Some reflections on holographic descriptions of Schwarzschild black holes

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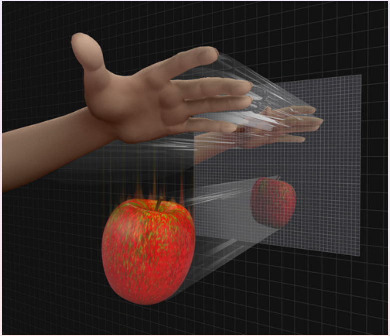
Plan to Talk

- Getting Started
  - A Naive Observation
  - Power Counting
  - The Large-N Limit
  - Microcanonical Systems
- System Requirements
  - Finite Size Effects
  - York's Isothermal Cavity
  - Thermodynamics
- Installation (?)
  - A Bizarre Speculation
  - Thermostatistics
- ~~Troubleshootings~~

Getting Started

- Holography

  $d+2$ dim gravity in bulk $\Leftrightarrow$ $d+1$ dim "matter" fields on boundary



- The boundary is sometimes called a "holographic screen"
- Example: $AdS_{d+2}/\mathrm{CFT}_{d+1}$

  $$AdS \text{ black holes in } d+2$$
  $$\Leftrightarrow \quad \mathrm{CFT}_{d+1} \text{ at finite temperature}$$

- Is there any holographic description of gravity in asymptotically dS or flat background? dS/CFT, matrix black holes, entropic forces $\cdots$

- Boundary thermodynamics $\longrightarrow$ bulk gravity (black holes...)

- A general question: When does thermodynamics apply to small systems?

  - What happens if a holographic screen becomes "tiny"? — may be related to "microscopic properties" of gravity

♣ A Naive Observation

- Black holes in AdS_{d+2}, with flat boundary $\mathbb{R}^1 \times \mathbb{R}^d$
  - Metric

$$ds^2 = L^2 \left[-r^2 f(r) dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2 f(r)} \right], \quad f(r) = 1 - \frac{r_+^2}{r^2}$$

  - Hawking temperature

$$\hat{\kappa} = \frac{d+1}{2} r^+ \quad \Rightarrow \quad \beta_H = \frac{2\pi}{\hat{\kappa}} \quad \Rightarrow \quad T_H = \frac{d+1}{4\pi} r_+$$

  - Thermodynamical relations

$$S \sim V_d T_H^d, \quad E \sim V_d T_H^{(d+1)}, \quad S \sim V_d^{1/(d+1)} E^{d/(d+1)}$$

- Heat capacity is positive

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = d \cdot S = (d+1)\frac{E}{T_H} > 0$$

- Thermodynamic stability: system + environment in equilibrium

$$\text{fluctuations}: \quad \text{environment} \quad \xrightarrow{\delta Q > 0} \quad \text{system}$$

$$\delta T \sim \delta Q / C_V > 0$$

- Boundary CFT_{d+1} has scaling invariance under

$$t \to \lambda t, \quad x^i \to \lambda x^i; \quad 1 \le i \le d$$

- If $T = 0$, no characteristic scales in the boundary theory
- Finite temperature $t \sim t + i\beta$ sets up a natural length scale, so a physical quantity of dimension L^α should scale as $T^{-\alpha}$

$$[\mathcal{X}] = L^\alpha \quad \Rightarrow \quad \mathcal{X} \sim T^{-\alpha}$$

- Putting the system into a box of volume $R^d \Rightarrow \mathcal{X} = T^{-\alpha} f(RT)$
- For extensive variables

$$f(RT) = c \cdot (RT)^d \Rightarrow \mathcal{X} = c \cdot V_d T^{d-\alpha}$$

- Applying to entropy and energy $\rightarrow$ reproduce the thermodynamics of *AdS* black holes

$$S = c_1 V_d T^d, \quad E = c_2 V_d T^{d+1} \Rightarrow S \sim V_d^{1/(d+1)} E^{d/(d+1)}$$

- The first law of thermodynamics

$$dE = TdS \Rightarrow c_2 = c_1 \cdot \frac{d}{d+1} \Rightarrow S = \frac{d+1}{d} \frac{E}{T}$$

- Schwarzschild Black Holes

$$ds^2 = -\left(1 - \frac{w_{d+1} M}{r^{d-1}}\right) dt^2 + \left(1 - \frac{w_{d+1} M}{r^{d-1}}\right)^{-1} dr^2 + r^2 d\Omega_d^2$$

$$w_{d+1} \equiv \frac{16\pi G_{d+2}}{d \cdot \mathrm{Vol}(S^d)}, \quad \mathrm{Vol}(S^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}$$

- Horizon and ADM energy

$$r_H = (w_{d+1} M)^{\frac{1}{d-1}}\,, \quad E = M$$

- Surface gravity and temperature

$$\hat{\kappa} = \frac{d-1}{2r_H}, \quad \beta_H = \frac{2\pi}{\hat{\kappa}} \quad \Rightarrow \quad T_H = \frac{d-1}{4\pi r_H}$$

- Dependence of energy on temperature

$$E = \frac{r_H^{d-1}}{w_{d+1}} = \frac{d \cdot \mathrm{Vol}(S^d)}{16\pi G_{d+2}} \left(\frac{d-1}{4\pi}\right)^{d-1} T_H^{-d+1}$$

- Bekenstein-Hawking entropy

$$S = \frac{\mathcal{A}_d}{4G_{d+2}} = \frac{\mathrm{Vol}(S^d)}{4G_{d+2}} r_H^d = \frac{\mathrm{Vol}(S^d)}{4G_{d+2}} \left(\frac{d-1}{4\pi}\right)^{d} T_H^{-d}$$

- Relation between entropy and energy

$$S \sim V_d^{-\frac{1}{d-1}} E^{\frac{d}{d-1}}, \quad V_d \propto \mathrm{Vol}(S^d)$$

- Heat capacity is negative, indicating thermodynamic instability

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = -d \cdot S = -(d-1)\frac{E}{T_H} < 0$$

- At a fixed point of RG flows (e.g. IR fixed point), there are models exhibiting anisotropic scaling behavior (Lifshitz Fixed Point)

$$t \to \lambda^z t, \quad x^i \to \lambda x^i; \quad 1 \le i \le d$$

- Dimensions

$$[\mathbf{x}] = L, \quad [\mathbf{k}] = L^{-1}, \quad [t] = L^z, \quad [\omega] = [E] = [T] = L^{-z}, \quad [c] = L^{1-z}$$

- Toy model action ($N = 1$, free theory)

$$S_0 = \int dt\, d^d\mathbf{x} \left[\dot{\phi}^2 - \phi(-\boldsymbol{\nabla}^2)^z \phi\right]$$

- Canonical dim: $1 = [S_0] = L^{z+d} \cdot L^{-2z} \cdot [\phi]^2 \;\Rightarrow\; [\phi] = L^{(z-d)/2}$

- When $T = 0$, scale invariance $\Rightarrow$ no characteristic scales in the theory
- At finite temperature, $t \sim t + i\beta$, $T = 1/\beta$

$$[T^{-1/z}] = [\beta^{1/z}] = L$$

- A physical quantity $\mathcal{X}$ of dimension $[\mathcal{X}] = L^{\alpha}$ should scale as $\mathcal{X} \sim T^{-\alpha/z}$
- If the system is put into a box with spacial volume R^d, then

$$\mathcal{X} = T^{-\alpha/z} f(RT^{1/z})$$

- For extensive variables

$$f(RT^{1/z}) = c \cdot R^d T^{d/z} \;\Rightarrow\; \mathcal{X} = c \cdot V_d T^{\frac{d-\alpha}{z}}$$

- Applying to entropy and energy ($\alpha = 0, -z$, respectively)

$$S = c_1 V_d T^{d/z}, \;\; E = c_2 V_d T^{(d+z)/z} \;\Rightarrow\; S \sim V_d^{z/(d+z)} E^{d/(d+z)}$$

- The first law of thermodynamics

$$dE = TdS \;\Rightarrow\; c_2 = c_1 \cdot \frac{d}{d+z} \;\Rightarrow\; S = \frac{d+z}{d}\frac{E}{T}$$

- Heat capacity

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = \frac{d}{z}S = \frac{d+z}{z}\frac{E}{T}$$

- $C_V < 0$ iff the critical exponent z is "unphysical": $-d < z < 0$
- Thermodynamics at $z = -1$ $\Rightarrow$ Schwarzschild black holes?
- Adding a mass term $-m^2\phi^2$ and a K.T. $-c^2\phi(-\nabla^2)\phi$ to the free action, one gets

$$\omega_{\mathbf{k}} = \sqrt{m^2 + c^2\mathbf{k}^2 + (\mathbf{k}^2)^z}$$

- For $z = -1$, the dispersion relation in the IR region $|\mathbf{k}| \sim 0$ looks somewhat "stange" $\omega_{\mathbf{k}} \sim 1/|\mathbf{k}|$

cf. stretched membrane [Miao, hep-th/0311105]

- If $[\mathbf{x}] = [t] = L$, we have to insert a dimensionful parameter $\zeta \sim m_p^{-2(z-1)}$ into the action, so that $[\dot{\phi}^2] = [\zeta \cdot \phi(-\nabla^2)^z \phi]$

- For $z > 1$, Kachru, Liu and Mulligan proposed a "gravity duals of Lifshitz-like fixed points" `arXiv:0808.1725 [hep-th]`
- Thermodynamics → Taylor, `arXiv:0812.0530` [hep-th]

$$ds^2 = L^2\left[-r^{2z}f(r)dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2 f(r)}\right], \quad f(r) \equiv 1 - \left(\frac{r_+}{r}\right)^{d+z}$$

$$\hat{\kappa} = \frac{d+z}{2} r_+^z, \quad \beta_H = \frac{2\pi}{\hat{\kappa}} \;\Rightarrow\; T_H = \frac{1}{\beta_H} = \frac{d+z}{4\pi} r_+^z$$

$$I_E = -\frac{L^d V_d}{16\pi G_{d+2}} r_+^{d+z}\beta_H = -\frac{(4\pi)^{d/z} L^d V_d}{4(d+z)^{(d+z)/z} G_{d+2}} \beta_H^{-d/z}$$

$$E = \frac{\partial I_E}{\partial \beta_H} = \frac{L^d d}{16\pi G_{d+2}} \left(\frac{4\pi}{d+z}\right)^{(d+z)/z} V_d T_H^{(d+z)/z}$$

$$S = \beta_H E - I_E = \frac{L^d}{4G_{d+2}} \left(\frac{4\pi}{d+z}\right)^{d/z} V_d T_H^{d/z} \sim V_d^{z/(d+z)} E^{d/(d+z)}$$

♣ Power Counting

- Nontrivial dynamics may come from interacting terms, e.g.

$$S_I = \int dt\, d^d\mathbf{x} \sum_{n\geq 3} g_n \phi^n$$

- Dimensions of the couplings

$$1 = [S_I] = L^{z+d} \cdot [g_n] \cdot L^{n(z-d)/2} \;\Rightarrow\; [g_n] = L^{-(z+d)-n(z-d)/2}$$

- Conditions for perturbatively renormalizable interactions $g_n\phi^n$:

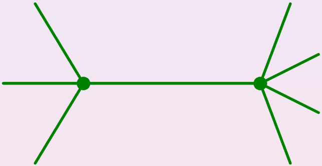
$$z + d + \frac{n(z-d)}{2} \geq 0$$

  - If $z \geq d$, the inequality holds for any integer $n \geq 0$, in this case all polynomial interactions are renormalizable
  - If $z < d$, renormalizablity imposes an upper bound on n

$$n \leq \frac{2(d+z)}{d-z} \;\Rightarrow\; n_{\max} = \frac{2(d+z)}{d-z} = 2 + \frac{4z}{d-z}$$

- When $z = -1$, $n_{\max} < 2$, no renormalizable interactions allowed?
  - For an ℓ-loop Feynman diagram with V_n n-valence vertices, together with I internal lines and E external edges:

$$V = \sum_{n \geq 3} V_n, \quad \sum_{n \geq 3} n V_n = 2I + E, \quad \ell - I + V = 1$$



- $n = 4,\ 5;\quad I = 1,\ E = 7$
- $V_4 = V_5 = 1 \ \Rightarrow\ V = 2$
- $\sum_n n V_n = 4 \cdot 1 + 5 \cdot 1 = 9$
- $2I + E = 9$

$$\mathcal{A} \sim \int \prod_{a=1}^{\ell} d\omega_a\, d^d\mathbf{k}_a \prod_{i=1}^{I} G(\omega_i, \mathbf{k}_i) \prod_{n \geq 3} \frac{g_n^{V_n}}{n!}$$

$$G(\omega, \mathbf{k}) \sim \frac{1}{\omega^2 - [m^2 + c^2\mathbf{k}^2 + (\mathbf{k}^2)^z]}, \quad [G] = L^{2z}$$

- If $z > 1$, the UV behavior of the propagator is dominated by

$$G(\omega, \mathbf{k}) \sim \frac{1}{\omega^2 - (\mathbf{k}^2)^z}$$

  - Degree of divergence δ:

$$\left[\prod_{a=1}^{\ell} d\omega_a d^d\mathbf{k}_a \prod_{i=1}^{I} G(\omega_a, \mathbf{k}_a)\right] = L^{-\delta} \begin{cases} \delta > 0 & \text{superficially divergent} \\ \delta = 0 & \text{possibly logarithmic} \\ \delta < 0 & \text{convergent} \end{cases}$$

  - Dimensional analysis gives (Visser, `arXiv:0902.0590 [hep-th]`)

$$\delta = (z + d)\ell - 2zI = \ell d - (I + V - 1)z$$

  - $I + V - 1$ always positive, so $z \uparrow \;\Rightarrow\; \delta \downarrow$
  - In particular if $z \geq d$, then

$$\delta \leq \ell d - (I + V - 1)d = -2(V - 1)d \leq 0 \quad \rightarrow \text{renormalizable for any } n$$

- For $z < 1$ (e.g. $z = -1$), $G(\omega, \mathbf{k}) \sim 1/(\omega^2 - c^2\mathbf{k}^2)$ at UV

♣ The Large-N Limit

- Motivated by the above, the model would consist of
  - A scalar field $\Phi(x)$ in $d+1$ dimensions, with N components

$$\Phi = (\phi_1, \phi_2, \cdots, \phi_N), \quad \Phi^2 \equiv \sum_{i=1}^{N} \phi_i^2 \ \rightarrow \ O(N)\text{-invariant variable}$$

  - A potential $V(\Phi)$ with $O(N)$ symmetry
  - Kinetic terms (with $z=-1$, thus quasi-local)

$$\mathcal{L}_0 = \frac{1}{2}\left[(\partial_t \Phi)^2 - c^2(\boldsymbol{\nabla}\Phi)^2 - m^2\Phi^2 - \zeta \cdot \Phi(-\boldsymbol{\nabla}^2)^z \Phi\right]$$

- Dimensions reset to $[t] = [\mathbf{x}] = L$; $[c] = 1$, $[\zeta] = L^{2(z-1)} \xrightarrow{z=-1} L^{-4}$
- Writing $V(\Phi) = NU(\Phi^2)$, the degree of U lowered; e.g.

$$\deg V = 4 \quad \Rightarrow \quad \deg U = 2$$

- Introduce two Lagrange multipliers $\lambda(x)$ and $\rho(x)$

$$\exp\left[i\int d^{d+1}x\, V(\Phi)\right] \propto \int [d\lambda][d\rho] \exp\left\{iN\int d^{d+1}x \left[\frac{1}{2}\lambda(\Phi^2 - \rho) + U(\rho)\right]\right\}$$

- In ϕ^4-theory, $\deg U = 2$, integration over ρ is gaussian and can be performed; this will result in "Hubbard-Stratonovich transformation" [Hubbard, Phys. Rev. Lett. 3 (1959) 77]
- The effective Lagrangian is a quardratic form in Φ

$$\mathcal{L} = -\frac{1}{2}\left[(\partial_\mu \Phi)^2 + (m^2+\lambda)\Phi^2 + \zeta\cdot\Phi(-\nabla^2)^z\Phi\right] + \frac{N}{2}\lambda\rho - NU(\rho)$$

$$\Rightarrow \quad S_{\rm eff} = N\int d^{d+1}x\left[\frac{1}{2}\lambda\rho - U(\rho)\right] + \frac{N}{2}\mathrm{Tr}\log\left[-\partial_\mu^2 + \zeta(-\nabla^2)^z + m^2 + \lambda\right]$$

- N plays the role of $1/\hbar$, so taking the large-N limit leads to a classical theory for λ, ρ

$$S_{\rm eff} = N\int d^{d+1}x\,\mathcal{L}_{\rm eff} \xrightarrow{\text{Wick rotation}} Ni\int_0^\beta dt\int d^d\mathbf{x}\,\mathcal{L}_{\rm eff}$$

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\lambda\rho - U(\rho) + \frac{1}{2}\int\frac{d\omega d^d\mathbf{k}}{(2\pi)^{d+1}}\log\left[-\omega^2 + c^2\mathbf{k}^2 + \zeta(\mathbf{k}^2)^z + m^2 + \lambda\right]$$

$$\omega_n \sim \frac{2\pi n}{i\beta} \;\Rightarrow\; \int\frac{d\omega}{2\pi}\mathcal{F}(-\omega^2,\cdots) \to \frac{1}{i\beta}\sum_{n\in\mathbb{Z}}\mathcal{F}\left(\frac{4\pi^2n^2}{\beta^2},\cdots\right) \quad \text{Matsubara frequencies}$$

- Classical equations of motion $\Leftrightarrow$ saddle point equations

$$\frac{1}{2}\lambda = U'(\rho), \quad \rho = \frac{1}{(2\pi)^{d+1}}\int \frac{d\omega d^d\mathbf{k}}{[-\omega^2 + c^2\mathbf{k}^2 + \zeta(\mathbf{k}^2)^z + m^2 + \lambda]}$$

$$\Rightarrow \quad \rho = \frac{1}{(2\pi)^{d+1}}\int \frac{d\omega d^d\mathbf{k}}{[-\omega^2 + c^2\mathbf{k}^2 + \zeta(\mathbf{k}^2)^z + m^2 + 2U'(\rho)]}$$

- Working out the saddle point λ_0, ρ_0 and insetting them back into $S_{\rm eff}$, one gets the free energy

$$F = N\frac{R^d}{T}f(T, c, \zeta, z, m^2) + \frac{1}{N}\text{-corrections} \qquad (N \propto \frac{1}{G_{d+2}}??)$$

$$f(T, c, \zeta, z, m^2) = \rho_0 U'(\rho_0) - U(\rho_0) + f_0(T, c, \zeta, z, m^2 + \lambda_0)$$

$$f_0(T, c, \zeta, z, u) = \frac{T}{2}\sum_n \int \frac{d^d\mathbf{k}}{(2\pi)^d} \log\left[(2\pi n T)^2 + c^2\mathbf{k}^2 + \zeta(\mathbf{k}^2)^z + u\right]$$

- In free theory, the potential vanishes $U \equiv 0 \Rightarrow \lambda_0 = 2U'(\rho_0) = 0$, hence the free energy becomes

$$F = N\frac{R^d}{T}f_0(T, c, \zeta, z, m^2)$$

- For N large, the $O(N)$ invariant quantities self-average and have small fluctuations (central limit theorem), e.g.

$$\langle\Phi^2(x)\Phi^2(y)\rangle \sim \langle\Phi^2(x)\rangle\langle\Phi^2(y)\rangle + \text{terms suppressed by } N^{-1}$$

- Thus large-N limit is essentially a mean-field theory [Zinn-Justin, QFT & critical phenomena, 1996]
- $1/N$ corrections to the critical exponent z could be computed see e.g. Shpot, Pis'mak and Diehl, cond-mat/0412405, arXiv:0802.2434

$$z \to z + \frac{z^{(1)}}{N} + O(N^{-2})$$

- This would give black hole entropy a logarithmic correction

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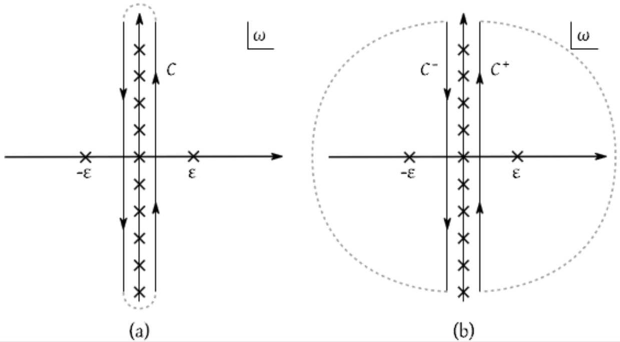
- A Little Computaion

$$\frac{\partial f_0}{\partial u} = \frac{T}{2}\sum_n \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{[(2\pi n T)^2 + c^2\mathbf{k}^2 + \zeta(\mathbf{k}^2)^z + u]}$$

$$\equiv \frac{T}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_n \frac{1}{(2\pi n T)^2 + \varepsilon^2(\mathbf{k}, u)} \quad \left(\varepsilon = \sqrt{c^2\mathbf{k}^2 + \zeta(\mathbf{k}^2)^z + u}\right)$$

- The sum has a contour integral representation

$$\sum_n \frac{1}{(2\pi n T)^2+\varepsilon^2} = -\oint_{\mathcal{C}} \frac{d\omega}{2\pi i}\frac{\varphi(\omega)}{\omega^2-\varepsilon^2} = \frac{1}{2\varepsilon}\left[\varphi(\varepsilon)-\varphi(-\varepsilon)\right] = \frac{\coth\left(\frac{\varepsilon}{2T}\right)}{2\varepsilon T}$$



$$\Rightarrow f_0(T,c,\zeta,z,u) = \frac{1}{2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\left[\sqrt{c^2\mathbf{k}^2+\zeta(\mathbf{k}^2)^z+u}\right.$$
$$\left. + 2T\log\left(1-e^{-\sqrt{c^2\mathbf{k}^2+\zeta(\mathbf{k}^2)^z+u}/T}\right)\right]$$

- For free field theories, $u = m^2$ does not depend on T, in this case the first term is nothing but the divergent vacuum energy density of $T = 0$ QFTs (recall $F = NR^d f_0/T$, $E = \partial F/\partial\beta$)
- The second term gives the finite-temperature contributions to the free energy

$$F(\beta) = NR^d \int \frac{d^d\mathbf{k}}{(2\pi)^d} \log\left(1 - e^{-\beta\sqrt{c^2\mathbf{k}^2+\zeta(\mathbf{k}^2)^z+m^2}}\right)$$

- Applying to the Lifshitz fixed point: $c = m = 0$

$$\begin{aligned} F(\beta) &= NR^d\,\mathrm{Vol}\left(S^{d-1}\right)\int_0^\infty \frac{dk}{(2\pi)^d} k^{d-1}\log\left(1 - e^{-\beta\sqrt{\zeta}k^z}\right) \\ &= NR^d\,\mathrm{Vol}\left(S^{d-1}\right)\left(\beta\sqrt{\zeta}\right)^{-\frac{d}{z}}\int_0^\infty \frac{ds}{(2\pi)^d} s^{d-1}\log\left(1 - e^{-s^z}\right) \end{aligned}$$

- The integral is convergent for $z > 0$, giving rise to a negative constant, denoted by $-I_d(z)$

$$I_d(d) = \frac{1}{(2\pi)^d}\frac{\pi^2}{6},\quad I_2(1) = \frac{\zeta(3)}{4\pi^2},\quad I_3(2) = \frac{\zeta(5/2)}{32\pi^{5/2}}\cdots$$

- Thermodynamic quantities have the expected scaling behavior

$$F = -\frac{N I_d(z) \mathrm{Vol}\left(S^{d-1}\right)}{\zeta^{d/2z}} R^d T^{d/z}$$

$$E = \frac{\partial}{\partial \beta} F(\beta) = \frac{d}{z} \cdot \frac{N I_d(z) \mathrm{Vol}\left(S^{d-1}\right)}{\zeta^{d/2z}} R^d T^{(d+z)/z}$$

$$S = \beta E - F = \frac{d+z}{z} \cdot \frac{N I_d(z) \mathrm{Vol}\left(S^{d-1}\right)}{\zeta^{d/2z}} R^d T^{d/z} = \frac{d+z}{d} \frac{E}{T}$$

- For $z < 0$, we find divergence at $s \equiv (\beta\sqrt{\zeta})^{1/z} k \sim \infty$
- One has to add an UV regulator $c^2 \mathbf{k}^2$ to the integral $\Rightarrow$ ~~scaling~~
- In particular, taking $z = -1$, $m = 0$ and $c = 1$

$$F(\beta) = N R^d \, \mathrm{Vol}\left(S^{d-1}\right) \int_0^\infty \frac{dk}{(2\pi)^d} k^{d-1} \log\left(1 - e^{-\beta\sqrt{k^2 + \frac{\zeta}{k^2}}}\right)$$

$$F'(\beta) = N R^d \, \mathrm{Vol}\left(S^{d-1}\right) \int_0^\infty \frac{dk\, k^{d-1}}{(2\pi)^d} \frac{\sqrt{k^2 + \zeta k^{-2}}}{e^{\beta\sqrt{k^2+\zeta k^{-2}}} - 1}$$

$$F''(\beta) = -N R^d \, \mathrm{Vol}\left(S^{d-1}\right) \int_0^\infty \frac{dk\, k^{d-1}}{(2\pi)^d} \frac{\left(k^2 + \zeta k^{-2}\right) e^{\beta\sqrt{k^2+\zeta k^{-2}}}}{(e^{\beta\sqrt{k^2+\zeta k^{-2}}} - 1)^2}$$

- The heat capacity is positive

$$C_V = \frac{\partial E}{\partial T} = -\beta^2 F''(\beta) > 0$$

- In a canonical ensemble, $C_V > 0$ even before thermodynamic limit is taken:

$$F(\beta) = -\log Z(\beta) \;\Rightarrow\; F''(\beta) = \frac{Z'(\beta)^2}{Z^2} - \frac{Z''(\beta)}{Z} = -\langle (\Delta E)^2 \rangle < 0$$

- The failure of reproducing thermal relations of Schwarzschild black holes is, of course, expectable:
  - Black holes (in flat spacetime) with $C_V < 0$ are not in thermal equilibrium with radiation, they can't be described by thermal stable states in any boundary theories
  - Gross-Perry-Yaffe instability: there are no ways of creating a translationally invariant state with finite energy density; hot flat space is unstable [Phys. Rev. D 25 (1982) 330]
- One could use microcanonical ensembles [Ann. Phys. 146, 419 (1983)]

♣ Microcanonical Systems

$$\omega(E) = \frac{1}{C}\int d^N\mathbf{q}d^N\mathbf{p}\,\delta(E - H(\mathbf{q},\mathbf{p})), \quad \Omega(E) = \int_{\tilde{E}\le E} d\tilde{E}\,\omega(\tilde{E}), \quad S = \log\Omega(E)$$

$$\langle A\rangle = \frac{1}{\omega(E)C}\int d^N\mathbf{q}d^N\mathbf{p}\,A(\mathbf{q},\mathbf{p})\,\delta(E - H(\mathbf{q},\mathbf{p})), \quad \frac{1}{T} = \left.\frac{\partial S}{\partial E}\right|_V, \quad \frac{1}{C_V} = \left.\frac{\partial T}{\partial E}\right|_V$$

$$\Rightarrow T = \frac{2\langle K\rangle}{3N}, \quad \frac{C_V}{N} = \left[\frac{2}{3} - N\left(\frac{\langle K^2\rangle - \langle K\rangle^2}{\langle K\rangle^2}\right)\right]^{-1}$$

- In most cases, $C_V < 0$ arises in "small" systems; under large volume limit $V \longrightarrow \infty$, the microcanonical and the (grand-) canonical ensembles are usually equivalent ["small" means the size L comparable to the range of interactions, cf. van Hove theorem]
- Two subsystems at the same microcanonical temperature:

$$\begin{aligned} S_{\text{tot}} &= S_1(E_1+\epsilon) + S_2(E_2-\epsilon) \\ &= S_1(E_1) + S_2(E_2) + \cancel{\left(\frac{1}{T} - \frac{1}{T}\right)}\epsilon - \frac{1}{2}\left(\frac{\epsilon}{T}\right)^2\left[\frac{1}{C_1} + \frac{1}{C_2}\right] \end{aligned}$$

- (Meta)stability corresponds to (local) maximum of the total entropy $\Rightarrow$

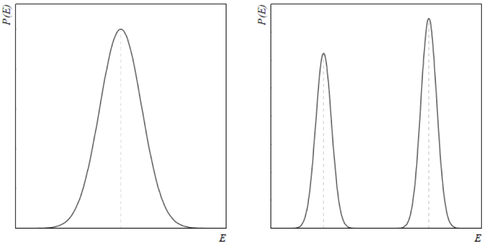
$$\frac{1}{C_1}+\frac{1}{C_2}>0 \begin{cases} C_1>0, \quad C_2>0; & \text{total system stable} \\ C_1<0, \quad C_2<0; & \text{unstable, runaway} \\ C_1>0, \quad C_2<0; & \text{depends on} \begin{cases} C_1+C_2<0 & \text{stable} \\ C_1+C_2>0 & \text{unstable} \end{cases} \end{cases}$$

- Canonical partition is a Laplace transform of microcanonical entropy

$$Z(\beta, F, \cdots) = \int_0^{\infty} dE\, dx \cdots e^{S(E)-\beta(E+Fx+\cdots)}$$

- In Verlinde's approach (arXiv:1001.078, [hep-th]), thermodynamic force F in a boundary system (defined on certain holographic screen) is interpreted as the bulk gravitational force; thermodynamic displacement x conjugate to F plays the role of an emergent bulk space coordinate

- During a first order phase transition at some $T = T_c$, there is an amount of energy (latent heat) E_ℓ released or absorbed by the system without changing temperature



The distribution

$$P(E) \sim \exp[S(E) - \beta E]$$

has separated peaks at E_1 and $E_2 = E_1 + E_\ell$, corresponding to two phases (say, liquid and gas)

- $P(E)$ is smooth microcanonically before taking $V \to \infty$
- Between the two pure phases, there must be a minimum of $\log P(E) = S(E) - \beta E$; in a neighborhood of this minimum

$$\frac{\partial^2}{\partial E^2} \log P(E) > 0 \;\Rightarrow\; \frac{\partial^2 S}{\partial E^2} > 0 \;\Rightarrow\; \frac{1}{C_V} = -T^2 \frac{\partial^2 S}{\partial E^2} < 0$$

- $\exists$ negative heat capacity is a generic signal of phase separation

- A toy model [due to Hüller, Z. Phys. B 95 (1994) 63]

$$S_1(E,N) = V \cdot s_V(\epsilon) + V^{\frac{d-1}{d}} \cdot s_{\partial V}(\epsilon), \quad \epsilon \equiv \frac{E}{V}, \;\; s(\epsilon) \equiv \frac{S(E,V)}{V}$$

$$s_V(\epsilon) = \beta_c \epsilon - \begin{cases} 0, & \text{if } -\epsilon_\ell < \epsilon < \epsilon_\ell \\ \alpha_4 \left(|\epsilon| - \epsilon_\ell\right)^4, & \text{if } |\epsilon| \geq \epsilon_\ell \end{cases}$$

$$s_{\partial V}(\epsilon) = -\alpha \cos \frac{\pi \epsilon}{\epsilon_\ell} \quad \rightarrow S_1 \text{ not extensive: } S_1(\lambda E, \lambda V) \neq \lambda S_1(E,V)$$

- The bulk specific entropy $s_V(\epsilon)$ obeys van Hove's condition

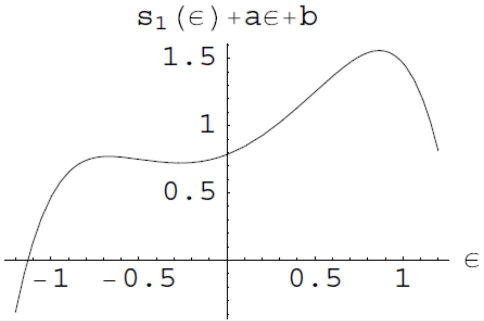
$$\frac{\partial^2 s_V}{\partial \epsilon^2} \leq 0, \;\; \forall \epsilon \;\Rightarrow\; (C_V)_{\text{bulk}} \geq 0$$

- In energy range $-\epsilon_\ell < \epsilon < \epsilon_\ell$, Hove's condition violated for the total specific entropy at finite V

$$\frac{\partial^2 s_1}{\partial \epsilon^2} > 0 \;\Rightarrow\; C_V < 0$$

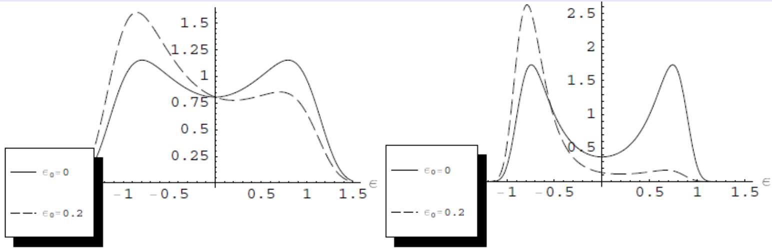
- Surface contribution disappears as $V \rightarrow \infty$

- The entropy becomes an extensive quantity in the thermodynamic limit
- The minimum ϵ_0 of $P(E) \propto e^{S_1(E)-\beta E}$ separating two peaks depends on β



- The critical value $\beta = \beta_c$ corresponds to $\epsilon_0 = 0$; when β is slightly larger than β_c, $\epsilon_0 > 0$

- At the critical temperature, two pure phases appear, located at the peaks of $P(E)$ with the same probability of occurrence

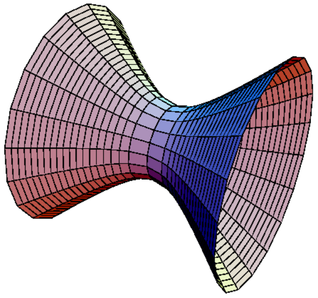


- Below critical temperature, the pure phase with smaller E is more stable than the other and forms the dominant phase
- When V increased (e.g. from $V \sim 10^2$ to $V \sim 10^3$), width of peaks become narrower, the non-dominant phase is much less important; in the thermodynamic limit $V \to \infty$, only the dominant phase remains at $\beta \neq \beta_c$
- At $\beta = \beta_c$, $P(E) \propto \delta(\epsilon + \epsilon_\ell) + \delta(\epsilon - \epsilon_\ell)$

System Requirements

♣ Finite Size Effects

- Finite size is essential to get a well-defined $C_V < 0$ in some energy region [When $V = \infty$, C_V diverse at the critical point]
- Example: *AdS* has a confining potential



- When black holes are "small" ($\lesssim L$), negative heat capacity may appear in thermodynamics
- $C_V < 0$ should be a signal of the existence of a certain first order phase transition (Hawking-Page)
- The physics on the CFT side is known pretty well by now

- r_+: the largest real root of $F(r) \equiv 1 - \frac{w_{d+1}M}{r^{d-1}} + \frac{r^2}{L^2} = 0$

- Surface gravity and Hawking temperature

$$\hat{\kappa} = \frac{1}{2}F'(r_+) = \frac{(d+1)r_+^2 + (d-1)L^2}{2L^2 r_+}, \quad \beta_H = \frac{\hat{\kappa}}{2\pi} = \frac{4\pi L^2 r_+}{(d+1)r_+^2 + (d-1)L^2}$$

- The Hawking temperature has a lower bound (for $d \geq 1$)

$$T_H = \frac{1}{4\pi L}\left[\frac{(d+1)r_+}{L} + \frac{(d-1)L}{r_+}\right] \geq \frac{\sqrt{d^2-1}}{2\pi L} \equiv T_{\rm min}$$

- Associated to each $T_H \geq T_{\rm min}$ there are two black holes

$$r_+ = \frac{2\pi L^2}{(d+1)}\left[T_H \pm \sqrt{T_H^2 - T_{\rm min}^2}\right] \quad \begin{cases} + : & \text{large black hole} \\ - : & \text{small black hole} \end{cases}$$

- At the minimal temperature, large and small black holes have the same size $r_+ = L\sqrt{(d-1)/(d+1)}$
- Large black hole is heavier: $r_+ \uparrow$, the ADM energy $E = M$ is monotonely increasing

- Heat capacity is computed by

$$
\begin{aligned}
C_V &= \frac{\partial E}{\partial T_H} = \frac{\partial E}{\partial r_+}\frac{\partial r_+}{\partial T_H} \\
&= \frac{\mathrm{Vol}\,(S^d)(r_+^{d-1} d)\,T_H}{4G_{d+2}} \cdot \frac{2\pi L^2}{(d+1)}\left[1 \pm \frac{T_H}{\sqrt{T_H^2 - T_{\min}^2}}\right] \\
&\Rightarrow \begin{cases} C_V > 0 & \text{for large black holes} \\ C_V < 0 & \text{for small black holes} \end{cases}
\end{aligned}
$$

- Since $C_V^{\mathrm{large}} + C_V^{\mathrm{small}} > 0$, small black holes cannot be in thermal equilibrium with the large ones, they will decay either to large black holes or to pure thermal *AdS*
- Bekenstein-Hawking entropy

$$
\begin{aligned}
S &= \frac{A}{4G_{d+2}} = \frac{\mathrm{Vol}\,(S^d) r_+^d}{4G_{d+2}} \\
&= \frac{\mathrm{Vol}\,(S^d)}{4G_{d+2}}\left(\frac{2\pi L^2}{d+1}\right)^d \left[T_H \pm \sqrt{T_H^2 - T_{\min}^2}\right]^d
\end{aligned}
$$

- Scaling behavior $S \propto T_H^d$ is violated by the finite size effect
- In holographic dual, S is not strictly extensive, at least in the strong coupling region
- According to Gibbons-Hawking, free energy F of AdS black holes can be computed by Euclidean Einstein-Hilbert action evaluated at the black hole solution
- One may compare this F to the free energy F_0 of the pure thermal AdS [Witten, hep-th/9803131]
- Regulating Euclidean actions by a large cavity of radius R; the Tolman (or local) temperatures at $r = R$ should be the same

$$\frac{T_H}{\sqrt{1 - \frac{w_{d+1} M}{R^{d-1}} + \frac{R^2}{L^2}}} = \frac{T_0}{\sqrt{1 + \frac{R^2}{L^2}}}$$

$$\Rightarrow \quad \beta_0 = \beta_H \sqrt{1 - \frac{L^2 r_+^{d-1} + r_+^{d+1}}{L^2 R^{d-1} + R^{d+1}}} = \beta_H \left[1 - \frac{L^2 r_+^{d-1} + r_+^{d+1}}{2R^{d+1}} + \cdots \right]$$

- The regulated spacetime volumes w/ and w/o black holes are

$$V(R) = \mathrm{Vol}\,(S^d) \int_0^{\beta_H} dt \int_{r_+}^{R} dr\, r^d = \frac{\mathrm{Vol}\,(S^d)}{d+1} \left(R^{d+1} - r_+^{d+1}\right) \beta_H$$

$$V_0(R) = \mathrm{Vol}\,(S^d) \int_0^{\beta_0} dt \int_0^{R} dr\, r^d = \frac{\mathrm{Vol}\,(S^d)}{d+1} R^{d+1} \beta_0$$

$$= \frac{\mathrm{Vol}\,(S^d)}{d+1} \left[R^{d+1} - \frac{L^2 r_+^{d-1} + r_+^{d+1}}{2} + \mathrm{O}(R^{-1}) \right] \beta_H$$

- Difference between the free energies

$$F - F_0 = \frac{d+1}{8\pi G_{d+2} L^2} \lim_{R\to\infty} [V(R) - V_0(R)]$$

$$= \frac{\mathrm{Vol}\,(S^d)}{4 G_{d+2}} \frac{(L^2 - r_+^2) r_+^d}{(d-1)L^2 + (d+1) r_+^2}$$

- Hawking-Page phase transition occurred at $r_+ = L$

$$\begin{cases} r_+ > L, & F < F_0, \quad \text{black hole phase dominant} \\ r_+ < L, & F_0 < F, \quad \text{pure thermal } AdS \text{ dominant} \end{cases}$$

- The corresponding critical temperature

$$T_c = \frac{1}{4\pi L}\left[\frac{(d+1)r_+}{L} + \frac{(d-1)L}{r_+}\right]_{r_+=L}$$
$$= \frac{d}{2\pi L} \quad \left(> T_{\min} = \frac{\sqrt{d^2-1}}{2\pi L}\right)$$

- Energy and entropy differences are computed by

$$E - E_0 = \frac{\partial}{\partial\beta}(F - F_0) = M, \quad S - S_0 = \frac{\mathrm{Vol}\,(S^d) r_+^d}{4G_{d+2}}$$

- The size of small black holes have an upper-bound

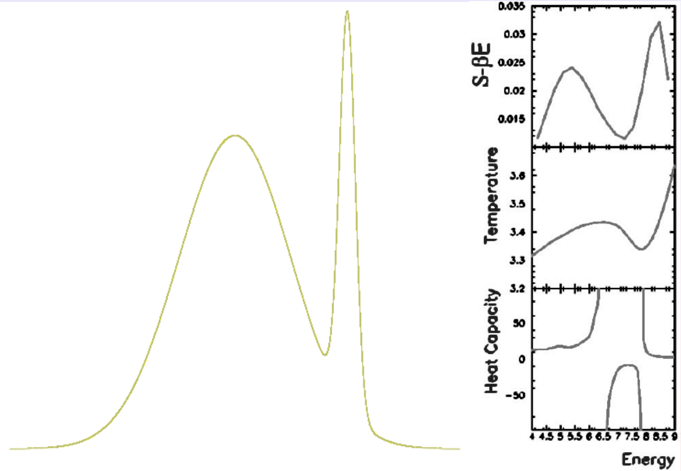
$$r_+ = \frac{2\pi L^2}{d+1}\left[\frac{T_{\min}^2}{T_H + \sqrt{T_H^2 - T_{\min}^2}}\right] \le \frac{2\pi L^2}{d+1} T_{\min} = L\sqrt{\frac{d-1}{d+1}} < L$$

$\Rightarrow \quad F - F_0 > 0 \;\Rightarrow$ small black holes never dominant over thermal AdS phase

- The phase transition between hot AdS and large black holes is first order: at transition point the latent heat M non-vanishing

- Behavior of the distribution function $P(E)$
  - $T_H \gg T_c$: A main peak located at $E \sim E_0 + M$, corresponding to the large black hole phase; other peaks (if any) are not significant
  - $T_H \gtrsim T_c$: two peaks appear, one of which located at $E \sim E_0$ (pure thermal AdS phase), the other located at $E \sim E_0 + M$ (large black hole phase), $P(E_0 + M) > P(E_0)$
  - $T_H = T_c$: two peaks have equal height $P(E_0 + M) = P(E_0)$
  - $T_H \lesssim T_c$: $P(E_0 + M) < P(E_0)$, the pure hot AdS phase is slightly dominant
  - $T_{\min} < T_H < T_c$: the dominant phase is pure thermal AdS
- The boundary CFT lives on $\mathbb{R} \times S^d$, finite size effects are not negligible $\longrightarrow$ insure the boundary CFT (e.g. $\mathcal{N} = 4$ SYM) undergoing a confinement/deconfinement phase transition against scale invariance; peaks described approximately by thermal stable states in boundary theory
- In the flat boundary $\mathbb{R}^1 \times \mathbb{R}^d$ limit, we will always be in the large black hole phase; Hawking-Page transition will never occur

- Small black holes have mass $\tilde{M} < M$, located at a minimum $E \sim E_0 + \tilde{M} \in [E_0, E + M]$ of $P(E)$, separating the two peaks



♣ York's Isothermal Cavity

- Schwarzschild black holes are similar to small *AdS* black holes
    - Both are not thermodynamic stable
    - Both have negative heat capacity
- The difference is that small *AdS* black holes separate two (meta) stable phases when $T_H \sim T_c$; in the Schwarzschild case there seems no such stable phases nearby [The fate of small black holes in AdS_5 is simply decaying to the (de)confinement phases of SYM; what about the fate of Schwarzschild black holes?]
- Following York [Phys Rev D33 2092 (1986)], consider $(d+2)$-dim Schwarzschild black holes confined to an isothermal cavity, and investigate its finite size effects [suppose $d > 1$]
- Let $L =$ radius of cavity, $T =$ uniform temperature on the wall; the size of this system is described by the geometric quantity $\mathrm{Vol}(S^d)L^d \equiv A$ (the invariant area of the wall)

- If there are no black holes in the cavity, we just get a thermal flat space (t, r, Ω_d), with $t \sim t + i\beta$, $\beta = 1/T$; the boundary of this spacetime (at $r = L$) is $S^1 \times S^d$ (Euclidean version)
- Putting a Schwarzschild black hole into the cavity, we have the Hawking temperature

$$T_H = \frac{d-1}{4\pi} \left(w_{d+1} M\right)^{-\frac{1}{(d-1)}}, \quad w_{d+1} \equiv \frac{16\pi G_{d+2}}{d \cdot \mathrm{Vol}\left(S^d\right)}$$

- Isothermal condition requires the temperature T on cavity's wall equal to the local Hawking temperature at $r = L$

$$\frac{T_H}{\sqrt{1 - \frac{w_{d+1} M}{L^{d-1}}}} = T \ \Rightarrow \ \left(w_{d+1} M\right)^{\frac{2}{d-1}} - \frac{\left(w_{d+1} M\right)^{\frac{d+1}{d-1}}}{L^{d-1}} = \left(\frac{d-1}{4\pi T}\right)^2$$

- Multiplying by L^{-2} to derive the "isothermal equation"

$$\xi^2 - \xi^{d+1} = \left(\frac{d-1}{4\pi L T}\right)^2, \qquad \xi \equiv \frac{\left(w_{d+1} M\right)^{\frac{1}{d-1}}}{L} = \left(\frac{w_{d+1} M}{L^{d-1}}\right)^{\frac{1}{d-1}}$$

- The existence of positive real roots $\xi = \xi_r > 0$ will force T to have a minimal value $T_{\min} > 0$; At temperature T below this bound, no black hole solutions with real mass $M = M(T) > 0$ allowed $\longrightarrow$ the cavity has to be in the thermal flat phase
  - Proof by contradiction: Take a set of temperatures arbitrarily close to zero; if the equation has a positive real root $\xi_r(T)$ for each T in such a set, then

$$\xi_r(T)^2 - \xi_r(T)^{d+1} \to +\infty \text{ as } T \to 0 \Rightarrow \begin{cases} \xi_r(T) \to +\infty \\ 1 - \xi_r(T)^{d-1} > 0 \end{cases} \text{impossible}$$

- In York's original work $d = 2$, the equation becomes cubic; a general cubic equation $\xi^3 + a\xi^2 + b\xi + c = 0$ has discriminant

$$\Delta = -4p^3 - 27q^2 \begin{cases} p = b - \dfrac{a^2}{3} \\ q = \dfrac{2a^3}{27} - \dfrac{ab}{3} + c \end{cases}$$

$p = q = 0 \Rightarrow$ all 3 roots are zero

$$\text{otherwise} \begin{cases} \Delta < 0 \Leftrightarrow 1 \text{ real root} \\ \Delta = 0 \Leftrightarrow 2 \text{ distinct real} \\ \Delta > 0 \Leftrightarrow 3 \text{ distinct real} \end{cases}$$

- If $\Delta < 0$, the three roots of a generic cubic equation are $\xi_r, \omega, \bar{\omega}$, where ξ_r is real and $\omega, \bar{\omega}$ are a pair of conjugate complex numbers; a relation between roots and coefficients gives

$$\xi_r |\omega|^2 = -c \quad \Rightarrow \quad \xi_r \text{ is positive iff } c < 0$$

- When $c > 0$, a necessary condition for the cubic equation having positive real roots is $\Delta \geq 0$
- Applying to the isothermal equation $\xi^3 - \xi^2 + (4\pi LT)^{-2} = 0$

$$\begin{cases} a = -1 \\ b = 0 \\ c = \left(\dfrac{1}{4\pi LT}\right)^2 \end{cases} \quad \Rightarrow \quad \Delta = \left(\frac{1}{2\pi LT}\right)^2 \left[1 - \left(\frac{\sqrt{27}}{8\pi LT}\right)^2\right]$$

- The isothermal equation allows positive real solutions only if $\Delta \geq 0$ or, equivalently

$$T \geq \frac{\sqrt{27}}{8\pi L} \equiv T_{\min} \quad \begin{cases} T = T_{\min}, & \text{2 distinct real } \xi_r = \xi_1, \xi_1, \xi_2 \\ T > T_{\min}, & \text{3 distinct real } \xi_r = \xi_1, \xi_2, \xi_3 \end{cases}$$

- When $T = T_{\min}$, $\xi_1^2\xi_2 = -c < 0 \Rightarrow \xi_2 < 0$, $2\xi_1 + \xi_2 = -a \Rightarrow \xi_1 > 0$, $\exists$ exactly one positive root ξ_1
- For $T > T_{\min}$, one deduces from $\xi_1\xi_2\xi_3 = -c < 0$ that the number of distinct positive roots is either 2 or 0; since $\xi_1 + \xi_2 + \xi_3 = -a = 1$, not all solutions are negative $\Rightarrow$ there must be precisely two different positive roots ξ_1, ξ_2
- Summary: given wall temperature T, there exists a minimal value $T_{\min} \sim 1/L$ such that
  - If $T < T_{\min}$, no black holes allowed, the system is in thermal flat phase
  - If $T > T_{\min}$, there are two black hole solutions with masses $M_1 \neq M_2$, the heavier black hole is the larger one
  - If $T = T_{\min}$, large and small black holes become degenerate, with the same mass $M_1 = M_2$
- Next we consider $d + 2$-dim

- In general dimensions, the minimal temperature $T_{\min}$ could be determined by the condition that two distinct positive roots of $P(\xi) = \xi^{d+1} - \xi^2 + c$ become degenerate at some $\xi_0 > 0$

$$\begin{cases} P'(\xi_0) = 0 \;\Rightarrow\; \xi_0 \left[(d+1)\xi_0^{d-1} - 2\right] = 0 \;\Rightarrow\; \xi_0 = \left(\dfrac{2}{d+1}\right)^{\frac{1}{d-1}} < 1 \\ P(\xi_0) = 0 \;\Rightarrow\; \left(\dfrac{d-1}{4\pi L T_{\min}}\right)^2 = \xi_0^2 - \xi_0^{d+1} = \left(\dfrac{2}{d+1}\right)^{\frac{2}{d-1}} - \left(\dfrac{2}{d+1}\right)^{\frac{d+1}{d-1}} \end{cases}$$

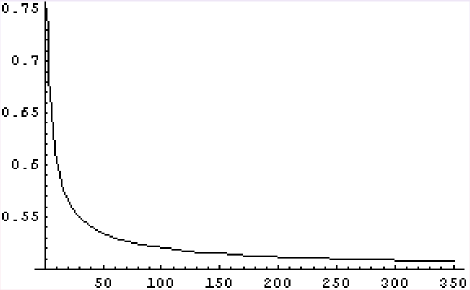
- The limiting temperature in $(d+2)$-dimensions

$$T_{\min} = \frac{d-1}{4\pi L}\left[\left(\frac{2}{d+1}\right)^{\frac{2}{d-1}} - \left(\frac{2}{d+1}\right)^{\frac{d+1}{d-1}}\right]^{-\frac{1}{2}}, \quad d = 2, 3, \cdots$$

$d =$	2	3	4	5	6
$LT_{\min}$:	$\frac{3\sqrt{3}}{8\pi} \approx .21$	$\frac{1}{\pi} \approx .32$	$\frac{5^{5/6}\sqrt{3}}{2^{1/3}(4\pi)} \approx .42$	$\frac{3^{3/4}}{\sqrt{2}\pi} \approx .51$	$\frac{7^{7/10}\sqrt{5}}{2^{1/5}(4\pi)} \approx .61$

- Compare this $T_{\min}$ to the minimal temperature of AdS black holes:

$$T_{\min}^{AdS} = \frac{\sqrt{d^2-1}}{2\pi L} \;\Rightarrow\; \frac{T_{\min}}{T_{\min}^{AdS}} = \frac{1}{2}\sqrt{\frac{d-1}{d+1}}\left[\left(\frac{2}{d+1}\right)^{\frac{2}{d-1}} - \left(\frac{2}{d+1}\right)^{\frac{d+1}{d-1}}\right]^{-\frac{1}{2}}$$



- Behavior of the ratio
  - Decreasing when d larger
  - Converging to $1/2$ at $d = \infty$

♣ Thermodynamics

- Thermodynamics can be constructed by the Gibbons-Hawking approach [Phys. Rev. D 15 (1977) 2752]

$$
\begin{aligned}
\mathcal{I} &= \mathcal{I}_1 - \mathcal{I}_0 \\
\mathcal{I}_1 &= -\frac{1}{16\pi G_{d+2}} \int_{\mathcal{M}} dt\, dr\, d^d\Omega \sqrt{g}\, R + \frac{1}{8\pi G_{d+2}} \oint_{\partial\mathcal{M}} dt\, d^d\Omega \sqrt{\gamma}\, \mathcal{K} \\
\mathcal{K} &= \text{trace of the extrinsic curvature tensor on } \partial\mathcal{M} = S^1 \times S^d \\
\gamma_{\alpha\beta} &= \text{the induced metric on } \partial\mathcal{M} \\
\mathcal{I}_0 &= \text{subtract term, i.e. } \mathcal{I}_1 \text{ evaluated on } \mathcal{M}_{\text{flat}}, \text{ with } \partial\mathcal{M}_{\text{flat}} = \partial\mathcal{M}
\end{aligned}
$$

- Since $R = 0$, the bulk action vanishes
- The period of Euclidean time t

$$
\beta_H = \frac{1}{T_H} = \frac{4\pi}{d-1} (w_{d+1} M)^{\frac{1}{d-1}} \quad \Rightarrow \quad \oint_{S^1} dt\, (\cdots) = \int_0^{\beta_H} dt\, (\cdots)
$$

- Schwarzschild metric in $d+2$ dim

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{w_{d+1}M}{r^{d-1}} & 0 & 0 \\ 0 & \left(1-\frac{w_{d+1}M}{r^{d-1}}\right)^{-1} & 0 \\ 0 & 0 & r^2\omega_{ab} \end{pmatrix}, \quad \omega_{ab} : \text{metric on unit sphere}$$

- The metric $\gamma_{\alpha\beta}$ on $S^1 \times S^d$ at $r = L$ is induced from $g_{\mu\nu}$

$$\gamma_{\alpha\beta} = \begin{pmatrix} 1-\frac{w_{d+1}M}{L^{d-1}} & 0 \\ 0 & L^2\omega_{ab} \end{pmatrix} \Rightarrow \sqrt{\gamma} = \left(1-\frac{w_{d+1}M}{L^{d-1}}\right)^{\frac{1}{2}} \underbrace{L^d \cdot \sqrt{\omega}}_{S^d \text{ at } r=L}$$

- $\sqrt{\gamma}$ contains a factor $\sqrt{g_{tt}}$, giving the proper length of S^1

$$\beta = \int_0^{\beta_H} \sqrt{g_{tt}}dt = \beta_H \left(1-\frac{w_{d+1}M}{L^{d-1}}\right)^{\frac{1}{2}} \to \text{inverse of the wall temperature}$$

- Trace of the second fundamental form at $r = L$

$$
\begin{aligned}
\mathcal{K} &= -\frac{1}{\sqrt{g}}\frac{\partial}{\partial r}\left[\sqrt{g}\left(1-\frac{w_{d+1}M}{r^{d-1}}\right)^{\frac{1}{2}}\right]_{r=L} \\
&= -\frac{d}{L}\left(1-\frac{w_{d+1}M}{L^{d-1}}\right)^{\frac{1}{2}} - \frac{(d-1)w_{d+1}M}{2L^{d}}\left(1-\frac{w_{d+1}M}{L^{d-1}}\right)^{-\frac{1}{2}} \\
&= -\left(1-\frac{w_{d+1}M}{L^{d-1}}\right)^{-\frac{1}{2}}\left[\frac{d}{L}-\frac{(d+1)w_{d+1}M}{2L^{d}}\right]
\end{aligned}
$$

- Integration over S^d results in the invariant "size" of the wall

$$
\oint_{S^d} d^d\Omega\, L^d \cdot \sqrt{\omega} = \mathrm{Vol}\,(S^d)L^d
$$

- Putting these things together

$$
\begin{aligned}
\mathcal{I}_1 &= -\frac{\mathrm{Vol}\,(S^d)L^d}{8\pi G_{d+2}}\left[\frac{d}{L}-\frac{(d+1)w_{d+1}M}{2L^{d}}\right]\beta_H \\
&= \frac{\mathrm{Vol}\,(S^d)L^d}{2(d-1)G_{d+2}}\left[\frac{d+1}{2}\cdot\left(\frac{w_{d+1}M}{L^{d-1}}\right)^{\frac{d}{d-1}} - d\cdot\left(\frac{w_{d+1}M}{L^{d-1}}\right)^{\frac{1}{d-1}}\right]
\end{aligned}
$$

- Similarly, $\mathcal{I}_0$ is computed by

$$\mathcal{I}_0 = \frac{1}{8\pi G_{d+2}} \int_0^{\beta} dt \oint_{S^d} d^d\Omega \sqrt{\gamma_0}\, \mathcal{K}_0$$

$$\beta = \frac{1}{T}, \quad \sqrt{\gamma_0} = L^d \cdot \sqrt{\omega}, \quad \mathcal{K}_0 = -\frac{1}{\sqrt{g_0}} \frac{\partial \sqrt{g_0}}{\partial r}\bigg|_{r=L} = -\frac{d}{L}$$

$$\Rightarrow \quad \mathcal{I}_0 = -\frac{d \cdot \mathrm{Vol}\,(S^d) L^d}{8\pi G_{d+2}} \frac{\beta}{L}$$

$$= -\frac{d \cdot \mathrm{Vol}\,(S^d) L^d}{2(d-1) G_{d+2}} \left(\frac{w_{d+1} M}{L^{d-1}}\right)^{\frac{1}{d-1}} \left(1 - \frac{w_{d+1} M}{L^{d-1}}\right)^{\frac{1}{2}}$$

- The total effective action $\mathcal{I} = \mathcal{I}_1 - \mathcal{I}_0$ may be interpreted as the free energy difference between black hole phase and the pure thermal flat space phase, which takes the form

$$\mathcal{I} = \frac{\mathrm{Vol}\,(S^d) L^d}{2(d-1) G_{d+2}} \mathcal{F}(\xi), \qquad \xi \equiv \left(\frac{w_{d+1} M}{L^{d-1}}\right)^{\frac{1}{d-1}}$$

$$\mathcal{F}(\xi) = \frac{d+1}{2} \xi^d + d \cdot \xi \left[\sqrt{1 - \xi^{d-1}} - 1\right]$$

- Isothermal equation

$$\xi^2 - \xi^{d+1} = \left(\frac{d-1}{4\pi LT}\right)^2 \;\Rightarrow\; \xi < 1 \;\Rightarrow\; \mathcal{F}(\xi) \text{ real-valued}$$

- For $\xi \sim 1^-$ and $0 < \xi \ll 1$

$$\mathcal{F}(\xi) \sim \begin{cases} \frac{d+1}{2} - d = -\frac{d-1}{2} < 0, & \text{black hole phase dominant} \\ \frac{d+1}{2}\xi^d + d \cdot \xi \left[\left(1 - \frac{1}{2}\xi^{d-1}\right) - 1\right] = \frac{1}{2}\xi^d > 0, & \text{thermal flat dominant} \end{cases}$$

- $\exists$ a phase transition point $\xi = \xi_c \neq 0$, at which $\mathcal{F}(\xi_c) = 0$

$$\frac{d+1}{2}\xi_c^{d-1} + d \cdot \left[\sqrt{1 - \xi_c^{d-1}} - 1\right] = 0$$

- Similar to the Hawking-Page phase transition

- There is a unique none-zero solution ξ_c, given by

$$\xi_c = \left[\frac{4d}{(d+1)^2}\right]^{\frac{1}{d-1}} \quad \Rightarrow \quad 0 < \xi_c < 1 \quad \Rightarrow$$

$$T_c = \frac{d-1}{4\pi L}\left\{\left[\frac{4d}{(d+1)^2}\right]^{\frac{2}{d-1}} - \left[\frac{4d}{(d+1)^2}\right]^{\frac{d+1}{d-1}}\right\}^{-\frac{1}{2}}$$

- If $T_c > T_{\min}$, the system will undergo a phase transition when temperature raised from $T_{\min} < T < T_c$ to $T > T_c$

$d =$	2	3	4	5	6
LT_{c} :	$\frac{27}{32\pi} \approx .27$	$\frac{2}{\sqrt{3}\pi} \approx .37$	$\frac{5^{5/3}}{2^{1/3}(8\pi)} \approx .46$	$\frac{3\sqrt{3}}{5^{1/4}(2\pi)} \approx .55$	$\frac{7^{2/5}}{2^{3/5}3^{1/5}(4\pi)} \approx .64$
$LT_{\min}$:	$\frac{3\sqrt{3}}{8\pi} \approx .21$	$\frac{1}{\pi} \approx .32$	$\frac{5^{5/6}\sqrt{3}}{2^{1/3}(4\pi)} \approx .42$	$\frac{3^{3/4}}{\sqrt{2}\pi} \approx .51$	$\frac{7^{7/10}\sqrt{5}}{2^{1/5}(4\pi)} \approx .61$

- In general one should be able to prove, for $d \geq 2$, that

$$\left(\frac{2}{d+1}\right)^{\frac{2}{d-1}} - \left(\frac{2}{d+1}\right)^{\frac{d+1}{d-1}} > \left[\frac{4d}{(d+1)^2}\right]^{\frac{2}{d-1}} - \left[\frac{4d}{(d+1)^2}\right]^{\frac{d+1}{d-1}}$$

- For a proof, consider the function $f(x) = x^2 - x^{d+1}$, $x \in \mathbb{R}^+$
  - There are only two extremal points of $f(x)$ in $[0, \infty)$, given by

$$f'(x) = 0 \;\Rightarrow\; 2x - (d+1)x^d = 0 \;\Rightarrow\; \begin{cases} x = 0 \\ x = \left(\frac{2}{d+1}\right)^{\frac{1}{d-1}} \end{cases}$$

  - The second order derivative of $f(x)$ at these extremal points

$$f''(x) = 2 - d(d+1)x^{d-1} = \begin{cases} 2 > 0, & x = 0 \\ -2(d-1) < 0, & x = \left(\frac{2}{d+1}\right)^{\frac{1}{d-1}} \end{cases}$$

  - Hence $x = 0$ is the minimal point, and $x = [2/(d+1)]^{1/(d-1)}$ the maximal one

- Since there are no other extremal points in $\mathbb{R}^+$, we conclude that

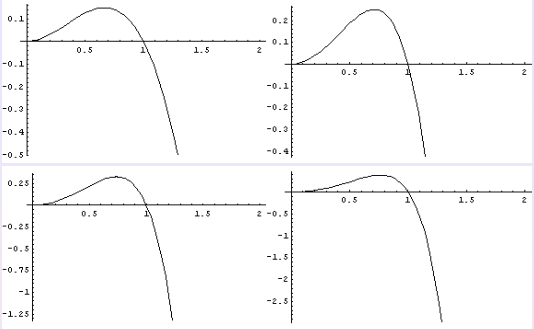
$$f\left[\left(\frac{2}{d+1}\right)^{\frac{1}{d-1}}\right] > f(x), \quad \forall x \in [0,\infty),\ x \neq \left(\frac{2}{d+1}\right)^{\frac{1}{d-1}}$$

- The inequality on the last page follows if we take

$$x = \left[\frac{4d}{(d+1)^2}\right]^{\frac{1}{d-1}} \in \mathbb{R}^+ \qquad \square$$

- The above proof, though quite simple, tells us something useful: If the wall has a temperature below its limiting value, $T < T_{\min}$, then there are no solutions $\xi \in \mathbb{R}^+$ of the isothermal equation

$$\xi^2 - \xi^{d+1} = \left(\frac{d-1}{4\pi LT}\right)^2 \qquad \text{[R.H.S exceeds the maximal value of } f(\xi)\text{]}$$



For each $T > T_{\min}$, positive real solutions $\xi_r(T)$ are located at the intersecting points of $y = f(\xi)$ and the line $y = [(d-1)/(4\pi LT)]^2 > 0$; there are exactly two such points $0 < \xi_1(T) < \xi_2(T) < 1$, associated to a small and a large black hole, respectively. Mass degeneration occurs when the line move up to the limiting position

- Question: which black hole, the large one or the small one, is thermodynamically (meta) stable?
- The system has a couple of parameters such as $\beta = 1/T$, $\beta_H = 1/T_H$, L, M etc., only two of them are independent
- E.g. given β, L, the mass of black holes is determined by solving the isothermal equation

$$\xi^2 - \xi^{d+1} = \left(\frac{d-1}{4\pi LT}\right)^2 \Rightarrow \xi = \xi(T, L) \Rightarrow M = \frac{L^{d-1}}{w_{d+1}} \cdot \xi^{d-1}$$

- We choose β and L as independent variables — they are temperature and size of the wall
- The thermodynamic energy of this system is computed by

$$E = \left.\frac{\partial \mathcal{I}}{\partial \beta}\right|_A , \qquad A = \mathrm{Vol}\,(S^d)L^d \text{ is the invariant area of the wall}$$

- Using the isothermal equation, one may write

$$\begin{aligned}
\mathcal{F}(\xi) &\equiv \frac{d+1}{2}\xi^d + d\cdot\xi\left[\sqrt{1-\xi^{d-1}}-1\right] \\
&= \frac{d+1}{2}\xi^d - d\cdot\xi + d\cdot\sqrt{\xi^2-\xi^{d+1}} \\
&= \frac{d+1}{2}\xi^d - d\cdot\xi + \frac{d(d-1)}{4\pi L}\beta
\end{aligned}$$

- Thus we find

$$\left.\frac{\partial \mathcal{F}(\xi)}{\partial \beta}\right|_A = d\cdot\left(\frac{d+1}{2}\xi^{d-1}-1\right)\cdot\left.\frac{\partial \xi}{\partial \beta}\right|_A + \frac{d(d-1)}{4\pi L}$$

- On the other hand, differentiating the isothermal equation yields

$$\Big[2\xi - (d+1)\xi^d\Big] \cdot \frac{\partial \xi}{\partial \beta}\bigg|_A = 2\left(\frac{d-1}{4\pi L}\right)^2 \beta$$

$$\Rightarrow \quad \left(\frac{d+1}{2}\xi^{d-1} - 1\right) \cdot \frac{\partial \xi}{\partial \beta}\bigg|_A = -\left(\frac{d-1}{4\pi L}\right)^2 \frac{1}{T\xi} = -\frac{d-1}{4\pi L}\frac{\sqrt{\xi^2 - \xi^{d+1}}}{\xi}$$

$$\Rightarrow \quad \frac{\partial \mathcal{F}(\xi)}{\partial \beta}\bigg|_A = \frac{d(d-1)}{4\pi L}\left[1 - \sqrt{1-\xi^{d-1}}\right] = \frac{d(d-1)}{4\pi L}\left(1 - \sqrt{1 - \frac{w_{d+1}M}{L^{d-1}}}\right)$$

- This gives a closed form expression for the thermodynamic energy

$$E = \frac{\mathrm{Vol}\,(S^d)L^d}{2(d-1)G_{d+2}}\,\frac{\partial \mathcal{F}(\xi)}{\partial \beta}\bigg|_A = \frac{d \cdot \mathrm{Vol}\,(S^d)L^{d-1}}{8\pi G_{d+2}}\left(1 - \sqrt{1 - \frac{w_{d+1}M}{L^{d-1}}}\right)$$

- When $L \to \infty$, the ADM energy $E = M$ recovered

- The entropy is determined by $S = \beta E - \mathcal{I}$; writing

$$E = \frac{\mathrm{Vol}\,(S^d) L^d}{2(d-1) G_{d+2}} \mathcal{E}, \quad S = \frac{\mathrm{Vol}\,(S^d) L^d}{2(d-1) G_{d+2}} \mathcal{S} \quad \Rightarrow \quad \mathcal{S} = \beta \mathcal{E} - \mathcal{F}$$

$$\mathcal{E} = \left. \frac{\partial \mathcal{F}(\xi)}{\partial \beta} \right|_A = \frac{d(d-1)}{4\pi L} \left(1 - \sqrt{1 - \xi^{d-1}}\right)$$

$$\beta \mathcal{E} = \frac{d(d-1)}{4\pi L T} \left(1 - \sqrt{1 - \xi^{d-1}}\right) = d\xi \sqrt{1 - \xi^{d-1}} \left(1 - \sqrt{1 - \xi^{d-1}}\right)$$
$$= d\xi \sqrt{1 - \xi^{d-1}} - d(\xi - \xi^d)$$

$$\beta \mathcal{E} - \mathcal{F} = d\xi \sqrt{1 - \xi^{d-1}} - d(\xi - \xi^d) - \left\{ \frac{d+1}{2} \xi^d + d\xi \left[\sqrt{1 - \xi^{d-1}} - 1 \right] \right\}$$
$$= \frac{d-1}{2} \xi^d \quad \Rightarrow \quad \mathcal{S} = \frac{d-1}{2} \left(\frac{w_{d+1} M}{L^{d-1}} \right)^{\frac{d}{d-1}}$$

- The final form of the entropy reads

$$S = \frac{\mathrm{Vol}\,(S^d)}{4 G_{d+2}} \left(w_{d+1} M\right)^{\frac{d}{d-1}} = \frac{\mathrm{Vol}\,(S^d) r_H^d}{4 G_{d+2}}$$

- The entropy increases as M becomes larger; so if we have two black holes of mass $M_1 < M_2$ at the same temperature, M_1 is thermodynamically unstable
- Thermal stability is determined by heat capacity
- In thermodynamics one usually consider partial derivatives at fixed space volume V, $(\partial/\partial X)|_V$, here we are interested in partial derivatives with wall area $A = \mathrm{Vol}(S^d)L^d$ fixed, this amounts to fixing the wall size L
- The heat capacity C_A is defined by

$$\begin{array}{cl} C_A & = T\dfrac{\partial S}{\partial T}\bigg|_A = \dfrac{\partial E}{\partial T}\bigg|_A = -\beta^2 \, \dfrac{\partial E}{\partial \beta}\bigg|_A \\ \downarrow & \\ \dfrac{\mathrm{Vol}(S^d)L^d}{2(d-1)G_{d+2}}\mathcal{C}_A & \Rightarrow \mathcal{C}_A = -\beta^2 \, \dfrac{\partial^2 \mathcal{F}}{\partial \beta^2}\bigg|_A \end{array}$$

- We have derived

$$\left.\frac{\partial \mathcal{F}}{\partial \beta}\right|_A = \frac{d(d-1)}{4\pi L}\left[1-\sqrt{1-\xi^{d-1}}\right], \quad \left.\frac{\partial \xi}{\partial \beta}\right|_A = -\frac{d-1}{4\pi L}\frac{\sqrt{1-\xi^{d-1}}}{\frac{d+1}{2}\xi^{d-1}-1}$$

$$\Rightarrow \quad \left.\frac{\partial^2 \mathcal{F}}{\partial \beta^2}\right|_A = \frac{d(d-1)^2}{8\pi L}\frac{\xi^{d-2}}{\sqrt{1-\xi^{d-1}}}\left.\frac{\partial \xi}{\partial \beta}\right|_A$$

$$= -\frac{d(d-1)}{2}\left(\frac{d-1}{4\pi L}\right)^2 \xi^{d-2}\left(\frac{d+1}{2}\xi^{d-1}-1\right)^{-1}$$

- This gives the heat capacity

$$\mathcal{C}_A = -\beta^2 \left.\frac{\partial^2 \mathcal{F}}{\partial \beta^2}\right|_A = \frac{d(d-1)}{2}\left(\xi^d-\xi^{2d-1}\right)\left(\frac{d+1}{2}\xi^{d-1}-1\right)^{-1}$$

$$= \frac{d(d-1)}{2}\frac{(w_{d+1}M)^{\frac{d}{d-1}}}{L^d}\left(1-\frac{w_{d+1}M}{L^{d-1}}\right)\left(\frac{d+1}{2}\frac{w_{d+1}M}{L^{d-1}}-1\right)^{-1}$$

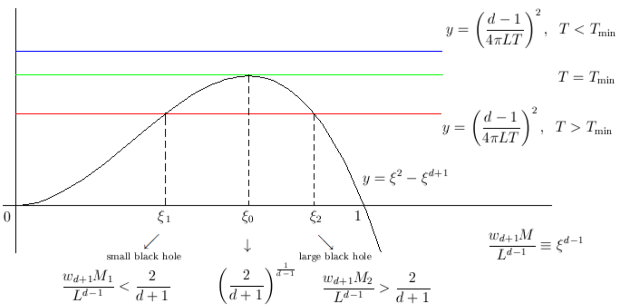
$$C_A = \frac{\mathrm{Vol}\,(S^d)L^d}{2(d-1)G_{d+2}}\mathcal{C}_A$$

$$= \frac{d\cdot \mathrm{Vol}\,(S^d)}{4G_{d+2}}(w_{d+1}M)^{\frac{d}{d-1}}\left(1-\frac{w_{d+1}M}{L^{d-1}}\right)\left(\frac{d+1}{2}\frac{w_{d+1}M}{L^{d-1}}-1\right)^{-1}$$

- Small and large black holes
  - Recall that

$$\frac{w_{d+1}M}{L^{d-1}} \equiv \xi^{d-1} \quad \rightarrow \quad \frac{w_{d+1}M_0}{L^{d-1}} = \frac{2}{d+2} \Leftrightarrow \xi_0 = \left(\frac{2}{d+2}\right)^{\frac{1}{d-1}}$$

  - The "critical" mass M_0 separating $C_A < 0$ and $C_A > 0$ corresponds precisely to the maximal point ξ_0 of the function $f(\xi) = \xi^2 - \xi^{d+1}$



  - Given $T > T_{\min}$, the isothermal equation has two solutions $0 < \xi_1 < \xi_0 < \xi_2 < 1$, associated with small and large black holes of masses $M_1 < M_0 < M_2$

- The function $f(\xi) = \xi^2 - \xi^{d+1}$ decreases more rapidly (when ξ leaves ξ_0 to the right) than it increases (when ξ approaches to ξ_0 from the left), it follows that

$$
\begin{aligned}
& 0 < \xi_2 - \xi_0 < \xi_0 - \xi_1 \ \Rightarrow\ 0 < \frac{d+1}{2}\xi_2^{d-1} - 1 < 1 - \frac{d+1}{2}\xi_1^{d-1} \\
\Rightarrow \quad & \left(\frac{d+1}{2}\xi_2^{d-1} - 1\right)^{-1} > \left(1 - \frac{d+1}{2}\xi_1^{d-1}\right)^{-1} \\
\Rightarrow \quad & \left(\frac{d+1}{2}\xi_1^{d-1} - 1\right)^{-1} + \left(\frac{d+1}{2}\xi_2^{d-1} - 1\right)^{-1} > 0 \\
\overset{\text{for } d \geq 2}{\Rightarrow} \quad & \xi_1^{d-2}\left(\frac{d+1}{2}\xi_1^{d-1} - 1\right)^{-1} + \xi_2^{d-2}\left(\frac{d+1}{2}\xi_2^{d-1} - 1\right)^{-1} > 0 \\
\Rightarrow \quad & C_A^{\text{small}} + C_A^{\text{large}} > 0
\end{aligned}
$$

- A system containing both large and small black holes (at the same temperature) is thermodynamically unstable, the small one has to decay [either to large black hole or to thermal flat space]

- Since $dE \neq TdS$, there should be a new variable entering into the first law of thermodynamics: surface pressure

$$\sigma \equiv - \left.\frac{\partial E}{\partial A}\right|_S \quad \Rightarrow \quad dE = TdS - \sigma dA$$

- We now choose L, M as two independent variables; since the entropy depends only on M (not on L), keeping S unchanged amounts to holding M as a constant

$$\begin{aligned}
\left.\frac{\partial E}{\partial L}\right|_S &= \frac{d \cdot \mathrm{Vol}\,(S^d)}{8\pi G_{d+2}} \frac{\partial}{\partial L} \left[L^{d-1} \left(1 - \sqrt{1 - \frac{w_{d+1} M}{L^{d-1}}} \right) \right]_{M=\mathrm{const.}} \\
&= \frac{d(d-1)\mathrm{Vol}\,(S^d) L^{d-2}}{8\pi G_{d+2}} \left[1 - \left(1 - \frac{w_{d+1} M}{2L^{d-1}} \right) \left(1 - \frac{w_{d+1} M}{L^{d-1}} \right)^{-\frac{1}{2}} \right] \\
\left.\frac{\partial A}{\partial L}\right|_S &= d \cdot \mathrm{Vol}\,(S^d) L^{d-1}
\end{aligned}$$

- The surface pressure is then given by

$$\sigma = \frac{d-1}{8\pi G_{d+2} L} \left[\left(1 - \frac{w_{d+1} M}{2L^{d-1}} \right) \left(1 - \frac{w_{d+1} M}{L^{d-1}} \right)^{-\frac{1}{2}} - 1 \right]$$

- One has $(1-x/2)^2 \geq 1-x;\ 0 < x < 1 \Rightarrow (1-x/2)(1-x)^{-1/2} > 1 \Rightarrow \sigma > 0$
- To study mechanical stability of the system, one needs to consider the isothermal compressibility

$$\kappa_T(A) \equiv \frac{1}{A}\left.\frac{\partial A}{\partial \sigma}\right|_T = \frac{d}{L}\cdot\left.\frac{\partial L}{\partial \sigma}\right|_T$$

$$= \frac{8\pi d \cdot G_{d+2}L}{d-1}\left(\frac{d+1}{2}\xi^{d-1}-1\right)\left[\left(\frac{d+1}{2}\xi^{d-1}-1\right)\left(1-\sqrt{1-\xi^{d-1}}\right)\right.$$

$$\left. + \frac{1}{2}\xi^{d-1}\sqrt{1-\xi^{d-1}}\right]^{-1} \Rightarrow \kappa_T(A) > 0 \ \text{ for large black holes}$$

- Number of states: Let $\tilde{\beta}$ be a saddle-point of $-\mathcal{I}(\beta, L) + \beta E$

$$\nu(E) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\beta\, \exp\left(-\mathcal{I}(\beta, L) + \beta E\right) \xrightarrow{\text{saddle-point}} \exp\left(-\mathcal{I}(\tilde{\beta}, L) + \tilde{\beta}E\right) = e^{S(E)}$$

- The thermodynamic behavior is very similar to AdS black holes

Installation (?)

- To construct a microscopic "boundary" description of Schwarzschild black holes, it seems necessary to confine such holes in an (isothermal) cavity, in order to stabilize the bulk system thermodynamically
- The entropy in the boundary theory should not be strictly extensive
- A first order phase transition should occur in the holographic dual, at some critical temperature
- Unlike the AdS case, there are subtleties to choose a holographic screen
  - The boundary of AdS at infinity has a nice property: each isometric transformation inside AdS space induces a conformal transformation on the boundary, this provides a natural way of constructing a "finite size" holographic screen
  - The isothermal wall of York's cavity itself is not a proper candidate for the holographic screen $\cdots$

♣ A Bizarre Speculation

- The boundary theory might obey Hill's nanothermodynamics T. L. Hill , Thermodynamics of Small Systems, Parts 1 and 2, (W. A. Benjamin and Co., 1964)], but not the ordinary thermodynamical laws
- In one-component nano-systems considered by Hill, the first law of the usual thermodynamics is still valid, but the entropy is not extensive in the number of particles
- Hill introduced a subdivision (entropic) potential $\mathcal{J}$ such that

$$\begin{cases} \mathcal{J} = S - \sum_\alpha F_\alpha x^\alpha \\ d\mathcal{J} = -\sum_\alpha x^\alpha dF_\alpha \end{cases}$$

- The usual Gibbs-Duhem relation $\sum_\alpha x^\alpha dF_\alpha = 0$ is generalized
- $\mathcal{J}$ is an intensive variable, conjugate to the number λ of "nano-systems"; $\mathcal{J}$ vanishes for a macroscopic (extensive) system $\Rightarrow$ entropic force $dS = \sum_\alpha F_\alpha dx^\alpha$

♣ Thermostatistics

- The usual thermostatistics is based on Gibbs-Shannon's entropy

$$S = -\sum_i p_i \log p_i$$

- This entropy obeys the extensive condition: if A, B are two independent systems, $p_{ij}^{A\oplus B} = p_i^A p_j^B$, then

$$S(A \oplus B) = S(A) + S(B)$$

- For isolated systems, the principle of extremum at equiprobability gives

$$p_i \sim \frac{1}{\Omega} \quad \Rightarrow \quad S \sim \log \Omega, \quad \Omega = \sum_i p_i$$

- Tsallis entropy

$$S = -k_B \frac{1 - \sum_i p_i^q}{1 - q}$$

- q: degree of nonextensivity

$$S(A + B) = S(A) + S(B) + (1 - q)S(A)S(B)$$