Multi-Loop Computation in Superstring Perturbation Theory

Chuan-Jie Zhu

Department of Physics , Renmin University of China
Beijing, P. R. China

Xidi, Huangshan, May 28–June 6th, 2010

Contents

1. Three quantization formalisms

2. Pure spinor and some basic formulas

3. Multi-loop superstring amplitudes

4. $\tilde{b}_B(z, u)$

5. Two-loop 4-particle amplitude

6 . Three-loop 4-particle amplitude

1. Three quantization formalisms

The formulation and (1st) quantization of superstring theory is a difficult problem. Either we keep manifest Lorentz covariance without manifest (spacetime) supersymmetry (NSR formalism) or we have manifest spacetime supersymmetry but using light-cone gauge (Green-Schwarz). Only in 2000 a new formalism called pure spinor formalism was constructed by Berkovits (hep-th/0001035, Super-Poincare Covariant Quantization of the Superstring).

1. The Neveu-Schwarz-Ramond formalism

The basic fields (open string or left-moving part of closed string) are:

$$X^\mu, \quad \psi^\mu, \quad (b,c), \quad (\beta,\gamma).$$

The central charges are

$$c = D + \frac{D}{2} + (-26) + 11 = 0 \Longrightarrow D = 10.$$

For multi-loop amplitude we have

$$A_{n,g} = \int \mathcal{D}X\mathcal{D}\psi\mathcal{D}b\mathcal{D}c\mathcal{D}\beta\mathcal{D}\gamma \cdots$$

$$\left| \int d^{3g-3}\tau \prod_{i=1}^{3g-3} \langle \mu_i, b \rangle \prod_{a=1}^{2g-2} (\delta(\beta(z_a)))J(\tilde{z}_a) \right|^2 e^{-S} V_1 \cdots V_n.$$

(b, c) **zero-modes and** (β, γ) **zero-modes give rise to the factor** $\prod_{i=1}^{3g-3} \langle \mu_i, b \rangle \prod_{a=1}^{2g-2} (\delta(\beta(z_a)))$**.**

Integration over super-moduli: super-current insertions $J(\tilde{z}_a)$.

What is more important is the fact that ψ^μ and the (β, γ) ghosts have fractional (odd-half integer) conformal weight on Riemann surface. So they have an extra property call spin structure. There must be a summation over spin structures. (Seiberg-Witten, 1986. GSO, 1976: GSO projection and SUSY. Modular invariance or global diffeomorphism.)

Problems: super-moduli integration lost manifest

“gauge invariance” (solved for 2-loop in 2001 by D’Hoker and Phong, leading to 1st computation of the manifest gauge parameter independent 2-loop 4-particle amplitude in 2002 by Zheng-Wu-Zhu, identically obtained by D’Hoker and Phong at the beginning of 2005.) and modular invariance (rules for summation over spin structures unknown).

Summary of the NSR formalism

- **Spacetime supersymmetric only after GSO proj.**
- **Higher loops: summation over spin structure and modular invariance not known. The measure is not manifestly gauge independent (total derivatives, spurious poles, etc.)**
- **Applied to multi-particle, higher-loop (2-loop, see below) and topological string theory amplitudes.**

2. The Green-Schwarz formalism

The basic fields are X^i and S^a, $a = 1, \cdots, 8.$

Conformal weight: 0. So there is no spin structure problem.

Also spacetime supersymmetry is manifest.

The biggest problem is that Lorentz covariance is not manifest (causing contact interactions, etc.)

3. PS: the pure spinor formulation

the basic fields are:

$$X^\mu, \quad (\theta^\alpha, p_\alpha), \quad (\lambda^\alpha, w_\alpha), \quad \alpha = 1, \cdots, 16.$$

The conformal weights are:

$$0, \quad (0, 1), \quad (0, 1).$$

$(\theta^\alpha, p_\alpha)$ **are fermionic and the rests are bosonic.**

λ^α **is called a pure spinor and must satisfy the**

constraints:

$$\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta = 0, \qquad \mu = 0, 1, \cdots, 9.$$

Only 11 independent components remain. (The dual field w_α has gauge transformations. Physical observable has gauge invariance.)

An explicit representation of the Γ matrix. An explicit solution of the pure spinor constraints.

0 central charges:

$$c = 10 + 16 \times (-2) + 11 \times 2 = 0.$$

(Remember $c = 2\epsilon(6(\lambda^2 - \lambda) + 1)$ **for a** $(\lambda, 1 - \lambda)$ **system.)**

Summary of Berkovits' pure spinor formalism

- **Lorentz covariant and manifestly spacetime supersymmetric (no summation over spin structures).**

- **All integer dimensional free fields on (ordinary) Riemann surface.**

- **Shortcoming: pure spinor constraints and very complicated composite $\tilde{b}$ fields.**

2. Pure spinor and some basic formulas

$$\Pi^m = \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta),$$

$$d_\alpha = p_\alpha - \frac{1}{2}\left(\partial X^m + \frac{1}{4}(\theta\gamma^m\partial\theta)\right)(\gamma_m\theta)_\alpha$$

The basic OPEs are:

$$X^m(z,\overline{z})X^n(w,\overline{w}) \longrightarrow -\frac{\alpha'}{2}\eta^{mn}\ln|z-w|^2,$$

$$p_\alpha(z)\theta^\beta(w) \longrightarrow \frac{\delta_\alpha^\beta}{z-w},$$

$$d_\alpha(z)d_\beta(w) \longrightarrow -\frac{\alpha'}{2}\frac{\gamma^m_{\alpha\beta}\Pi_m}{z-w},$$

$$d_\alpha(z)\Pi^m(w) \longrightarrow \frac{\alpha'}{2}\frac{(\gamma^m\partial\theta)_\alpha}{z-w}.$$

Furthermore, if $V(y,\theta)$ is a generic superfield then

its OPE's with d_α and Π^m are computed as follows

$$d_\alpha(z)V(y,\theta) \longrightarrow \frac{\alpha'}{2}\frac{D_\alpha V(y,\theta)}{z-y},$$

$$\Pi^m(z)V(y,\theta) \longrightarrow \frac{\partial^m V(y,\theta)}{z-y},$$

Here the supersymmetric derivative D_α is given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m.$$

Why we need the pure spinor field λ?

The Lorentz currents from the fermionic variables

$$\Sigma^{mn} = \frac{1}{2}(p\gamma^{mn}\theta)$$

give the following OPEs:

$$\Sigma^{mn}(w)\Sigma^{pq}(z) = \frac{1}{4}\frac{p(\gamma^{mn}\gamma^{pq} - \gamma^{pq}\gamma^{mn})\theta}{w-z} + \frac{1}{4}\left(\frac{\mathrm{tr}(\gamma^{mn}\gamma^{pq})}{(w-z)^2}\right)$$

$$= \frac{\eta^{p[n}\Sigma^{m]q} - \eta^{q[n}\Sigma^{m]p}}{w-z} + 4\frac{\eta^{m[q}\eta^{p]n}}{(w-z)^2}$$

with $\gamma^{mn}\gamma^{pq} - \gamma^{pq}\gamma^{mn} = 2\eta^{np}\gamma^{mq} - 2\eta^{nq}\gamma^{mp} + 2\eta^{mq}\gamma^{np} - 2\eta^{mp}\gamma^{nq}$ **and** $\mathrm{tr}(\gamma^{mn}\gamma^{pq}) = -32\delta^{mn}_{pq}$**.**

(Recalling that in the RNS formalism the OPE of the Lorentz currents for the fermionic variables $\Sigma_{\mathrm{RNS}} = \psi^m\psi^n$ **satisfies**

$$\Sigma^{mn}_{RNS}(w)\Sigma^{pq}_{RNS}(z) \to \frac{\eta^{p[n}\Sigma^{m]q}_{RNS} - \eta^{q[n}\Sigma^{m]p}_{RNS}}{w-z} + \frac{\eta^{m[q}\eta^{p]n}}{(w-z)^2}$$

The different double pole coefficients would make the computations of scattering amplitudes not agree with each other.)

To insure Lorentz covariance, w_α only appears in:

$$J = w_\alpha \lambda^\alpha, \qquad N_{mn} = \frac{1}{2} w_\alpha (\gamma_{mn})^\alpha{}_\beta \lambda^\beta$$

$$N^{kl}(y) N^{mn}(z) \to \frac{\delta^{m[l} N^{k]n}(z) - \delta^{n[l} N^{k]m}(z)}{y - z}$$

$$-3 \frac{\delta^{kn} \delta^{lm} - \delta^{km} \delta^{ln}}{(y-z)^2},$$

$$N^{mn}(y)\lambda^{\alpha}(z) \to \frac{1}{2}\frac{(\gamma^{mn})^{\alpha}{}_{\beta}\lambda^{\beta}(z)}{(y-z)}.$$

$$Q = \frac{1}{2\pi i}\oint \lambda^{\alpha} d_{\alpha}$$

satisfies $Q^2 = 0$ **due to the pure spinor constraints.**

Therefore we can define the unintegrated and integrated massless vertex operators for the super-

Yang-Mills states as follows

$$V = \lambda^\alpha A_\alpha(x,\theta)$$

$$U = \mathrm{e}^{ik\cdot X}(\partial\theta^\alpha A_\alpha(\theta)+\Pi^m A_m(\theta)+d_\alpha W^\alpha(\theta)+\frac{1}{2}N^{mn}\mathcal{F}_{mn}(\theta))$$

where the superfields A_α, A_m, W^α and $\mathcal{F}_{mn}$ describe the super-Yang-Mills theory in D=10.

(In the RNS formalism the unintegrated vertex operator satisfies $QU = \partial V$, as one can check by

recalling that $U = \{\oint b, V\}$ **and** $T = \{Q, b\}$**.)**

3. Multi-loop superstring amplitudes

Amplitudes can be easily constructed. For bosonic string theory we have

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\text{top.}} \int \frac{\mathcal{D}X\mathcal{D}g}{\text{Vol.(Diff} \times \text{Weyl)}} \mathrm{e}^{-S_X-\lambda\chi} \times \prod_{i=1}^{n} \int \mathrm{d}^2\sigma_i (\det g(\sigma_i))^{1/2} V_{j_i}(k_i,\sigma_i)$$

$V_j(k,\sigma)$ **is a vertex operator describing a specific** particle.

For superstring theory in the pure spinor formalism the proposal is (hep-th/0406055):

$$\mathcal{A} = \int d^2\tau_1 ... d^2\tau_{3g-3} \langle \mid \prod_{P=1}^{3g-3} \int d^2 u_P \mu_P(u_P) \widetilde{b}_{B_P}(u_P, z_P)$$

$$\times \prod_{P=3g-1}^{10g} Z_{B_P}(z_P) \prod_{R=1}^{g} Z_J(v_R)$$

$$\times \prod_{I=1}^{11} Y_{C_I}(y_I) \;|^2 \; \prod_{T=1}^{N} \int d^2 t_T U_T(t_T) \; \rangle$$

where

- **"picture-lowering" operator:** $Y_C = C_\alpha \theta^\alpha \delta(C_\beta \lambda^\beta)$
- **"picture-raising" operator:**

$$Z_B = \frac{1}{2} B_{mn} \lambda \gamma^{mn} d \, \delta(B^{pq} N_{pq}), \qquad Z_J = \lambda^\alpha d_\alpha \delta(J)$$

More importantly is the composite "b ghost" field: $\tilde{b}_B(z, u)$.

4. $\tilde{b}_B(z,u)$

By construction we have:

$$\{Q, b(z)\} = T(z)$$

$$\{Q, \tilde{b}_B(z,w)\} = T(z) Z_B(w)$$

$$\tilde{b}_B(z,w) = b_B(z) + T(z) \int_z^w \mathrm{d}u \, B_{pq} \partial N^{pq}(u) \delta(BN(u))$$

$$T = -\frac{1}{2} \partial X^a \partial X_a - p_\alpha \partial \theta^\alpha + T_{\lambda,w}.$$

To construct $b_B(z)$, we introduce the following sequence of operators:

$$Z_B \equiv \lambda^\alpha Z_\alpha,$$

$$\{Q, Z_\alpha\} = \lambda^\beta Z_{\beta\alpha},$$

$$[Q, Z_{\beta\alpha}] = \lambda^\gamma Z_{\gamma\beta\alpha},$$

$$\{Q, Z_{\gamma\beta\alpha}\} = \lambda^\delta Z_{\delta\gamma\beta\alpha} + \partial\lambda^\delta \Upsilon_{\delta\gamma\beta\alpha},$$

where the operators $Z_{\beta\alpha}$, $Z_{\gamma\beta\alpha}$, $Z_{\delta\gamma\beta\alpha}$ and $\Upsilon_{\delta\gamma\beta\alpha}$ are Γ_5-*traceless* (=0 when saturated with Γ_5 between 2 adjacent indices). Concretely, the full expression of the operators Z's and $\Upsilon_{\alpha_1\cdots\alpha_4}$ in the case of Z_B takes the form:

$$
\begin{aligned}
Z_\alpha &= \frac{1}{2}B_{ab}(\Gamma^{ab}d)_\alpha\delta(B_{cd}N^{cd}) \\
&\equiv \frac{1}{2}(Bd)_\alpha\delta(BN),
\end{aligned}
$$

$$Z_{\beta\alpha} = -\frac{1}{2}(\Gamma^c\Gamma^{ba})_{\beta\alpha}\Pi_c B_{ab}\delta(BN) - \frac{1}{4}(Bd)_\alpha(Bd)_\beta\partial\delta(BN),$$

$$\begin{aligned} Z_{\gamma\beta\alpha} &= -\frac{1}{2}[(\Gamma^c\Gamma^{ba})_{\beta\alpha}(\Gamma_c\partial\theta)_\gamma B_{ab}\delta(BN) \\ &\quad +\frac{1}{2}(\Gamma^c\Gamma^{ba})_{\beta\alpha}(Bd)_\gamma\Pi_c B_{ab}\partial\delta(BN) \\ &+ \frac{1}{2}(\Gamma^c\Gamma^{ba})_{\gamma[\beta}(Bd)_{\alpha]}\Pi_c B_{ab}\partial\delta(BN) \end{aligned}$$

$$+\frac{1}{4}(Bd)_\alpha(Bd)_\beta(Bd)_\gamma\partial^2\delta(BN)],$$

$$
\begin{aligned}
Z_{\delta\gamma\beta\alpha} &= -\frac{1}{4}[(((\Gamma^c\Gamma^{ba})_{\beta\alpha}(Bd)_{[\delta}(\Gamma_c\partial\theta)_{\gamma]} \\
&\quad -(\Gamma^c\Gamma^{ba})_{\gamma[\beta}(Bd)_{\alpha]}(\Gamma_c\partial\theta)_\delta)B_{ab}\partial\delta(BN) \\
&- ((\Gamma^f\Gamma^{ed})_{\delta[\gamma}(\Gamma^c\Gamma^{ba})_{\beta]\alpha} \\
&\quad +(\Gamma^f\Gamma^{ed})_{\delta\alpha}(\Gamma^c\Gamma^{ba})_{\gamma\beta})\Pi_c B_{ab}\Pi_f B_{de}\partial\delta(BN) \\
&- \frac{1}{2}((\Gamma^c\Gamma^{ba})_{\beta\alpha}(Bd)_\gamma(Bd)_\delta + (\Gamma^c\Gamma^{ba})_{\gamma[\beta}(Bd)_{\alpha]}(Bd)_\delta
\end{aligned}
$$

$$+ \quad \frac{1}{2}(\Gamma^c\Gamma^{ba})_{\delta[\alpha}(Bd)_\beta(Bd)_{\gamma]})\Pi_c B_{ab}\partial^2\delta(BN)$$
$$- \quad \frac{1}{4}(Bd)_\alpha(Bd)_\beta(Bd)_\gamma(Bd)_\delta\partial^3\delta(BN)],$$

$$\Upsilon_{\delta\gamma\beta\alpha} = -\frac{1}{2}(\Gamma_c)_{\delta\gamma}(\Gamma^c\Gamma^{ba})_{\beta\alpha}B_{ab}\delta(BN).$$

On the other hand we introduce another set of operators G^α, $\cdots$:

$$\{Q, G^\alpha\} = \lambda^\alpha T.$$

$$[Q, H^{\alpha\beta}] = \lambda^\alpha G^\beta + \cdots,$$

$$\{Q, K^{\alpha\beta\gamma}\} = \lambda^\alpha H^{\beta\gamma} + \cdots,$$

$$[Q, L^{\alpha\beta\gamma\delta}] = \lambda^\alpha K^{\beta\gamma\delta} + \cdots,$$

where the dots denote Γ_1-traceless terms. The chain

of equations finishes at the level of the last equation, since, for dimensional reasons, $\lambda^\alpha L^{\beta\gamma\delta\epsilon}$ vanishes modulo Γ_1-traceless terms so that

$$L^{\alpha\beta\gamma\delta} = \lambda^\alpha S^{\beta\gamma\delta} + \cdots,$$

and

$$[Q, S^{\alpha\beta\gamma}] = K^{\alpha\beta\gamma} + \lambda^\alpha T^{\beta\gamma} + \cdots,$$

for a suitable field $T^{\beta\gamma}$. Then, according to Berkovits

(hep-th/0406055), b_B is given by:

$$b_B = b_1 + b_2 + b_3 + b_4^{(a)} + b_4^{(b)},$$

where

$$\begin{aligned} b_1 &= G^{\alpha} Z_{\alpha}, \\ b_2 &= H^{\alpha\beta} Z_{\alpha\beta}, \\ b_3 &= -K^{\alpha\beta\gamma} Z_{\alpha\beta\gamma}, \\ b_4^{(a)} &= -L^{\alpha\beta\gamma\delta} Z_{\alpha\beta\gamma\delta}, \\ b_4^{(b)} &= -S^{\alpha\beta\gamma} \partial\lambda^{\delta} \Upsilon_{\delta\alpha\beta\gamma}. \end{aligned}$$

Firstly we have:

$$
\begin{aligned}
b_4^{(b)} &= B_{ab}[-T^{\omega\lambda}N^{ab} - \frac{1}{4}J\partial N^{ab} \\
&\quad + \frac{1}{4}N^{ab}\partial J + \frac{1}{2}N^{a}{}_{c}\partial N^{bc}]\delta(BN).
\end{aligned}
$$

The other needed fields are:

$$
G^{\alpha} =: \frac{1}{2}\Pi^{a}(\Gamma_{a}d)^{\alpha} : -\frac{1}{4}N_{ab}(\Gamma^{ab}\partial\theta)^{\alpha} - \frac{1}{4}J\partial\theta^{\alpha} - \frac{1}{4}\partial^{2}\theta^{\alpha},
$$

$$H^{\alpha\beta} = H^{(\alpha\beta)} + H^{[\alpha\beta]},$$

where

$$H^{(\alpha\beta)} = \frac{1}{16}\Gamma_a^{\alpha\beta}(N^{ab}\Pi_b - \frac{1}{2}J\Pi^a + 2\partial\Pi^a),$$

$$H^{[\alpha\beta]} = \frac{1}{96}\Gamma_{abc}^{\alpha\beta}(\frac{1}{4}d\Gamma^{abc}d + 6N^{ab}\Pi^c).$$

$$
\begin{aligned}
K^{\alpha\beta\gamma} &= -\frac{1}{48}\Gamma_a^{\alpha\beta}(\Gamma_b d)^\gamma N^{ab} - \frac{1}{192}\Gamma_{abc}^{\alpha\beta}(\Gamma^a d)^\gamma N^{bc} \\
&+ \frac{1}{192}\Gamma_a^{\beta\gamma}[(\Gamma_b d)^\alpha N^{ab} + \frac{3}{2}(\Gamma^a d)^\alpha J - 6(\Gamma^a \partial d)^\alpha] \\
&- \frac{1}{192}\Gamma_{abc}^{\beta\gamma}(\Gamma^a d)^\alpha N^{bc}
\end{aligned}
$$

But we only give the totally antisymmetric part of L**:**

$$
L^{[\alpha\beta\gamma\delta]} = -\frac{1}{3072}(\Gamma_{abc})^{[\alpha\beta}(\Gamma^{ade})^{\gamma\delta]}N^{bc}N_{de}.
$$

Schematically,we have:

$$
\begin{aligned}
b_B \;=\;& B(dd\Pi + dN\partial\theta + N\partial N + T_{\lambda,w}N + N\Pi\Pi)\delta(BN) \\
& + BB(dddd + ddN\Pi + NN\Pi\Pi + NNd\partial\theta)\partial\delta(BN) \\
& + BBB(ddddN + ddNN\Pi)\partial^2\delta(BN)\cdots \\
& + BBBB(ddddNN)\partial^3\delta(BN)
\end{aligned}
$$

5. Two-Loop 4-particle amplitude

The 2-loop 4-particle amplitude obtained by Zheng-Wu-Zhu in 2002 (hep-th/0212191, 198, 219) is:

$$\mathcal{A}_{II} \sim \int \frac{1}{T^5} \frac{\prod_{i=1}^{6} \mathrm{d}^2 a_i}{\mathrm{d}V_{pr} |\prod_{i<j} a_{ij}|^2} \prod_{i=1}^{4} \frac{\mathrm{d}^2 z_i}{|y(z_i)|^2} \prod_{i<j} \mathrm{e}^{-k_i \cdot k_j \langle X(z_i) X(z_j) \rangle}$$

$$\times \left| s(z_1 z_2 + z_3 z_4) + t(z_1 z_4 + z_2 z_3) + u(z_1 z_3 + z_2 z_4) \right|^2$$

$$\mathrm{d}V_{pr} = \frac{\mathrm{d}^2 a_i \mathrm{d}^2 a_j \mathrm{d}^2 a_k}{|a_{ij} a_{ik} a_{jk}|^2}, \quad T = \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2 |z_1 - z_2|^2}{|y(z_1) y(z_2)|^2},$$

$$\langle X(z_i) X(z_j) \rangle \equiv G(z_i, z_j) = -\ln |E(z_i, z_j)|^2$$

$$+2\pi(\mathrm{Im}\,\Omega)^{-1}_{IJ}\left(\mathrm{Im}\int_{z_i}^{z_j}\omega_I\right)\left(\mathrm{Im}\int_{z_i}^{z_j}\omega_J\right)$$

A better but equivalent form derived by D' Hoker and Phong (hep-th/0501197):

$$\mathbf{A}_{II}(\epsilon_i,k_i) = \frac{K\overline{K}}{2^{12}\pi^4}\int\frac{|\prod_{I\leq J}d\Omega_{IJ}|^2}{(\det\mathrm{Im}\,\Omega)^5}$$
$$\times\int_{\Sigma^4}|\mathcal{Y}_S|^2\exp\Big(-\sum_{i<j}k_i\cdot k_j\,G(z_i,z_j)\Big)$$
$$\mathcal{Y}_S = +(k_1-k_2)\cdot(k_3-k_4)\,\Delta(z_1,z_2)\Delta(z_3,z_4)+\cdots$$

$$\propto s(z_1 z_2 + z_3 z_4) + t(z_1 z_4 + z_2 z_3) + u(z_1 z_3 + z_2 z_4)$$

$$\Delta(z, w) \equiv \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)$$

Berkovits et. al. also obtained the same results by using pure spinor formalism (hep-th/0503197, 0509234).

6 . Three-Loop 4-particle amplitude

Integration over θ^α and d_α requires at least $16 + 16g$ zero modes to give a non-vanishing result.

For the 4-particle amplitude we are just on the verge of getting a non-vanishing amplitude.

There are $8g + 3$ d-field from Z_B and Z_J. So at least $8g - 3$ d-field must come from $\tilde{b}_B$ and the vertex operators.

There are no terms in $\tilde{b}_B$ with 3 d-fields. Terms with 4 d-fields contains a derivative on the delta function

(δ', $N\delta''$, or $NN\delta^{(3)}$.) Because

$$\int dx\delta'(x) = 0,$$

and

$$\int dxx\delta'(x) = -1,$$

we need some N fields from $\tilde{b}_B$ or the vertex operators.

Taken $g = 2$. Only 3 $\tilde{b}_B$ and 4 vertex operators. Need 13. 12 or 10, 8, $\cdots$. 8 and less are excluded. 10