Disk relation in string theory and field theory

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Outline

- **Introduction**
- **BCFW, KK-BCJ and KLT relations**
- **Disk relation in string theory**
- **Disk relation in field theory**
- **Conclusion**

1. Introduction

- **String theory as a theory containing both gauge field and gravity**

$$\left.\begin{array}{c}\text{strong intraction}\\ \text{weak interaction}\\ \text{electromaganetic interaction}\end{array}\right\}\ \left.\begin{array}{c}\text{gauge field}\Rightarrow\text{open string}\\ \\ \text{gravity}\Rightarrow\text{closed string}\end{array}\right\}\text{string theory.}$$

Thus the relations between gauge field and gravity correspond to the relations between open string and closed string.

- **The relations between gauge field and gravity**

  AdS/CFT(Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231) correspondence and KLT (Kawai, Lewellen and Tye, NPB **269** (1986) 1) relation are two examples for the relationships between gauge field and gravity.

Duality	gravity $\Leftrightarrow$ gauge field
AdS/CFT correspondence	weak $\Leftrightarrow$ strong
KLT relation	weak $\Leftrightarrow$ weak

- **KLT relation**

The trivial relation between free closed string states and free open string states

$$|N_L, N_R\rangle \otimes |p\rangle = |N_L\rangle \otimes |N_R\rangle \otimes |p\rangle\,. \qquad (1)$$

KLT relation factorize the closed string amplitudes on S_2 into two open string amplitudes on D_2 (except for phase factors)

$$\mathcal{M}_{S_2}(1_c, ..., M_c) \sim \kappa^{M-2} \sum_{P_L, P_R} \mathcal{A}^{(M)}(P_L)\bar{\mathcal{A}}^{(M)}(P_R)e^{i\pi F(P_L, P_R)}. \qquad (2)$$

KLT relation is a factorization relation (except for a phase factor), since $\sum_{P_L} \mathcal{A}^{(M)}(P_L)$ and $\sum_{P_R} \bar{\mathcal{A}}^{(M)}(P_R)$ are the amplitudes for left- and right-moving open strings.

- **The applications of KLT relation**
  - In string theory $(\mathcal{A}_{D_2}^{\text{(open)}})_L \times (\mathcal{A}_{D_2}^{\text{(open)}})_R \Rightarrow \mathcal{M}_{S_2}^{\text{(closed)}}$
  - Field theory limit $(\mathcal{A}_{\text{tree}}^{\text{(gluon)}})_L \times (\mathcal{A}_{\text{tree}}^{\text{(gluon)}})_R \Rightarrow \mathcal{M}_{\text{tree}}^{\text{(graviton)}}$
  - KLT relations $+$ unitarity$\rightarrow$ The ultraviolet properties of $N = 8$ supergravity (Bern, Dixon and Roiban, PLB **644** (2007) 265).
  - KLT relation can also be used to give the amplitudes for gravitons coupled to matter (Bern, De Freitas and Wong, PRL **84** (2000) 3531).

- **Does KLT factorization relation hold for the amplitudes on any worldsheet?**

  KLT factorization relation holds for the amplitudes on S_2 in string theory. Does it hold on worldsheet with other topologies?

  We find the KLT factorization relation does not hold on D_2. On D_2, the left- and right-moving sectors are connected into a single one. New relation is given as[1]

$$\mathcal{A}(\sigma(1_o), ..., \sigma(N_o), (N+1)_c, ..., (N+M)_c) = g^{N-2}\kappa^M \sum_{P''} e^{i\pi\Theta'(P'')}\mathcal{A}^{(N,2M)}(P''), \tag{3}$$

[1] We call this relations disk relation.

- **Disk relation for amplitudes in minimal coupling theory of gauge field and gravity**

Disk relation versus KLT relation:

Topology	Relations	Incorporation of gauge degree of freedom	Theory
S_2	KLT relation	Lorentz singlets in one sector of closed strings	Heterotic theories
D_2	Disk relation	Chan-Paton factor at the ends of open strings	Type I theory

KLT relation can give the tree amplitudes for minimal coupling theory of gauge field and gravity. Does disk relation also hold for gauge-gravity minimal coupling theory? We find disk relation also holds for tree amplitudes in minimal coupling theory of gauge field and gravity.

We find Three- and four-point amplitudes for gauge-gravity minimal coupling satisfy disk relation.

Disk relation also hold for the MHV amplitudes for gravitons minimal coupled to gluons. We expect there is a general expression of disk relation.

- **The general form of disk relation**

We construct the disk relations for amplitudes with N gluons and 1 graviton by BCFW (Britto, Cachazo, Feng, NPB**715** (2005) 499; Britto, Cachazo, Feng and Witten, PRL**94** (2005) 181602) recursion relation and KK-BCJ (Kleiss and Kuijf, NPB **312** (1989) 616; Zhu, PRD **22**, (1980) 2266; Bern, Carrasco and Johansson, PRD **78** (2008) 085011) relation. The disk relation for N gluons and M gravitons can be constructed similarly.

- **Relations among amplitudes**

BCFW	$\mathcal{M}^{(N)} \Rightarrow \mathcal{M}^{(M)}(N < M)$	guage field, gravity, gauge-gravity coupling
KK-BCJ	$\mathcal{A}^{(N)} \Rightarrow \mathcal{A}^{(N)}$	gauge field, gauge-matter coupling
KLT	$\mathcal{A}_L^{(N)} \times \mathcal{A}_R^{(N)} \Rightarrow M^{(N)}$	gravity, gravity-matter coupling
Disk	$\mathcal{A}^{(N+2M)} \Rightarrow \mathcal{A}^{(N,M)}$	gauge-gravity coupling

2. BCFW, KK-BCJ and KLT relations

- **BCFW recursion relation in field theory**

Complex momenta

$$p_1(z) = p_1 + zq, p_2(z) = p_2 - zq, \tag{4}$$

where $q^2 = q \cdot p_{1,2} = 0$ so that $p_{1,2}$ are on-shell momenta. Cauchy's residue theorem

$$\oint \frac{\mathcal{M}(z)}{z} = \sum_{z_I} \text{Res}\left(\frac{\mathcal{M}(z_I)}{z_I}\right) = 0. \tag{5}$$

If $\mathcal{M}(z) \to 0$ as $z \to \infty$, the amplitudes are characterized by their poles. Poles$\longrightarrow$ factorization channel $\Rightarrow$ residual at poles$\longrightarrow$ products of lower point amplitudes.

BCFW recursion relation constructs M-point on-shell tree amplitudes by N-point($N < M$) on-shell amplitudes

$$\mathcal{M}^{(N+2)}(p_{1,2}, k_i)$$
$$= \sum_{I_1, h} \mathcal{M}^{(N_1+2)}(p_1(z_{I_1}), k_{i_1}, h) \frac{1}{(p_1 + K_{I_1})^2} \mathcal{M}^{(N_2+2)}(p_2(z_{I_1}), k_{i_2}, -h). \tag{6}$$

where $K_{I_1} = \sum_{i \in I_1} k_i$.

The diagrammatic form of BCFW relation is

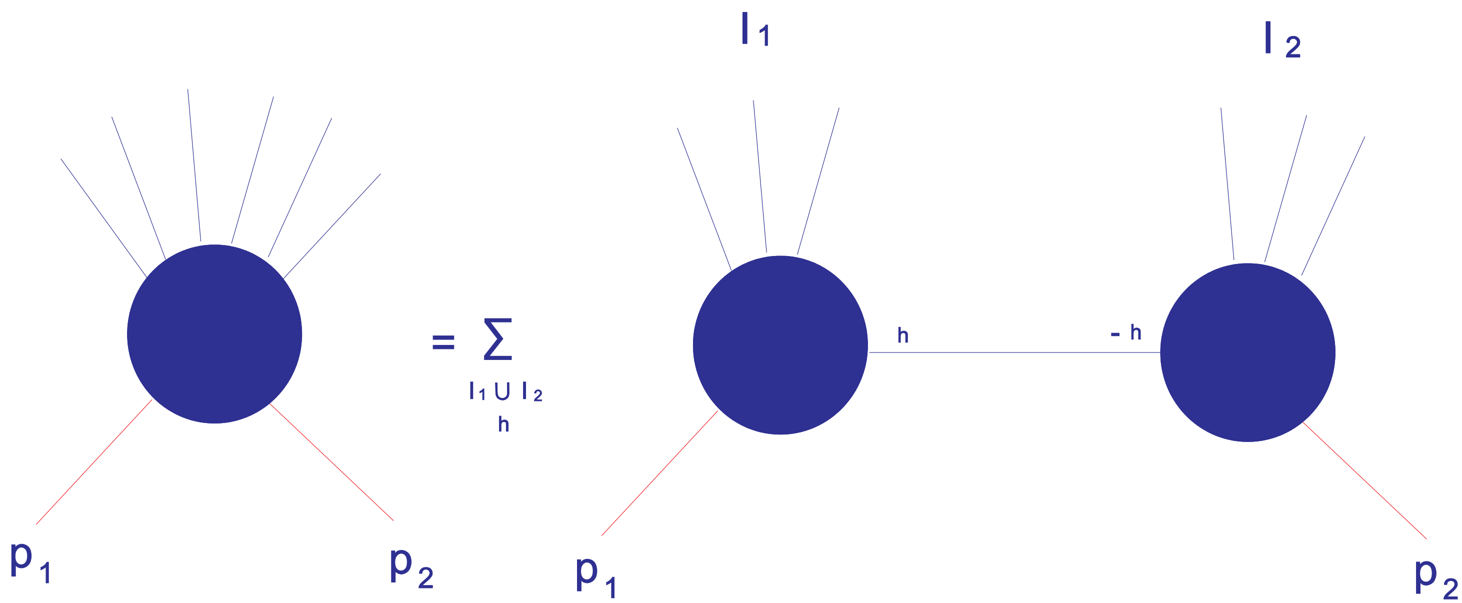

Figure 1: BCFW recursion relation

The amplitudes with at least one gluon leg or one graviton leg satisfy BCFW recursion relation (Cheung, JHEP **1003** (2010) 098).

- **KK-BCJ relation**

KK relation and BCJ relation are the relations among tree partial amplitudes for N gluons. KK relation is

$$\begin{aligned}&\mathcal{A}(\beta_1, ..., \beta_r, 1, \alpha_1, ..., \alpha_s, N)\\ =&(-1)^r \left[\sum_{\sigma \in OP\{\alpha\}\cup\{\beta^T\}} \mathcal{A}(1, \sigma, N)\right].\end{aligned} \tag{7}$$

$OP\{\alpha\} \cup \{\beta^T\}$ are all permutations of $\{\alpha\} \cup \{\beta^T\}$ that maintains the order of the individual elements belonging to each set. $\{\beta^T\}$ represents the set $\{\beta\}$ with the ordering reversed. KK relation express the tree partial amplitudes by $(N-2)!$ amplitudes. For

$r=1$ KK relation gives $U(1)$ decoupling identity.

BCJ relation is

$$\sum_{\sigma\in OP\{\alpha\}\cup\{\beta^T\}}\left[\sum_{1\leq i<j\leq r}s_{\beta_i,\beta_j}+\sum_{i=0}^{s}\sum_{j=1}^{r}(\alpha_i,\beta_j)\right]\mathcal{A}(1,\sigma,N)=0. \tag{8}$$

Here

$$(\alpha,\beta)=\begin{cases} s_{\alpha\beta} & (x_\beta>x_\alpha)\\ 0 & (x_\beta<x_\alpha).\end{cases} \tag{9}$$

BCJ relation give a further reduction of the tree partial amplitudes. With KK relation and BCJ relation, the number of independent amplitudes can be reduced to $(N-3)!$.

KK and BCJ relations can be derived from string theory by discussions on the monodromy of the amplitudes briefly (Bjerrum-Bohr, Damgaard and Vanhove, PRL **103** (2009) 161602). N-point open string amplitude on D_2 is[2]

$$\mathcal{A}(a_1, ..., a_N) = \int \prod_{i=1}^{N} dx_i \frac{|x_a - x_b||x_b - x_c||x_c - x_a|}{dx_a dx_b dx_c} \times \prod_{i=1}^{N-1} \theta(x_{a_{i+1}} - x_{a_i}) \prod_{1 \leq s < r \leq N} |x_s - x_r|^{2\alpha' k_r \cdot k_s} F_N. \tag{10}$$

[2] F_N denote the part do not contribute the branch cut points. We can set $x_1 = x_a = 0$, $x_{N-1} = x_b = 1$ and $x_N = x_c = \infty$.

The integral contours can be deformed to the region $]0, \infty[$

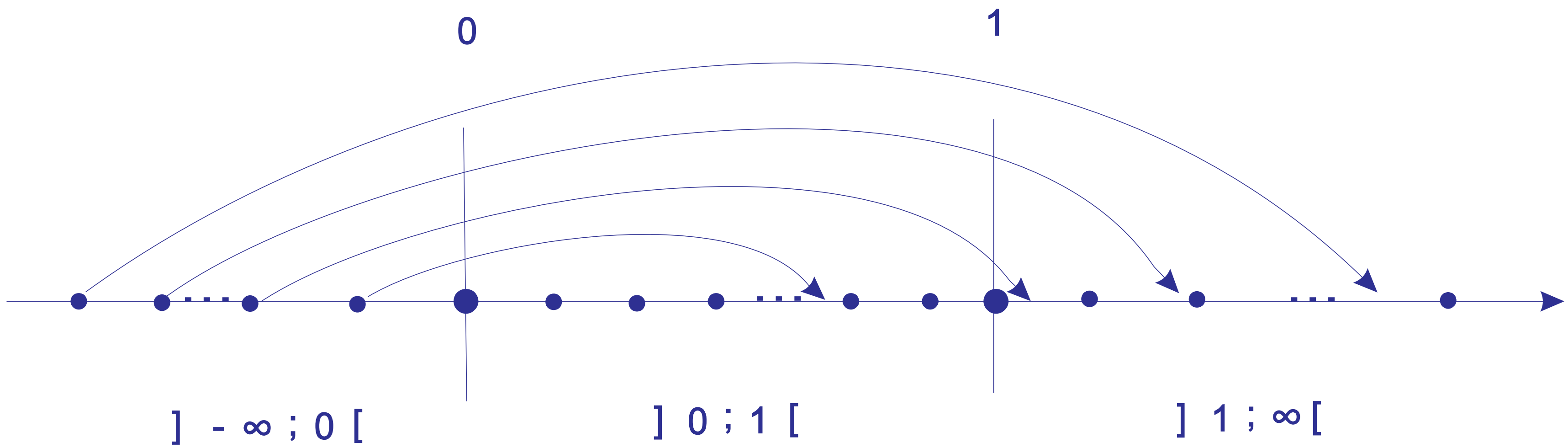

Figure 2: Contour deformation

When the contour of x integral passes through a branch cut point y we have

$$(x-y)^{\alpha} = (y-x)^{\alpha} \times \left\{ \begin{array}{l} e^{+i\pi\alpha}(\text{for clockwise rotation}) \\ e^{-i\pi\alpha}(\text{for counterclockwise }) \end{array} \right. . \qquad (11)$$

After deforming all the contours of x_{β_i} in the amplitude $\mathcal{A}(\beta_1, ..., \beta_r, 1, \alpha_1, ..., \alpha_s, N)$ to those in $]0, \infty[$, the real part of the amplitude gives $\mathcal{A}(\beta_1, ..., \beta_r, 1, \alpha_1, ..., \alpha_s, N)$ as

$$\begin{aligned} &\mathcal{A}(\beta_1, ..., \beta_r, 1, \alpha_1, ..., \alpha_s, N) \\ =&(-1)^r \Re \left[\prod_{1\leq i<j\leq r} e^{2i\pi\alpha'(k_{\beta_1}\cdot k_{\beta_j})} \sum_{\sigma\in OP\{\alpha\}\cup\{\beta^T\}} \prod_{i=0}^{s}\prod_{j=1}^{r} e^{2i\pi\alpha'(\alpha_i,\beta_j)} \mathcal{A}(1,\sigma,N) \right]. \end{aligned} \qquad (12)$$

The imaginary part vanishes

$$0=\Im\left[\prod_{1\leq i<j\leq r}e^{2i\pi\alpha'(k_{\beta_1}\cdot k_{\beta_j})}\sum_{\sigma\in OP\{\alpha\}\cup\{\beta^T\}}\prod_{i=0}^{s}\prod_{j=1}^{r}e^{2i\pi\alpha'(\alpha_i,\beta_j)}\mathcal{A}(1,\sigma,N)\right]. \tag{13}$$

The two relations given above can express the open string partial amplitudes on D_2 by $(N-3)!$ amplitudes.
Taking the field theory limits $\alpha'\to 0$, the real part condition gives the KK relation and the imaginary part condition gives the BCJ relation.

- **KLT relation**

KLT relation is the relation between closed string amplitudes on S_2 and open string amplitudes on D_2. Closed string amplitudes have the form

$$\mathcal{M}_{S_2}(1_c,...,M_c)=\kappa^{M-2}\frac{1}{V^{S_2}_{CKG}}\int d^2z_1d^2z_2...d^2z_M\langle\mathcal{V}_{1_c}(z_1,\bar{z}_1)...\mathcal{V}_{M_c}(z_M,\bar{z}_M)\rangle_{S_2}. \quad (14)$$

In bosonization formalism, the closed string vertex operator can be given as (Friedan, Martinec and Shenker,NPB,**271** (1986) 93, Kostelecky, Lechtenfeld, Lerche, Samuel and Watamura, NPB, **288**

(1987) 173) [3]

$$
\begin{aligned}
\mathcal{V}_c(\omega,\bar\omega) = &: \exp\left(q\phi_6 + \tilde q \tilde\phi_6\right) \\
&\exp\left(i\lambda\circ\phi + i\sum_{i=1}^{m}\varepsilon^i\circ\partial\phi_i + i\tilde\lambda\circ\tilde\phi + i\sum_{i=1}^{\tilde m}\bar\varepsilon^i\circ\bar\partial\tilde\phi_i\right) \\
&\exp\left(ik\cdot X + i\sum_{i=1}^{n}\epsilon^i\cdot\partial X + i\sum_{j=1}^{\tilde n}\bar\epsilon^j\cdot\bar\partial X\right)(\omega,\bar\omega) : |_{multilinear}.
\end{aligned} \tag{15}
$$

[3] λ and $\tilde\lambda$ are vectors in weight space. q and $\tilde q$ are superconformal ghost number. e. g. Fig. 3 gives the weight vector for $O(10)$.

Some primary fields of the spinning string

Operator	o(10) representation	λ	p	h
$T_{(-1)}$	1	$(0,-1)$	0	1/2
ψ^{μ}	10	$(0\cdots\pm1\cdots0,0)$	0	1/2
$\psi^{\mu}_{(+1)}$	10	$(0\cdots\pm1\cdots0,1)$	0	-1
$\psi^{\mu}_{(-1)}$	10	$(0\cdots\pm1\cdots0,-1)$	0	1
$J^{\mu\nu}$	45	$(0\cdots\pm1\cdots\pm1\cdots0,0)$	1/2	1
$S^{\dot{\alpha}}$	16_c	$\frac{1}{2}(\pm\pm\pm\pm\pm,0)_-$	0	5/8
S^{β}	16_s	$\frac{1}{2}(\pm\pm\pm\pm\pm,0)_+$	0	5/8
$S^{\dot{\alpha}}_{(+1/2)}$	16_c	$\frac{1}{2}(\pm\pm\pm\pm\pm,+)_-$	$-1/8$	0
$S^{\beta}_{(-1/2)}$	16_s	$\frac{1}{2}(\pm\pm\pm\pm\pm,+)_+$	7/8	0
$S^{\dot{\alpha}}_{(-1/2)}$	16_c	$\frac{1}{2}(\pm\pm\pm\pm\pm,-)_+$	$-1/8$	1
$S^{\beta}_{(+1/2)}$	16_s	$\frac{1}{2}(\pm\pm\pm\pm\pm,-)_-$	$-5/8$	1
$J^{\mu\nu}_{(-1)}$	45	$(0\cdots\pm1\cdots\pm1\cdots0,-1)$	1/2	3/2
$S^{\dot{\alpha}}_{(+3/2)}$	16_c	$\frac{1}{2}(\pm\pm\pm\pm\pm,+3)_-$	0	-2

The operators are in bosonized form given by $O = \exp(\lambda\cdot\phi)\exp[i\pi(\lambda\cdot M\partial\phi_0) + i\pi p]$. The subscripts + or − on spinor weights mean an even or odd total number of minus signs, respectively. Note that for $J^{\mu\nu}$ we do not write Cartan-subalgebra parts. The phases p are conventional. Except for those of $J^{\mu\nu}$ they are set to zero in this work; the ones given were used in refs. [64, 65]. The conformal weight is denoted by h

Figure 3: An example of weight vector given in Kostelecky, Lechtenfeld, Lerche, Samuel and Watamura, NPB, **288** (1987) 173

Any closed string amplitude on S_2 can be given as the bosonized form

$$\mathcal{M}_{S_2}(1_c, ..., M_c) = \kappa^{M-2} \int \prod_{i=1}^{M} d^2 z_i \frac{|z_a - z_b|^2 |z_b - z_c|^2 |z_c - z_a|^2}{dz_a^2 dz_b^2 dz_c^2}$$

$$\prod_{s>r} (z_s - z_r)^{\frac{\alpha'}{2} k_r \cdot k_s + \lambda_r \circ \lambda_s - q_r q_s} (\bar{z}_r - \bar{z}_s)^{\frac{\alpha'}{2} k_r \cdot k_s + \tilde{\lambda}_r \circ \tilde{\lambda}_s - \tilde{q}_r \tilde{q}_s}$$

$$\times \exp \left[-\sum_{s>r} \left(\sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \left(-\frac{\alpha'}{2} \right) \epsilon_r^{(i)} \cdot \epsilon_s^{(j)} - \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} \varepsilon_r^{(i)} \circ \varepsilon_s^{(j)} \right) (z_s - z_r)^{-2} + c.c. \right]$$

$$\times \exp \sum_{r \neq s} \left[\left(\sum_{i=1}^{n_s} \left(-\frac{\alpha'}{2} \right) k_r \cdot \epsilon_s^{(i)} - \sum_{i=1}^{m_s} \lambda_r \circ \varepsilon_s^{(i)} \right) (z_r - z_s)^{-1} + c.c. \right] \Big|_{multilinear}. \tag{16}$$

$z_r = x_r + iy_r$, $d^2 z_r = dx_r dy_r$. When we consider the y integral, the singular point are on the imaginary axis. We can deform the y_r

integral contours from the real axis to the imaginary axis (Fig.4).

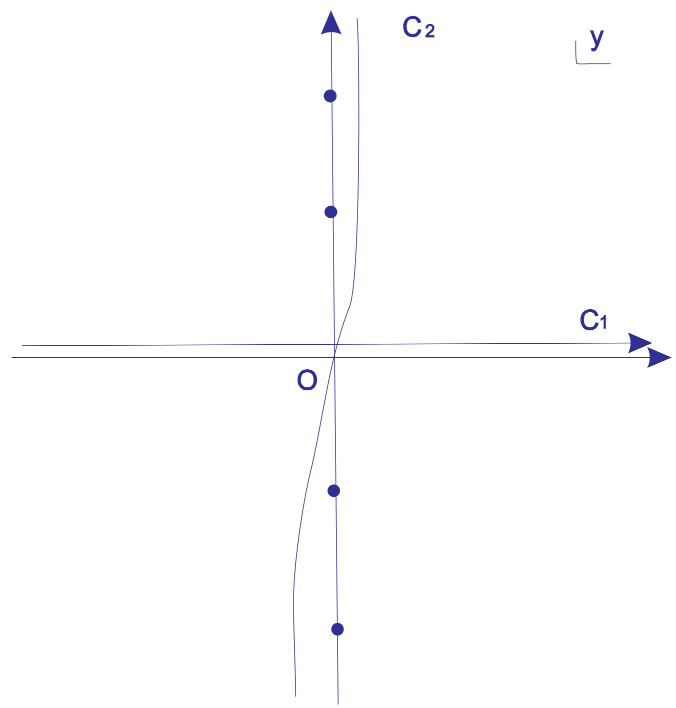

Figure 4: y integral contour deformation.

After this deformation $y_r \to y'_r = iy_r$, we define the real variables as

$\xi \equiv x + iy'$, $\eta \equiv x - iy'$. The closed string amplitude becomes

$$\begin{aligned}
&\mathcal{M}_{S_2}(1_c, ..., M_c) \\
= \; & \left(\frac{i}{2}\right)^{M-3} \kappa^{M-2} \int \prod_{i=1}^{M} d\xi_i \frac{|\xi_a - \xi_b||\xi_b - \xi_c||\xi_c - \xi_a|}{d\xi_a d\xi_b d\xi_c} \\
\times \; & \prod_{s>r} (\xi_s - \xi_r)^{\frac{\alpha'}{2} k_r \cdot k_s + \lambda_r \circ \lambda_s - q_r q_s} \\
\times \; & \exp\left[-\sum_{s>r}\left(\sum_{i=1}^{n_r}\sum_{j=1}^{n_s}\left(-\frac{\alpha'}{2}\right)\epsilon_r^{(i)} \cdot \epsilon_s^{(j)} - \sum_{i=1}^{m_r}\sum_{j=1}^{m_s} \varepsilon_r^{(i)} \circ \varepsilon_s^{(j)}\right)(\xi_s - \xi_r)^{-2}\right] \\
\times \; & \exp \sum_{r \neq s}\left[\left(\sum_{i=1}^{n_s}\left(-\frac{\alpha'}{2}\right) k_r \cdot \epsilon_s^{(i)} - \sum_{i=1}^{m_s} \lambda_r \circ \varepsilon_s^{(i)}\right)(\xi_r - \xi_s)^{-1}\right] \\
\times \; & \int \prod_{i=1}^{M} d\eta_i \frac{|\eta_a - \eta_b||\eta_b - \eta_c||\eta_c - \eta_a|}{d\eta_a d\eta_b d\eta_c} \qquad (17)
\end{aligned}$$

$$
\begin{aligned}
\times & \prod_{s>r} (\eta_s - \eta_r)^{\frac{\alpha'}{2} k_r \cdot k_s + \tilde{\lambda}_r \circ \tilde{\lambda}_s - \tilde{q}_r \tilde{q}_s} \\
\times & \exp\left[- \sum_{s>r} \left(\sum_{i=1}^{\tilde{n}_r} \sum_{j=1}^{\tilde{n}_s} \left(-\frac{\alpha'}{2} \right) \bar{\epsilon}_r^{(i)} \cdot \bar{\epsilon}_s^{(j)} - \sum_{i=1}^{\tilde{m}_r} \sum_{j=1}^{\tilde{m}_s} \bar{\varepsilon}_r^{(i)} \circ \bar{\varepsilon}_s^{(j)} \right) (\eta_s - \eta_r)^{-2} \right] \\
\times & \exp \sum_{r \neq s} \left[\left(\sum_{i=1}^{\tilde{n}_s} \left(-\frac{\alpha'}{2} \right) k_r \cdot \bar{\epsilon}_s^{(i)} - \sum_{i=1}^{\tilde{m}_s} \tilde{\lambda}_r \circ \bar{\varepsilon}_s^{(i)} \right) (\eta_r - \eta_s)^{-1} \right] |_{multilinear}.
\end{aligned}
$$

ξ integrals $\Rightarrow$ left moving sector, η integrals $\Rightarrow$right-moving sector.

This amplitude is just a product of two open string amplitudes corresponding to the left- and right-moving sectors except for a phase factor.

We should take a phase factor out since there are $|x_s - x_r|^{\frac{\alpha'}{2} k_r \cdot k_s}$ in open string amplitudes. After taking the absolute value of

$(\xi_s - \xi_r)^{\frac{\alpha'}{2}k_r\cdot k_s}$, $(\eta_s - \eta_r)^{\frac{\alpha'}{2}k_r\cdot k_s}$, the integrals in closed string amplitudes must be performed in the correct branch. The phase factor is defined as $e^{i\pi F(P_L,P_R)}$, where

$$F(P_L, P_R) = \sum_{s>r} f(k_i \cdot k_j; (\xi_s - \xi_r), (\eta_s - \eta_r)), \tag{18}$$

$$f(k_s \cdot k_r; \xi, \eta) = \begin{cases} 0 & (\xi\eta > 0) \\ \frac{\alpha'}{2}k_s \cdot k_r & (\xi\eta < 0) \end{cases}. \tag{19}$$

Since the phase factor only dependent on the permutations of the external legs of the open strings in the left- and right-moving sectors, in any given permutations P_L and P_R, the phase factor decouple from the integrals.

At last, the relation between closed string amplitude on S_2 and open string amplitudes on D_2 are given

$$\begin{aligned}&\mathcal{M}_{S_2}(1_c, ..., M_c)\\&=\left(\frac{i}{2}\right)^{M-3}\kappa^{M-2}\sum_{P_L,P_R}\mathcal{A}^{(M)}(P_L)\bar{\mathcal{A}}^{(M)}(P_R)e^{i\pi F(P_L,P_R)}.\end{aligned} \tag{20}$$

In KLT relation, the left- and right-moving waves of the closed strings do not interact with each other. Then the open strings in the left- and right-moving sectors are independent of each other (see Fig. 5). Thus the KLT relation is a factorization relation.

Figure 5: closed string amplitudes can be factorized into open string amplitudes corresponding to left- and right-moving sectors.

- **Reduction of KLT relation**

KLT relation on S_2 can be reduced by considering the relations among products of open string amplitudes. After the reduction, the number of terms of KLT relation can be reduced to

$$(M-3)!(\frac{1}{2}(M-3))!(\frac{1}{2}(M-3))!,\ M \text{ is odd,} \tag{21}$$

$$(M-3)!(\frac{1}{2}(M-4))!(\frac{1}{2}(M-2))!,\ M \text{ is even.} \tag{22}$$

The phase factors can be reduced to sine functions. Then KLT relation has the form

$$\begin{aligned}\mathcal{M}_{S_2}(1_c, 2_c, ..., M_c) = (-1)^{(M-3)}\kappa^{M-2}\mathcal{A}_L(1,2,...,M)\sum_{perms} f(i_1,...,i_j)\bar{f}(l_1,...,l_{j'})\\ \times \mathcal{A}_R(i_1,...,i_j,1,M-1,l_1,...,l_{j'},M)\\ +\mathcal{P}(2,...,M-2),\end{aligned} \tag{23}$$

where the sum is over all permutations

$\{i_1, ..., i_j\} \in \mathcal{P}(2, ..., \frac{1}{2}(M-3)+1)$ and $\{l_1, ..., l_{j'}\} \in \mathcal{P}(\frac{1}{2}(M-3)+2, ..., M-2)$ (M is odd) or $\{i_1, ..., i_j\} \in \mathcal{P}(2, ..., \frac{1}{2}(M-4)+1)$ and $\{l_1, ..., l_{j'}\} \in \mathcal{P}(\frac{1}{2}(M-4)+2, ..., M-2)$ (M is even). $+\mathcal{P}$ signifies a sum over the preceding expression for all permutations of legs $2, ..., M-2$.

$$f(i_1, ..., i_j) = \sin\left(\frac{1}{2}\alpha' k_1 \cdot k_{i_j}\right) \prod_{m=1}^{j-1} \sin\left[\frac{1}{2}\alpha'\left(k_1 \cdot k_{i_m} + \sum_{k=m+1}^{j} g(i_m, i_k)\right)\right],$$

$$\bar{f}(l_1, ..., l_{j'}) = \sin\left(\frac{1}{2}\alpha' k_{l_1} \cdot k_{M-1}\right) \prod_{m=1}^{j'} \sin\left[\frac{1}{2}\alpha'\left(k_{1_m} \cdot k_{M-1} + \sum_{k=1}^{m-1} g(l_k, l_m)\right)\right], \tag{24}$$

and

$$g(i,j) = \begin{cases} k_i \cdot k_j & (i > j) \\ 0 & (\text{others}) \end{cases} . \tag{25}$$

- **KLT relation in field theory**

Since $\kappa \sim \frac{1}{\alpha'} g^2$, when we take $\alpha' \to 0$, we get the field theory limit of KLT relation. KLT relation can express M-graviton tree amplitudes by products of two M-point pure gluon amplitudes (Berends, Giele and Kuijf, PLB **211** (1988) 91). In this factorization form we have $h^{\pm} \to g^{\pm} g^{\pm}$.

KLT relation can also express amplitudes for gravitons minimally coupled to gluons. In this situation we have $h^{\pm} \to g^{\pm} g^{\pm}$ and

$g^{\pm} \to g^{\pm} s$. The amplitude can be factorized into products of pure-gluon amplitudes and amplitudes for scalar coupled to gluons.

3. Disk relation in string theory

- **Does KLT factorization relation hold for any topology?**

KLT factorization relation →closed string amplitudes on S_2 relation

⇔ (open string amplitudes on $D_2)^2$.

What about higher-order closed string amplitudes?
Do the left- and right-moving sectors of closed strings independent of each other on other topologies such as [4] D_2, RP_2 , T_2,...?

[4] Closed string amplitudes on D_2 and RP_2 are tree amplitudes. Since the vacuum on D_2 contribute $g^{-2} \sim \kappa^{-1}$, these amplitudes are higher-order tree amplitudes.

- **Disk relation**

D_2 has a boundary, the correlation function is

$$\langle 0|\mathcal{V}_{1_c}(\omega_1,\bar{\omega}_1)...\mathcal{V}_{M_c}(\omega_M,\bar{\omega}_M)|B\rangle. \tag{26}$$

$\mid B\rangle \equiv B \mid 0\rangle$ and

$$B = \exp\big(\sum_{n=1}^{\infty} a_n^{\dagger} \cdot \tilde{a}_n^{\dagger}\big) \otimes \exp\big(\sum_{n=1}^{\infty} b_n^{\dagger} \circ \tilde{b}_n^{\dagger}\big) \otimes \exp\big(\sum_{n=1}^{\infty} c_n^{\dagger} \tilde{c}_n^{\dagger}\big) \tag{27}$$

is the bosonized boundary operator to create the Neumann boundary condition. B does not commute with the annihilation modes and commutes with zero modes as well as creation modes. This means only annihilation modes are reflected at the boundary (See Fig. 6).

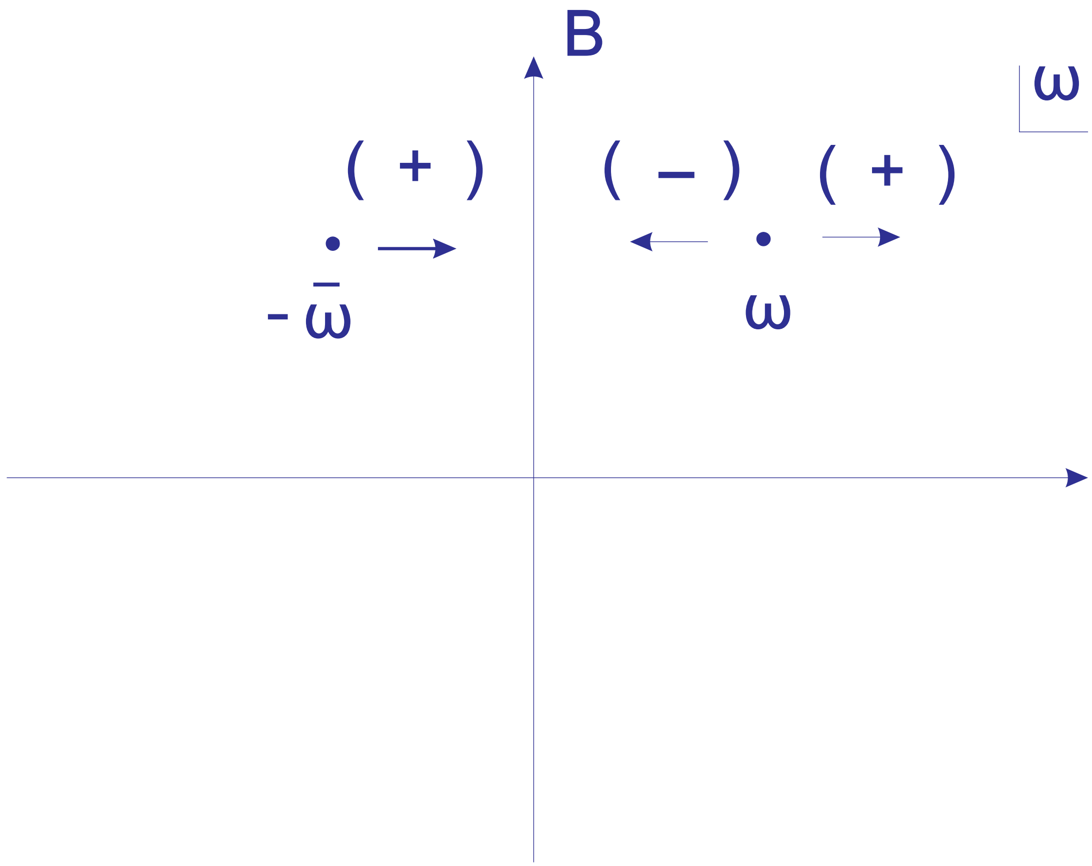

Figure 6: Only annihilation modes are reflected at the boundary.

After refection, the annihilation modes in left- (right-) moving sector turn to the creation modes in right- (left-) moving sector. Thus, there must be interactions between the two sectors. The left- and right-moving sectors are not independent of each other.
In $z = e^{\omega}$ coordinate, the M closed string amplitude can be given as

$$
\begin{aligned}
&\mathcal{M}_{D_2}(1_c, \ldots, M_c) \\
&= \kappa^{M-1} \int_{|z|<1} \prod_{i=1}^{M} d^2 z_i \frac{|1 - z_o \bar{z}_o|^2}{2\pi d^2 z_o} \\
&\times \prod_{s>r} (z_s - z_r)^{\frac{\alpha'}{2} k_r \cdot k_s + \lambda_r \circ \lambda_s - q_r q_s} (\bar{z}_r - \bar{z}_s)^{\frac{\alpha'}{2} k_r \cdot k_s + \tilde{\lambda}_r \circ \tilde{\lambda}_s - \tilde{q}_r \tilde{q}_s} \\
&\prod_{r,s} \left(1 - (z_r \bar{z}_s)^{-1}\right)^{\frac{\alpha'}{2} k_r \cdot k_s + \lambda_r \circ \tilde{\lambda}_s - q_r \tilde{q}_s} \\
&\times \exp \sum_{r=1}^{M} \left(\sum_{i=1}^{n_r} \sum_{j=1}^{\tilde{n}_s} \left(-\frac{\alpha'}{2} \right) \epsilon_r^{(i)} \cdot \bar{\epsilon}_r^{(j)} - \sum_{i=1}^{m_r} \sum_{j=1}^{\tilde{m}_s} \varepsilon_r^{(i)} \circ \bar{\varepsilon}_r^{(j)} \right) (1 - |z_r|^2)^{-2}
\end{aligned}
$$

$$\times \exp \sum_{s>r} \left[\left(\sum_{i=1}^{\tilde{n}_r} \sum_{j=1}^{n_s} \left(-\frac{\alpha'}{2} \right) \bar{\epsilon}_r^{(i)} \cdot \epsilon_s^{(j)} - \sum_{i=1}^{\tilde{m}_r} \sum_{j=1}^{m_s} \bar{\varepsilon}_r^{(i)} \circ \varepsilon_s^{(j)} \right) (1 - \bar{z}_r z_s)^{-2} + c.c. \right]$$

$$\times \exp \left[-\sum_{s>r} \left(\sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \left(-\frac{\alpha'}{2} \right) \epsilon_r^{(i)} \cdot \epsilon_s^{(j)} - \sum_{i=1}^{m_r} \sum_{j=1}^{m_s} \varepsilon_r^{(i)} \circ \varepsilon_s^{(j)} \right) (z_s - z_r)^{-2} + c.c. \right]$$

$$\times \exp \sum_{r \neq s} \left[\left(\sum_{i=1}^{n_s} \left(-\frac{\alpha'}{2} \right) k_r \cdot \epsilon_s^{(i)} - \sum_{i=1}^{m_s} \lambda_r \circ \varepsilon_s^{(i)} \right) \right.$$

$$\left. \times \left((z_r - z_s)^{-1} + (\bar{z}_r{}^{-1} - z_s)^{-1} \right) + c.c. \right]$$

$$\times \exp \sum_{r=1}^{N} \left[\left(\left(-\frac{\alpha'}{2} \right) k_r \cdot \sum_{i=1}^{n_r} \epsilon_r^{(i)} - \lambda_r \circ \sum_{i=1}^{m_r} \varepsilon_r^{(i)} \right) \right.$$

$$\left. \times \left((\bar{z}_r{}^{-1} - z_r)^{-1} + z_r{}^{-1} \right) + c.c. \right] \Big|_{multilinear}. \tag{28}$$

Continuating the fundamental region to the whole complex plan, following similar steps as in KLT relation case, using the conformal invariance in one sector, we express the M-point closed string amplitudes on D_2 by $2M$-point open string amplitudes on D_2.

$$\mathcal{M}_{D_2}(1_c, ..., M_c) = \left(\frac{i}{4}\right)^{M-1} \kappa^{M-1} \sum_P \mathcal{A}^{(2M)}(P) e^{i\pi\Theta(P)}. \tag{29}$$

$\Theta(P)$ is

$$\begin{aligned}\Theta(P) = &\sum_{1\le r<s\le M} (2\alpha' k'_s \cdot k'_r)\theta[-(\xi_s - \xi_r)(\xi_{s+M} - \xi_{r+M})] \\ &+ \sum_{1\le r<s\le M} (2\alpha' k'_s \cdot k'_r)\theta[-(\xi_s - \xi_{r+M})(\xi_{s+M} - \xi_r)] \\ &+ \sum_{1\le r\le M} (2\alpha' {k'}_r^2)\theta(\xi_{r+N} - \xi_r).\end{aligned} \tag{30}$$

k' denote the momentum of open string, it is just half of the momentum of the corresponding closed string. P are the permutations of the $2M$ open strings.
When we consider the amplitudes with N open strings with Chan-Paton degree of freedom and M closed strings on D_2, after a similar discussion, The mixed amplitude becomes

$$\begin{aligned}&\mathcal{M}_{D_2}(1_o^{a_1},...,N_o^{a_N},(N+1)_c,...,(N+M)_c)\\=&\sum_{\sigma}Tr\left(T^{a_{\sigma(1)}}...T^{a_{\sigma(N)}}\right)\mathcal{A}(\sigma(1_o),...,\sigma(N_o),(N+1)_c,...,(N+M)_c),\end{aligned} \tag{31}$$

and

$$\begin{aligned}&\mathcal{A}(\sigma(1_o),...,\sigma(N_o),(N+1)_c,...,(N+M)_c)\\=&g^{N-2}\kappa^M\sum_{P'}e^{i\pi\Theta'(P')}\mathcal{A}^{(N,2M)}(P'),\end{aligned} \tag{32}$$

Here T^a denote the Chan-Paton factor. P' denote the noncyclic permutations which preserve the open string order. $\Theta'(P)$ is

$$
\begin{aligned}
\Theta(P) =& \sum_{t,r}(2\alpha' k_t \cdot k'_r)\theta[-(x_t-\xi_r)(x_t-\xi_{r+M})] \\
&+ \sum_{N+1\le r<s\le N+M}(2\alpha' k'_s \cdot k'_r)\theta[-(\xi_s-\xi_r)(\xi_{s+M}-\xi_{r+M})] \\
&+ \sum_{N+1\le r<s\le N+M}(2\alpha' k'_s \cdot k'_r)\theta[-(\xi_s-\xi_{r+M})(\xi_{s+M}-\xi_r)] \\
&+ \sum_{1\le r\le M}(2\alpha' k'^2_r)\theta(\xi_{r+N}-\xi_r),
\end{aligned}
\tag{33}
$$

(32) is the disk relation for partial amplitudes.

• Reduction of disk relation

With KK-BCJ relation, we can reduce disk relation into $(N+2M-3)!$ terms (Stieberger, arXiv:0907.2211 [hep-th]).

$$\mathcal{A}(1_o, 2_o, 3_c) \sim \kappa \sin(2\pi\alpha' k_1 \cdot k_2)\mathcal{A}(1_o, 2_o, 3_o, 4_o), \tag{34}$$

$$\mathcal{A}(1_o, 2_o, 3_o, 4_c) \sim \kappa\ g \sin(2\pi\alpha' k_1 \cdot k_3)\mathcal{A}(1_o, 5_o, 2_o, 4_o, 3_o), \tag{35}$$

$$\begin{aligned}\mathcal{A}(1_o, 2_o, 3_c, 4_c) \sim\ & \kappa^2 \sin(\frac{\pi}{2}\alpha' s_{12}) \sin(\pi\alpha' s_{12})\mathcal{A}(1_o, 6_o, 3_o, 5_o, 4_o, 2_o) \\ & - \kappa^2 \sin(\frac{\pi}{2}\alpha' s_{12}) \sin(\pi\alpha' s_{13})\mathcal{A}(1_o, 3_o, 5_o, 4_o, 2_o, 6_o).\end{aligned} \tag{36}$$

4. Disk relation in field theory

- **The field theory limit of disk relation**

The field theory limit of disk relation can give the amplitudes for gauge-gravity coupling. $\kappa \sim \frac{1}{\alpha'} g^2$, $\alpha' \to 0$ we get the field theory limit of disk relations[5]

$$\mathcal{A}(1_g, 2_g, 3_h) \sim s_{12} \mathcal{A}(1_g, 2_g, 3_g, 3_g), \tag{37a}$$

$$\mathcal{A}(1_g, 2_g, 3_g, 4_h) \sim s_{13} \mathcal{A}(1_g, 4_g, 2_g, 4_g, 3_g), \tag{37b}$$

[5] here we denote both of the two gluons corresponding to i_h i_g, because the two gluons take the same momentum and helicity.

$$\mathcal{A}(1_g, 2_g, 3_h, 4_h) \sim s_{12}^2 \mathcal{A}(1_g, 4_g, 3_g, 4_g, 3_g, 2_g) - s_{12}s_{13}\mathcal{A}(1_g, 3_g, 4_g, 3_g, 2_g, 4_g). \tag{37c}$$

Through direct calculation, we find the three- and four-point disk relations give the right amplitudes in minimal coupling theory of gauge field and gravity.

- **Disk relation for the MHV amplitudes with one graviton minimal coupled to N gluons**

The expression of $\mathcal{A}(1_g^-, 2_g^+, ..., i_g^-, ..., N_g^+, (N+1)_h^+, ..., (N+M)_h^+)$

is[6]

$$\mathcal{A}(1_g^-, 2_g^+, ..., i_g^-, ..., N_g^+, (N+1)_h^+, ..., (N+M)_h^+)$$
$$\sim i\frac{\langle 1i\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle N1\rangle}S(1, i, \{h^+\}, \{g^+\}), \tag{38}$$

where

$$S(i, j, \{h^+\}, \{g^+\}) = \left(\prod_{m\in\{h^+\}}\frac{d}{da_m}\right)$$

[6] $|i^\pm\rangle \equiv |k_i^\pm\rangle \equiv u_\pm(k_i) = v_\mp(k_i), \langle i^\pm| \equiv \langle k_i^\pm| \equiv \bar{u}_\pm(k_i) = \bar{v}_\pm(k_i)$, where u and v are the positive and negative energy solutions of Dirac equation (Xu, Zhang and Chang, NPB, **291** (1987) 392), respectively. $\langle ij\rangle \equiv \langle i^-|j^+\rangle = \sqrt{|s_{ij}|}e^{i\phi_{ij}}$, $[ij] \equiv \langle i^+|j^-\rangle = \sqrt{|s_{ij}|}e^{-i(\phi_{ij}+\pi)}$.

$$\times \prod_{l\in\{g^+\}} \exp\left[\sum_{n_1\in\{h^+\}} a_{n_1}\frac{\langle li\rangle\langle lj\rangle[ln_1]}{\langle n_1 i\rangle\langle n_1 j\rangle\langle ln_1\rangle}\right.$$

$$\left.\left.\times \exp\left[\sum_{n_2\in\{h^+\},n_2\neq n_1} a_{n_2}\frac{\langle n_1 i\rangle\langle n_1 j\rangle[n_1 n_2]}{\langle n_2 i\rangle\langle n_2 j\rangle\langle n_1 n_2\rangle}\exp\left[\ldots\right]\right]\right]\right|_{a_j=0}. \qquad (39)$$

This expression is given in Selivanov, Phys. Lett. B **420** (1998) 274. When $M=1$, we have

$$\mathcal{A}(1_g^-, 2_g^+, \ldots, i_g^-, \ldots, N_g^+, (N+1)_h^+)$$

$$\sim i\frac{\langle 1i\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle N1\rangle}\sum_{l\in\{g^+\}}\frac{\langle l1\rangle\langle li\rangle[l,N+1]}{\langle N+1,1\rangle\langle N+1,i\rangle\langle l,N+1\rangle}$$

$$\sim i\sum_{l\in\{g^+\}}\frac{\langle 1i\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle N1\rangle}\langle l,N+1\rangle[l,N+1]$$

$$\times \frac{\langle 1l \rangle}{\langle 1, N+1\rangle\langle N+1, l\rangle} \frac{\langle li \rangle}{\langle l, N+1\rangle\langle N+1, i\rangle}. \tag{40}$$

$\langle l, N+1\rangle[l, N+1] = -s_{l,N+1}$. For $1 < l < i$

$$\frac{\langle 1l \rangle}{\langle 1, N+1\rangle\langle N+1, l\rangle} = \sum_{r=1}^{l-1} \frac{\langle r, r+1\rangle}{\langle r, N+1\rangle\langle N+1, r+1\rangle}, \tag{41}$$

$$\frac{\langle li \rangle}{\langle l, N+1\rangle\langle N+1, i\rangle} = \sum_{t=l}^{i-1} \frac{\langle t, t+1\rangle}{\langle t, N+1\rangle\langle N+1, t+1\rangle}. \tag{42}$$

For $i < l \leq N$

$$\frac{\langle il \rangle}{\langle i, N+1\rangle\langle N+1, l\rangle} = \sum_{r=i}^{l-1} \frac{\langle r, r+1\rangle}{\langle r, N+1\rangle\langle N+1, r+1\rangle}, \tag{43}$$

$$\frac{\langle l1\rangle}{\langle l,N+1\rangle\langle N+1,1\rangle}=\sum_{t=l}^{N}\frac{\langle t,t+1\rangle}{\langle t,N+1\rangle\langle N+1,t+1\rangle}. \tag{44}$$

The amplitude becomes

$$\begin{aligned}&\mathcal{A}(1_g^-,2_g^+,\ldots,i_g^-,\ldots,N_g^+,(N+1)_h^+)\\&\sim i\left(\sum_{1<l<i}s_{l,N+1}\sum_{r=1}^{l-1}\sum_{t=l}^{i-1}+\sum_{i<l\leq N}s_{l,N+1}\sum_{r=i}^{l-1}\sum_{t=l}^{N}\right)\\&\cdot\frac{\langle 1i\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle N1\rangle}\frac{\langle r,r+1\rangle}{\langle r,N+1\rangle\langle N+1,r+1\rangle}\frac{\langle t,t+1\rangle}{\langle t,N+1\rangle\langle N+1,t+1\rangle}.\end{aligned} \tag{45}$$

$i\frac{\langle 1i\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle N1\rangle}$ is MHV tree amplitude for N gluons.
$\frac{\langle r,r+1\rangle}{\langle r,N+1\rangle\langle N+1,r+1\rangle}$ insert a gluon corresponding to graviton into the

position between r and $r+1$. Thus the amplitude can be given as

$$\mathcal{A}(1_g^-, 2_g^+, ..., i_g^-, ..., N_g^+, (N+1)_h^+) \sim \sum_{l \in \{g^+\}} s_{l,N+1} \sum_P \mathcal{A}_{MHV}^{N+2}(P). \tag{46}$$

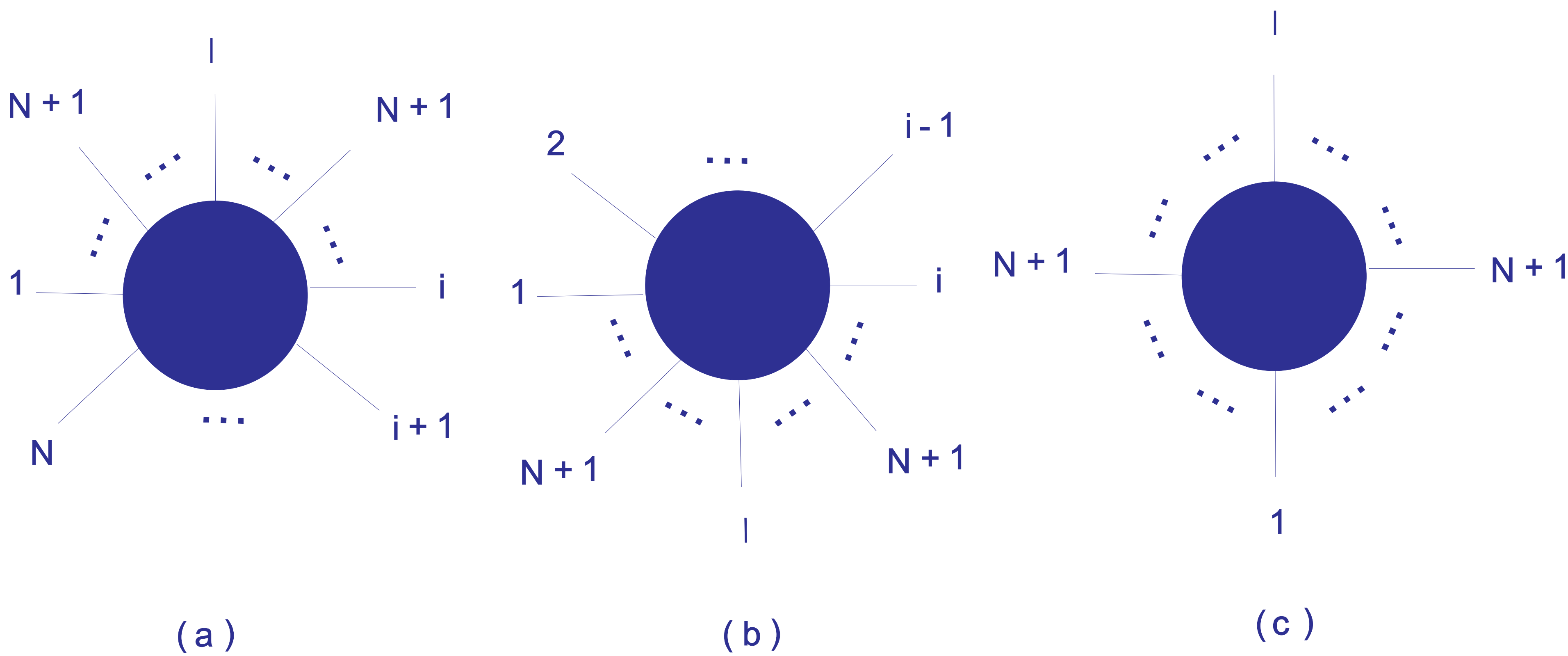

Figure 7: Positions of the two gluons corresponding to the graviton for (a) $1 < l < i$, (b) $i < l \leq N$ and (c) the expression independent of helicity configuration.

P are the insertions of the two gluons corresponding to the graviton. The insertions preserve the relative positions of the gluons. One is inserted into the positions between 1 and l while the other on is inserted into positions between l and i (See Fig. 7(a) and (b)).

In fact, with some properties of spinor helicity formalism, we can sum over all l instead of $\{g^+\}$ in (40). Then the relation can be given as

$$\mathcal{A}(1_g, 2_g, ..., N_g, (N+1)_h) \sim \sum_{1<l\leq N} s_{l,N+1} \sum_{P'} \mathcal{A}^{N+2}(P'). \tag{47}$$

Here, P' are the insertions preserving the order of the gluons. One gluon corresponding to $(N+1)_h$ is inserted into the positions between 1 and l while the other one is inserted into the positions between l and 1 (See Fig. 7(c)).

- **Disk relation for the amplitudes with one graviton minimal coupled to N gluons –arbitrary helicity configuration**

We use BCFW recursion relation to prove the disk relations (47) for one graviton minimally coupled to N gluons are right for arbitrary helicity configuration. Using BCJ relation for four-point amplitudes,we can give three point amplitude (37a) as

$$\mathcal{A}(1_g, 2_g, 3_h) \sim s_{23}\mathcal{A}(1_g, 3_g, 2_g, 3_g). \tag{48}$$

This amplitude is nonzero and satisfies the relation (47).

The BCFW expression of $\mathcal{A}(1_g, 2_g, ..., N_g, (N+1)_h)$ is

$$\begin{aligned}
&\mathcal{A}(1_g, 2_g, ..., N_g, (N+1)_h) \\
&= \sum_i \sum_H \mathcal{A}((i+1)_g, ..., N_g, 1_g, \hat{P}_1^H, (N+1)_h) \frac{1}{P_1^2} \mathcal{A}(\hat{2}_g, 3_g, ..., i_g, (-\hat{P}_1)^{-H}) \\
&+ \sum_i \sum_H \mathcal{A}((i+1)_g, ..., N_g, 1_g, \hat{P}_2^H) \frac{1}{P_2^2} \mathcal{A}(\hat{2}_g, 3_g, ..., i_g, (-\hat{P}_2)^{-H}, (N+1)_h).
\end{aligned} \tag{49}$$

where $\hat{P}_1 = \hat{k}_2 + k_3 + ... + k_i$ and $\hat{P}_2 = \hat{k}_2 + k_3 + ... + k_i + k_{N+1}$. If we have the relation (47) for $N' < N$, we have

$$\mathcal{A}((i+1)_g, ..., N_g, 1_g, \hat{P}_1^H, (N+1)_h) = \sum_{l \in I_1} s_{l,N+1} \sum_P \mathcal{A}^{\text{gluon}}(P). \tag{50}$$

$$\mathcal{A}(\hat{2}_g, 3_g, ..., i_g, (-\hat{P}_2)^{-H}, (N+1)_h) = \sum_{l \in I_2} s_{l,N+1} \sum_P \mathcal{A}^{\text{gluon}}(P). \qquad (51)$$

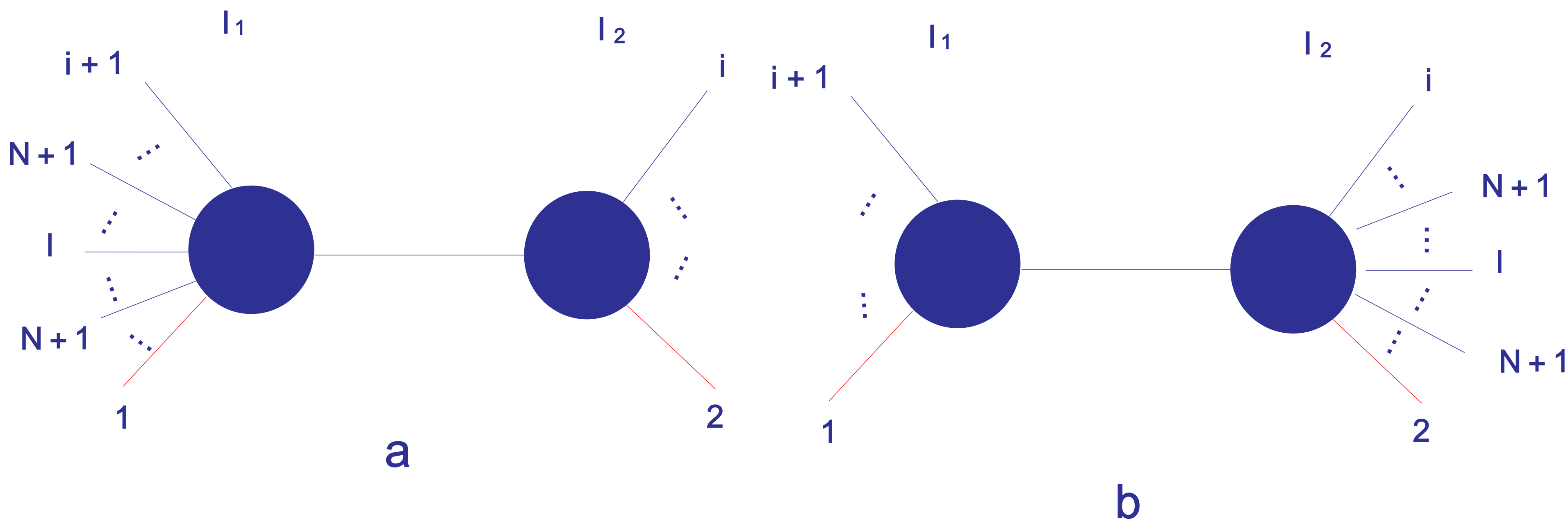

Figure 8:

The first and second terms in (49) corresponding to Fig.8(a), (b).

The diagrams contribute to $\sum_{1<l\leq N} s_{l,N+1} \sum_{P'} \mathcal{A}^{N+2}(P')$ in BCFW expression are given in Fig. 9, Fig. 10, Fig. 11.

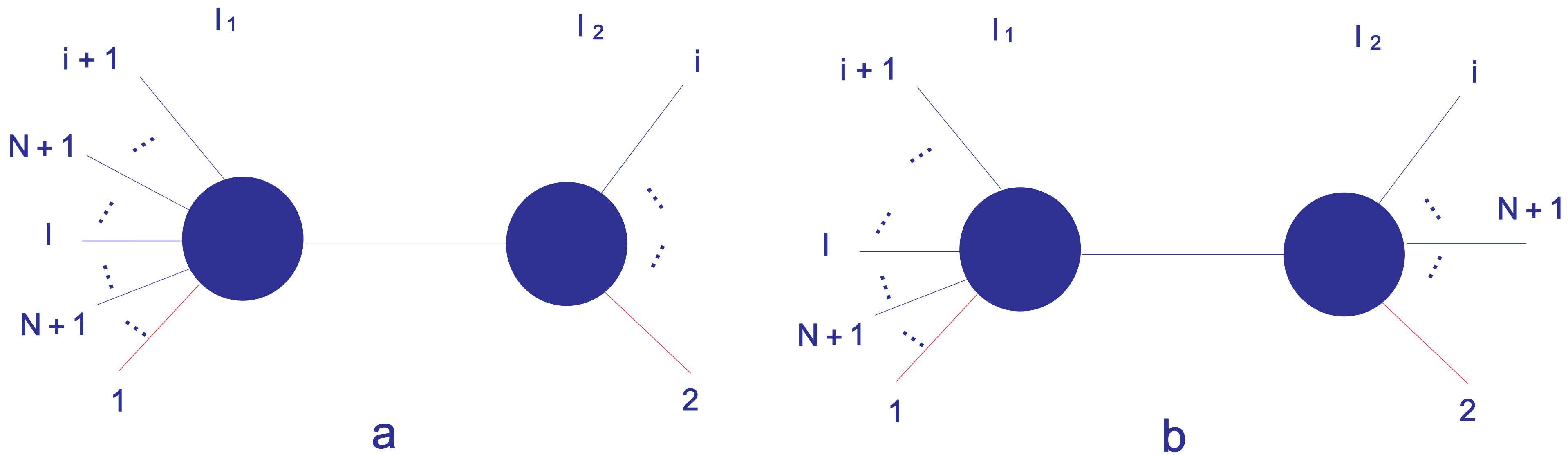

Figure 9:

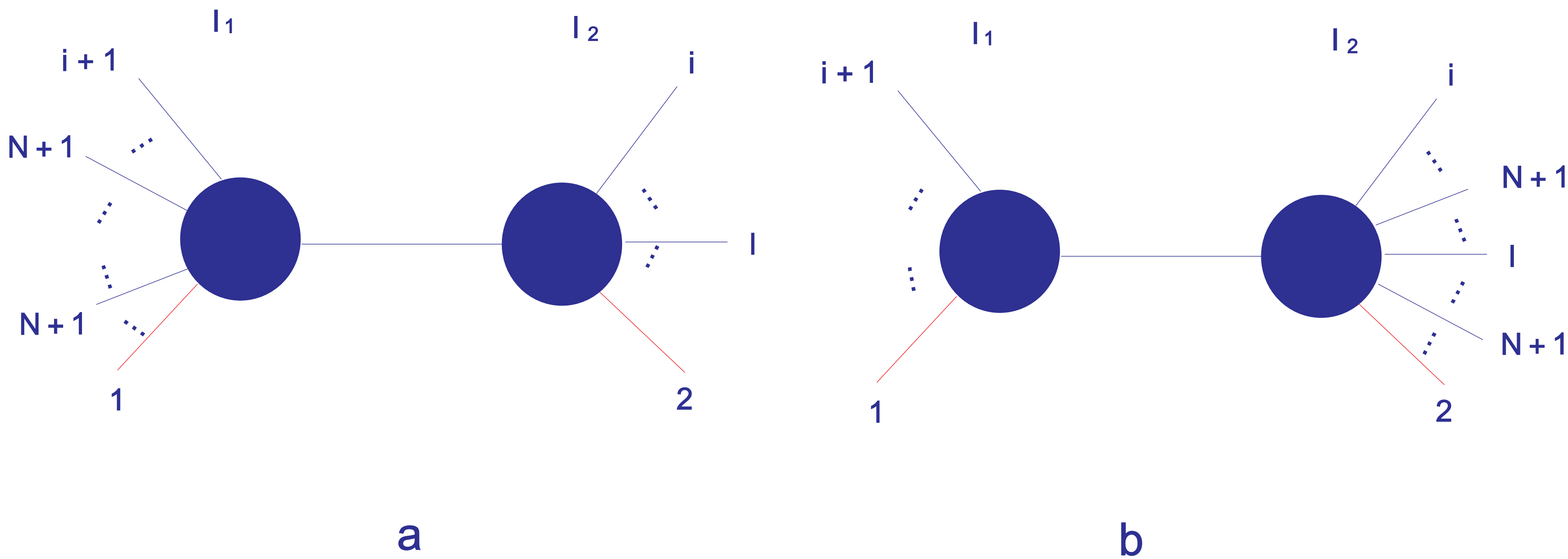

Figure 10:

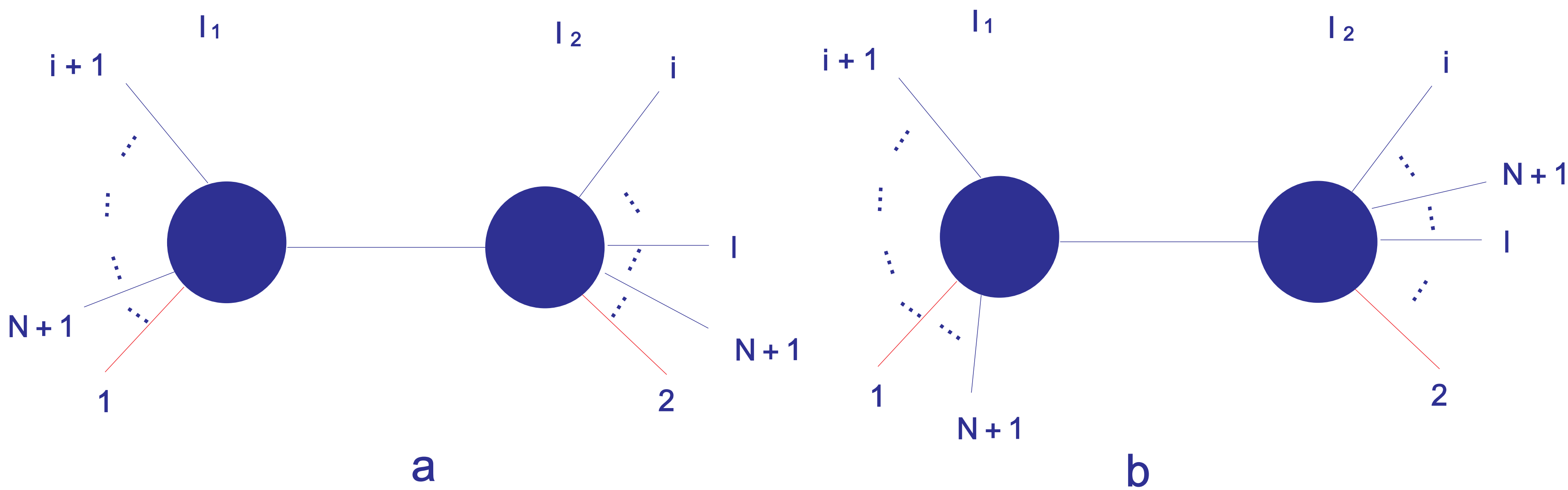

Figure 11:

Fig. 9 (a), Fig. 10 (a) give the same contributions to Fig. 8 (a). Fig. 10 (b) give the same contribution to Fig. 8 (b). Fig. 9 (b) and Fig. 11 (a), (b) cancel out due to KK-BCJ relation. Thus we prove the disk relation (47).

5. Conclusions

- KLT factorization relation does not hold on D_2.
- Amplitudes with N open strings and M closed strings on D_2 can be expressed by sum of partial amplitudes for $N + 2M$ open strings on D_2.
- Disk relation can be reduced by KK-BCJ relations to $(N + 2M - 3)!$ terms.
- The field theory limit of disk relation give the amplitudes in

minimal coupling theory of gauge field and gravity. This is based on the disk structure in string theory.

- Does KLT factorization relation hold on T_2?

- The disk relation may be used to study the ultraviolet properties of gravity.

- What is the relation between the actions of gauge field and gravity?