarrow$ Exact $SU(3) \times SU(2) \times U(1) \times U(1)_{B-L}$ spectrum
  - [Bouchard-Cvetic-Donagi] equivariant $\pi_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $SU(5)$ bundle, $\rightsquigarrow$ Exact $SU(3) \times SU(2) \times U(1)$ spectrum
  - No exotics; no anti-generation; 1 pair of Higgs; (RH neutrino)

Observatio Curiosa

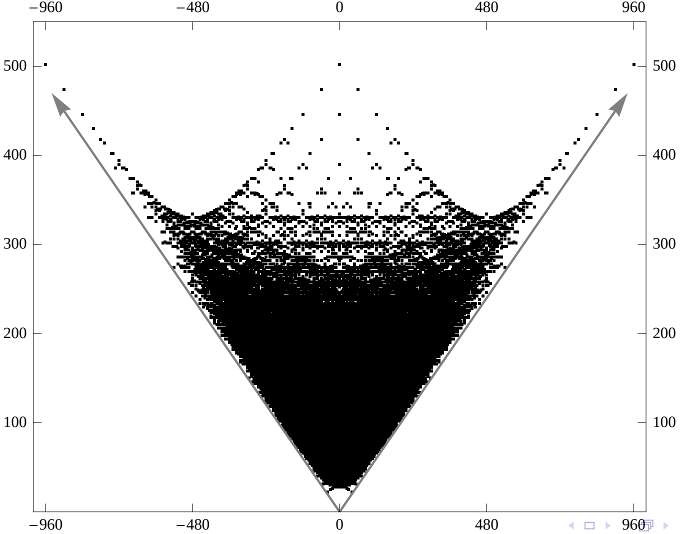
- $X_0^{19,19}$ is a CICY!
- Penn group *purely abstract*, but $X_0^{19,\,19} = \begin{pmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{pmatrix}$, Tian-Yau: $\begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$
- TRANSPOSES!!
- Why should the best manifold from 80's be so-simply related to the best manifold from completely different data-set and construction 20 years later ??
- Two manifolds (and quotients) are conifold transitions and vector bundles thereon transgress to one another (Candelas-de la Ossa-YHH-Szendroi)
- **Connectedness of the Heterotic Landscape?**
  - All CICY's are related by conifold transitions
  - Reid Conjecture: All CY3 are connected
  - Proposal: All (stable) vector bundles on all CY3 transgress

Algorithmic Compactification

- With advances in **computer algebra** and **algorithmic geometry**, can one examine the space of (heteroric) compactifications
- Turn to the largest CY database known (Kreuzer-Skarke)
  - Hypersurfaces in toric varieties (thus includes single equation in single weighted projective space and the 5 transpose-cyclic CICYs)
  - Double hypersurface in progress
  - construct stable $SU(n)$ bundles
  - discrete symmetres $\rightsquigarrow$ Quotients and Wilson lines?
- Start with warm-up datasets
  - Cyclic manifolds
  - CICYs
  - small $h^{1,1}$ KS toric manifolds

The CY3 Landscape: A Georgia O'Keefe Plot

20 years of research by mathematicians and physicists, about 500 million CY3:



Horizontal

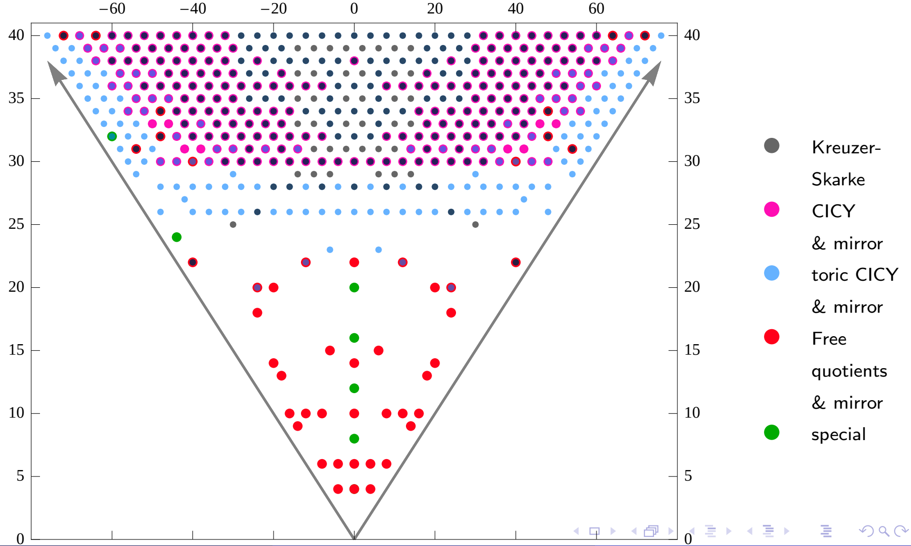
$\chi =$

$2(h^{1,1} - h^{2,1})$

Vertical

$h^{1,1} + h^{2,1}$

The Special Corner of the Landscape



We live in the Corner?

- Above $h^{1,1} + h^{2,1} = 25$ almost every site is occupied
- Below, comparatively sparse by orders of magnitude,

known MSSM models live there

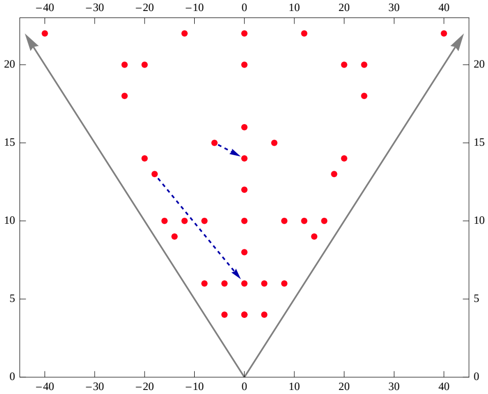
All CY3 with

$h^{1,1} + h^{2,1} < 22$

Transgressions:

Top Arrow: $N^{6,9} \to X^{7,7}$

Bottom Arrow: $N^{2,11} \to X^{3,3}$



- **Programme by P. Candelas w/ V. Braun, R. Davies** et al.

Monad Bundles over Large Sets: The Oxford Programme

- L. Anderson, J. Gray, YHH, S.-J. Lee, A. Lukas
- Highly programmable: explicit coordinates, integer lattices, combinatorics
- MONAD BUNDLES:
  - Defined by a short exact sequence of vector bundles (free resolution)

$$0 \to V \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad B = \bigoplus_{i=1}^{r_B} \mathcal{O}(b_r^i) \ , \quad C = \bigoplus_{j=1}^{r_C} \mathcal{O}(c_r^j)$$

  - DEF: $V = \text{ker}(g) = \text{im}(f)$, $\quad \text{rk}(V) = \text{rk}(B) - \text{rk}(C)$
  - Map g: matrix of polynomials. (e.g. on $\mathbb{P}^n$ the ij-th entry is a homogeneous polynomial of degree $c_i - b_j$)
  - The monad construction is a powerful and general way of defining vector bundles; e.g. every bundle on $\mathbb{P}^n$ can be written as a (generalised) monad

Summary of Constraints

- Bundle-ness: $b_r^i \leq c_r^j$ for all i, j
- The map g can be taken to be generic so long as exactness of the sequence
- Unitarity: $c_1(V) = 0 \Leftrightarrow \sum_{i=1}^{r_B} b_i^r - \sum_{j=1}^{r_C} c_j^r = 0$
- Anomaly cancellation:

  $c_2(TX) - c_2(V) = c_2(TX) - \frac{1}{2}(\sum_{i=1}^{r_B} b_s^i b_t^i - \sum_{i=1}^{r_C} c_s^j c_i^t) J^s J^t \geq 0$
- 3 Generations: $c_3(V) = \frac{1}{3}(\sum_{i=1}^{r_B} {b_r}^i {b_s}^i {b_t}^i - \sum_{j=1}^{r_C} {c_r}^j {c_s}^j {c_t}^j) J^r J^s J^t = 3k$ with k a divisor of $\chi(X)$
- Must prove Stability

Low-Energy SUSY: Stability of Bundles

- How to find bundles admitting HYM connection to give SUSY?
- Theorem [Donaldson-Uhlenbeck-Yau]: On each (poly-)stable holomorphic vector bundle V, $\exists$ unique Hermitian-YM connection satisfying HYM.
  - Analysis (hard PDE) $\rightsquigarrow$ algebra
  - generalises Calabi-Yau theorem
- DEF: slope$(V) = \frac{\deg(V)}{\text{rk}(V)}$, where $\deg(V) = \int_X c_1(V) \wedge J^2$
- DEF: V stable if every $W \subset V$ has slope$(W) <$ slope(V)
- Our bundles have $c_1(V) = 0$ so our bundles are stable if all sub-sheafs have strictly negative slope
- also, stability $\rightsquigarrow H^0(X,V) = H^0(X,V^*) = H^3(X,V) = H^3(X,V^*) = 0$

Inevitability of Computational Algebraic Geometry

- Computer search indispensable; computer algebra (M2, Singular) crucial
- In Conjunction with Standard Techniques $X \hookrightarrow A$
  - Normal Bundle is just embedding data (Q: what about non-CI?)

    $\mathcal{N}_X = \bigoplus_{j=1}^{K} \mathcal{O}(q_1^j, \ldots, q_m^j)$
  - Koszul Sequence:

    $0 \to V \otimes \wedge^K N_X^* \to V \otimes \wedge^{K-1} N_X^* \to \ldots \to V \otimes N_X^* \to V \to V|_X \to 0$
  - Spectral Sequence:

    $E_1^{p,q} := H^q(A, V \otimes \wedge^{-p} N_X^*) \qquad p = -K, \ldots, 0;\ q = 1, \ldots, \dim(A) = \sum_{i=1}^{m} n_i$

    $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$ for $r = 1, 2, \ldots$ gives

    $H^n(X, V|_X) \simeq \bigoplus_{p+q=n} E_\infty^{p,q}$
- Higher exterior powers more painful

A Needle in the Haystack

- Anderson-Gray-YHH-Lukas
- establish *G-equivariant structure* on the bundles and compute equivariant cohomology coupled to Wilson line action
- Find one monad which gives the *exact spectrum of MSSM*:
- On bi-cubic $X = \left[\begin{array}{c|c} \mathbb{P}^2_{x_i} & 3 \\ \mathbb{P}^2_{y_i} & 3 \end{array}\right]^{2,83}$, quotiented by $\mathbb{Z}_3 \times \mathbb{Z}_3$ with $\alpha = \exp(2\pi i/3)$:

$$\begin{array}{rcl} {\mathbb{Z}_3}^{(1)} & : & x_k \to x_{k+1}, \quad y_k \to y_{k+1} \\ {\mathbb{Z}_3}^{(2)} & : & x_k \to \alpha^k x_k, \quad y_k \to \alpha^{-k} y_k \ . \\ p_{(3,3)} & = & A_1^{k,\pm} \sum_j x_j^2 x_{j\pm1} y_{j+k}^2 y_{j+k\pm1} + A_2^k \sum_j x_j^3 y_{j+k}^3 + A_3 x_1 x_2 x_3 \sum_j y_j^3 \\ & & + A_4 y_1 y_2 y_3 \sum_j x_j^3 + A_5 x_1 x_2 x_3 y_1 y_2 y_3 \end{array}$$

- quotient is a $(2,11)$ manifold with $\mathbb{Z}_3 \times \mathbb{Z}_3$-Wilson line breaking $SO(10) \to SU(3) \times SU(2) \times U_Y(1) \times U(1)_{B-L}$

- In the corner of the CY3 plot
- semi-positive monad:

$$0 \to V \to \mathcal{O}_X(1,0)^{\oplus 3} \oplus \mathcal{O}_X(0,1)^{\oplus 3} \xrightarrow{f} \mathcal{O}_X(1,1) \oplus \mathcal{O}_X(2,2) \to 0$$

$$\tilde{f}^T = \begin{pmatrix} -2y_0 & -x_2 y_1^2 + 2x_0 y_1 y_2 - x_1 y_2^2 \\ -2y_2 & x_1 y_0^2 + 2x_2 y_0 y_1 - x_0 y_1^2 \\ -2y_1 & -x_2 y_0^2 + 2x_1 y_0 y_2 - x_0 y_2^2 \\ -x_0 & -2x_1 x_2 y_0 + x_0 x_1 y_1 + x_2^2 y_1 + 2x_1^2 y_2 - 2x_0 x_2 y_2 \\ -x_2 & x_1^2 y_0 + x_0 x_2 y_0 + 2x_0^2 y_1 - 2x_1 x_2 y_1 - 2x_0 x_1 y_2 \\ -x_1 & -2x_0 x_1 y_0 + 2x_2^2 y_0 - 2x_0 x_2 y_1 + x_0^2 y_2 + x_1 x_2 y_2 \end{pmatrix}$$

- Choose Wilsonline with $\alpha_{1,2}$ characters for $\mathbb{Z}_3^{(1),(2)}$:

$$\mathbb{Z}_3^{(1)} = \text{Diag}(\alpha_1^2 \mathbb{I}_{10}, \mathbb{I}_5, \alpha_1) \ , \qquad \mathbb{Z}_3^{(2)} = \text{Diag}(\alpha_2 \mathbb{I}_6, 1, \alpha_2^2 \mathbb{I}_3, \mathbb{I}_2, \alpha_2^2, \mathbb{I}_3, 1)$$

- Breaking Pattern:

$$\begin{aligned} \mathbf{16} &= \alpha_1^2 \alpha_2 (\mathbf{3}, \mathbf{2})_{(1,1)} \oplus \alpha_1^2 (\underline{\mathrm{f}}1, \mathbf{1})_{(6,3)} \oplus \alpha_1^2 \alpha_2^2 (\overline{\mathbf{3}}, \mathbf{1})_{(-4,-1)} \\ &\quad + \alpha_2^2 (\overline{\mathbf{3}}, \mathbf{1})_{(2,-1)} \oplus (\mathbf{1}, \mathbf{2})_{(-3,-3)} \oplus \alpha_1 (\mathbf{1}, \mathbf{1})_{(0,3)} \\ \mathbf{10} &= \alpha_1 (\mathbf{1}, \mathbf{2})_{(3,0)} \oplus \alpha_1 \alpha_2 (\mathbf{3}, \underline{\mathrm{f}}1)_{(-2,-2)} \oplus \alpha_1^2 (\mathbf{1}, \mathbf{2})_{(-3,0)} \oplus \alpha_1^2 \alpha_2^2 (\mathbf{3}, \mathbf{1})_{(2,2)} \end{aligned}$$

- No anti-generations, no exotic particles, 1 pair of MSSM Higgs:

Cohomology	Representation	Multiplicity	Name
$[\alpha_1^2\alpha_2 \otimes H^1(X,V)]^{inv}$	$(\mathbf{3},\mathbf{2})_{1,1}$	3	l eft-handed quark
$[\alpha_1^2 \otimes H^1(X,V)]^{inv}$	$(\mathbf{1},\mathbf{1})_{6,3}$	3	left-handed anti-lepton
$[\alpha_1^2\alpha_2^2 \otimes H^1(X,V)]^{inv}$	$(\overline{\mathbf{3}},\mathbf{1})_{-4,-1}$	3	left-handed anti-up
$[\alpha_2^2 \otimes H^1(X,V)]^{inv}$	$(\overline{\mathbf{3}},\mathbf{1})_{2,-1}$	3	left-handed an ti-down
$[H^1(X,V)]^{inv}$	$(\mathbf{1},\mathbf{2})_{-3,-3}$	3	left-handed lepton
$[\alpha_1 \otimes H^1(X,V)]^{inv}$	$(\mathbf{1},\mathbf{1})_{0,3}$	3	left-handed anti-neutrino
$[\alpha_1 \otimes H^1(X,\wedge^2 V)]^{inv}$	$(\mathbf{1},\mathbf{2})_{3,0}$	1	up Higgs
$[\alpha_1^2 \otimes H^1(X,\wedge^2 V)]^{inv}$	$(\mathbf{1},\mathbf{2})_{-3,0}$	1	down Higgs

- Marching On...
  - Classify discrete symmetries of CICYS: Braun
  - Stability analysis for the large dataset: Anderson-Gray-Lukas-Ovrut
  - Constructing monads up to $h^{1,1} = 3$ of the Kreuzer-Skarke list:

    YHH-Kreuzer-Lee-Lukas

Vacuum Geometry:

from Branes to Bottom-Up

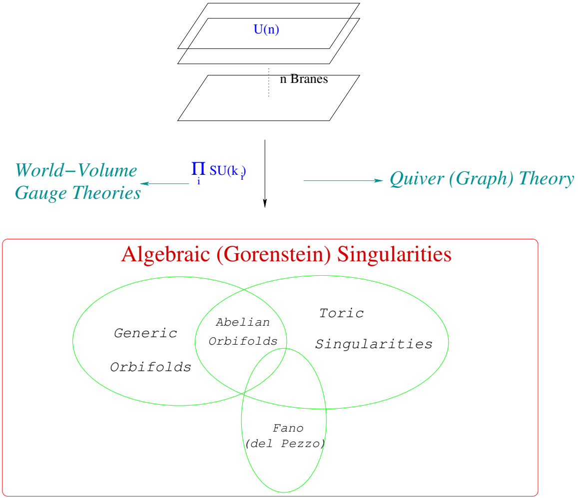
Type II Perspective

- AdS/CFT naturally $\rightsquigarrow$ classes of SUSY theories with product gauge groups
- Brane World: live on D3-brane $\perp$ 6D affine variety $\mathcal{M}$ (Douglas,Moore, Morrison et al)
  - world-volume: 3+1D SUSY QUIVER THEORY
  - TRANSVERSE: local (affine, singular) Calabi-Yau 3-fold (cone over Sasaki-Einstein);
  - D3 probes the geometry of $\mathcal{M}$
- a bijection:

1. Vacuum Moduli Space of w.v. $\stackrel{\text{D,F-flat}}{\longrightarrow}$ $\mathcal{M}$
2. Gauge Theory $\stackrel{\text{Geom. Eng.}}{\longleftarrow}$ Algebraic Geometry of $\mathcal{M}$

The Brane Engineering Programme

Stack of n parallel branes $\perp$ a Calabi-Yau singularity $\mathcal{M}$:



Orbifolds:

$\mathbb{C}^3/(\Gamma \subset SU(3))$

Toric:

e.g., conifold, $Y^{p,q}$, $L^{p,q}$ ...

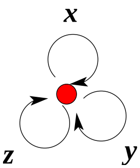
$=$ Dimers

Fano (del Pezzo):

$dP_{0,\ldots,8}$

Progenitor: $\mathcal{M} = \mathbb{C}^3$

- Original AdS/CFT: N D3-branes in flat space
- w.v. theory $U(N)$ with 3 adj. fields (x, y, z) and interaction $W = \mathrm{tr}(x[y, z])$



NODE = Gauge Group

ARROW = Bi−fundamental (Adj) Field

- QUIVER = Finite graph (label = rk(gauge factor)) + relations from SUSY
  - Matter Content: Nodes + arrows
  - Relations (F-Terms): $D_i W = 0 \leadsto [x, y] = [y, z] = [x, z] = 0$
- Gauge Invariants: $\mathrm{tr}(x^i y^j z^k)$, $i, j, k \in \mathbb{Z}_{\geq 0}$ $(R = i + j + k)$

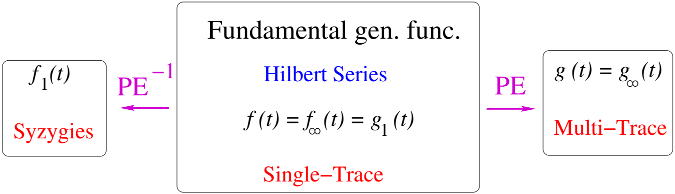
Method and Results

- Orbifolds: Pioneered by Douglas-Moore; Lawrence-Nekrasov-Vafa
  - $\Gamma \subset SU(2)$, ADE (McKay Correspondence); $\Gamma \subset SU(3)$ (Hanany-YHH)
    Projection of parent $\mathbb{C}^3$ theory
  - *Geometry of $\mathcal{M}$ and w.v. physics* $\sim Rep(\Gamma) = \{r_i\}$

    $\mathcal{R} \otimes \mathbf{r}_i = \bigoplus_j a_{ij}^{\mathcal{R}} \mathbf{r}_j$, $a_{ij}^{\mathcal{R}} \rightsquigarrow$ adjacency matrix of quiver
  - $\Delta(27)$: Berenstein-Jejjala-Leigh, SM on a brane
- Toric Calabi-Yau: Aspinwall, Beasley, Greene, Morrison et al
  - $\mathcal{M} \sim$ Combinatorical Data in $\mathbb{Z}^2$-lattice (CY3 $\rightsquigarrow$ planar toric diag)
  - Systematic algorithm: Feng-Hanany-YHH (Inverse Algorithm) 0003085
  - Dimers/Bipartite Tilings: w/ Hanany, Kennaway, Franco, Sparks; Dessin d'Enfants: w/ Hanany, Jejjala, Pasukonis, Ramgoolam, Rodriguez-Gomez;
- del Pezzo cones: Exceptional Collections (Hanany-Herzog-Vegh, Wijnholt)

The Plethystic Programme

- Benvenuti-Feng-Hanany-YHH
- Geometry $\leftarrow$ Gauge Theory $\rightarrow$ Combinatorics



- Given geometry (at least for CI), no need to know quiver theory $\rightsquigarrow$ direct counting, e.g., $x^2 + y^2 + z^2 + w^k = 0$
- **Applicable to all gauge theories**, not just D-brane probes: Lagrangian $\Rightarrow$ F,D-Flat $\Rightarrow$ Vacuum Moduli Space $\mathcal{M} \Rightarrow f_1 = \text{Syzygy}(\mathcal{M}) \Rightarrow$ $f = PE[f_1] \Rightarrow g = PE^2[f_1]$

A Bottom-Up View: Vacuum Moduli Space

$$S = \int d^4x \, [\int d^4\theta \; \Phi_i^\dagger e^V \Phi_i + \left(\frac{1}{4g^2} \int d^2\theta \; \mathrm{tr} \mathcal{W}_\alpha \mathcal{W}^\alpha + \int d^2\theta \; W(\Phi) + \mathrm{h.c.} \right)]$$

W = superpotential $\qquad V(\phi_i, \bar{\phi_i}) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{g^2}{4} (\sum_i q_i |\phi_i|^2)^2$

- VACUUM $\sim \boxed{V(\phi_i, \bar{\phi_i}) = 0} \Rightarrow \begin{cases} \frac{\partial W}{\partial \phi_i} = 0 & \text{F-TERMS} \\ \sum_i q_i |\phi_i|^2 = 0 & \text{D-TERMS} \end{cases}$
- D-flat is just gauge fixing; $\mathcal{M} =$ F-flat$//G^{\mathbb{C}}$
  1. Find all gauge invariant operators $D = \{GIO\}$
  2. Solve F-flat $\frac{\partial W}{\partial \phi_i} = 0$ and back-substitute into D
- $\mathcal{M}$ **:= vacuum moduli space = space of solutions to F and D-flatness**

  = CY3 as a complex variety = Space of loops/F-term Rel. (for 1 brane)
- for n-branes, get $Sym^n \mathcal{M} = \mathcal{M}^n / \Sigma_n$

A Computational Approach

- Algorithm (Gray-YHH-Jejjala-Nelson)
  1. n-fields: start with polynomial ring $\mathbb{C}[\phi_1, \ldots, \phi_n]$
  2. $D =$ set of k GIO's: a ring map $\mathbb{C}[\phi_1, \ldots, \phi_n] \xrightarrow{D} \mathbb{C}[D_1, \ldots, D_k]$
  3. Now incorporate superpotential: F-flatness

     $\langle f_{i=1,\ldots,n} = \frac{\partial W(\phi_i)}{\partial \phi_i} = 0 \rangle \simeq$ **ideal** of $\mathbb{C}[\phi_1, \ldots, \phi_k]$
  4. Moduli space = image of the ring map

     $\frac{\mathbb{C}[\phi_1,\ldots,\phi_n]}{\{F=\langle f_1,\ldots,f_n \rangle\}} \xrightarrow{D=GIO} \mathbb{C}[D_1, \ldots, D_k], \quad \mathcal{M} \simeq \mathrm{Im}(D)$
- Image is an ideal of $\mathbb{C}[D_1, \ldots, D_k]$, i.e.,

  $\mathcal{M}$ explicitly realised as an affine variety in $\mathbb{C}^k$
- Another computational challenge, STRINGVACUA: Mathematica package using Singular (Gray-Ilderton-YHH-Lukas)

Vacuum Geometry and Phenomenology

- $\mathcal{M}$ itself may have SPECIAL STRUCTURE (beyond gauge-inv and discrete symmetries) indications of new physics?
- sQCD(N_f,N_c) well-known $\dim \mathcal{M} = \begin{cases} N_f^2 & N_f < N_c \\ 2N_cN_f - (N_c^2 - 1) & N_f \geq N_c \end{cases}$
  - New: moduli space = CY of high dim; Hilbert series $g^{(N_f, N_c)}(t, \tilde{t}) =$
    $\sum\limits_{n_1, n_2, \ldots, n_k, \ell, m \geq 0} [n_1, n_2, \ldots, n_k, \ell_{N_c;L}, 0, \ldots, 0; 0, \ldots, 0, m_{N_c;R}, n_k, \ldots, n_2, n_1] \; t^a \; \tilde{t}^b$
    - e.g. 2-colours: $g^{(N_f, N_c=2)}(t) = {}_2F_1(2N_f - 1, 2N_f; 2; t^2)$
    - e.g. $N_f = N_c$: $g^{N_f = N_c}(t) = \frac{1-t^{2N_c}}{(1-t^2)^{N_c^2}(1-t^{N_c})^2}$
- MSSM: 991 Gauge invariants (a challenge)
  - Electro-weak sector: $\dim_{\mathbb{C}} = 3$, an affine cone over the Veroenese surface

Numerical Algebraic Geometry

- w/ D. Mehta and J. Hauenstein
- *Disadvantage of Gröbner Basis*
  - Exponential running time and memory usage
  - Non-parallelizable
- When asking relatively simple questions such as dimension, primary decomposition (branches of moduli space), numerical solutions of 0-dimesional ideals:

  homotopy continuation method
  - Highly parallelizable (by primary components)
  - numerically efficient
  - implementation: Bertini

Cloud Computing: A String Cartography

- NSF+Microsoft Grant, with Gray-Jejjala-Nelson
- w/ B. Jurke & J. Simon: Swiss-Cheese Scan
- To attack: MSSM vacuum geometry (numerical?) w/ D. Mehta
- String Vacua in the Cloud
  - Microsoft Azure pletform, announced Feb. 2010
  - European Environment Agency's Climate Monitoring: "Eye on Earth"
  - 500K core-hours and 1.5TB
- cf. Candelas et al. (1990's): CERN supercomputer, punch-cards $\Rightarrow 10^4$ CICYs
- cf. Kreuzer et al. (2000): SGI origin 2000, ~ 30 processors @ $\sim$ 6 months running time $\Rightarrow 10^{10}$ toric hypersurfaces