Deformation of Codimension-2 Surfaces and Horizon Thermodynamics

曹利明 (Li-Ming Cao)

日本近畿大学物理系

2011年7月

主要内容

Event horizon

Black hole region, $\mathcal{B}$, of an asymptotically flat spacetime is defined as

$$\mathcal{B} = \mathcal{M} - I^{-}(\mathcal{I}^{+})\,.$$

- The event horizon $\mathcal{H}$ is defined as the boundary of $\mathcal{B}$. So $\mathcal{H}$ is the boundary of the past of $\mathcal{I}^{+}$.
- Other spacetimes with different asymptotical behavior. For example, asymptotically AdS spacetime
- We do not know how to define the event horizon if we can not get the information of the future infinity of a spacetime.

The definition of event Horizon

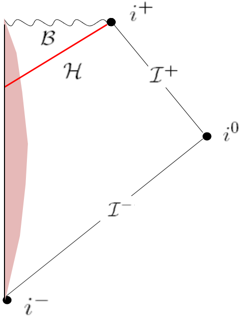


Figure: The definition of event horizon

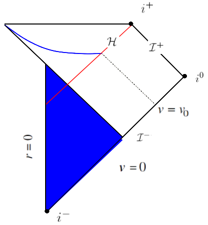
Fail to describe strong gravity property

A special Vaidya spacetime (Ashtekar and Krishnan 2004):

$$g = -f(v,r)dv^2 + 2dvdr + r^2 d\Omega_2^2\,, \tag{1}$$

with

$$f(v,r) = \begin{cases} 1 & v < 0; \\ 1 - m(v)/r & 0 \le v < 0 \le v_0, \dot{m} > 0; \\ 1 - m_0/r & v > v_0, m_0 \text{ is a constant.} \end{cases}$$



Killing horizon

A null hypersurface $\mathcal{K}$ is a Killing horizon of a Killing vector field ξ if, on $\mathcal{K}$, ξ is normal to $\mathcal{K}$.

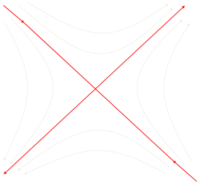


Figure: Killing Horizon

However, in reality, black holes in our universe and our universe itself are all dynamical. There are no stationary Killing vectors associated with the spacetimes.

Killing horizon

Stationary black hole mechanism:

- Zeroth Law: Surface gravity is a constant on Killing horizon

$$\xi_b \nabla^a \xi^b = -\kappa \xi^a \,, \tag{2}$$

- First law: Killing horizon

$$\delta M = \frac{\kappa}{8\pi} \delta A + \cdots \tag{3}$$

- Second law: Event horizon with some energy condition

$$\delta A \geq 0 \,. \tag{4}$$

The thermodynamics of stationary black holes

Hawking radiation: Quantum particles creation effects. a black hole radiates particles to infinity with a perfect body spectrum with a temperature

$$T = \frac{\kappa}{2\pi} \, . \tag{5}$$

This is just "Hawking temperature". The entropy of the system is given by

$$S = \frac{A}{4G} \, , \tag{6}$$

which is called "Bekenstein-Hawking" entropy.

Apparent horizon

The definition of an apparent horizon depends on the slicing of a spacetime.

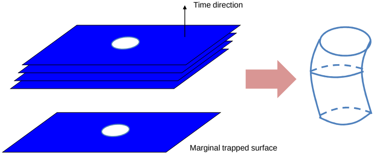


Figure: Apparent horizon

Trapped region and marginal trapped surface

Trapped region inside a slice: Defined by the extrinsic geometry of an 2-surface

$$\theta^{(\ell)}\,, \qquad \theta^{(n)} \tag{7}$$

For a sphere embedded in a flat spacetime, one gets

$$\theta^{(\ell)} = \frac{2}{r} > 0\,, \qquad \theta^{(n)} = -\frac{2}{r} < 0\,. \tag{8}$$

Expansions

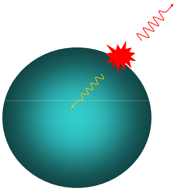


Figure: Two expansions

However, for a curved spacetime, one may find some region in a slice of the spacetime which satisfies:

$$\theta^{(\ell)} < 0\,, \qquad \theta^{(n)} < 0\,. \tag{9}$$

Quasi-local horizons

Recent years, people have proposed several quasi-local horizons:

- Trapping horizon (Hayward, 1994)
- Isolated horizon (Ashtekar, 1999)
- Dynamical horizon (Ashtekar and Krishnan, 2002)
- Slowly evolving horizon (Booth)

These horizons are generalization of the apparent horizon, and their definitions do not depend on the slicing of the spacetime.

Similar to the definition of the apparent horizon, these horizons can be defined by the extrinsic geometry of codimension-2 spacelike surface.

Thermodynamics of quasi-local horizons

Balance equations (Similar to the first law of the stationary black hole):

$$\int_S (\text{“surface gravity”})\delta\epsilon_S = \int_S \Big[(\text{matter})+(\text{gravitational radiation})\Big]\,. \tag{10}$$

The relation between energy flux and the geometric variation of marginal trapped surface (or generalized apparent horizon).

The energy flux and the variation of the marginal trapped surfaces are linked together by focusing and cross focusing equations.

Rindler horizon in Minkowski spacetime

It's a special Killing horizon: The Killing horizon corresponding to the Lorentz boost of the Minkowski spacetime.

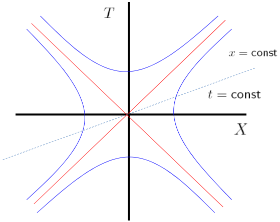


Figure: Rindler horizon in flat spacetime

Unruh temperature

The orbits of the Lorentz boost are the worldlines of an uniformly accelerating observers.

An uniformly accelerating observer can observe the therm spectrum of vacuum fluctuation, and the temperature is proportional to its acceleration:

$$T = \frac{a}{2\pi} \tag{11}$$

W. G. Unruh, Phys. Rev. D 14, 870 (1976)

Local Rindler horizon of a curved spacetime

For a curved spacetime, we can construct the Rindler horizon in the local Riemann normal coordinates, and the corresponding horizon is called "local Rindler horizon".

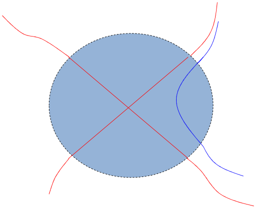


Figure: Local Rindler horizon

The work of Jacobson et. al.

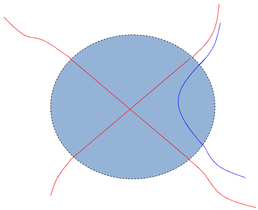
Based on the local Rindler horizon, and considering Clausius relation

$$\delta Q = T\delta S \tag{12}$$

Jacobson gets Einstein equation

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab}\,. \tag{13}$$

T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995)



The work of Jacobson

The heat flux δQ is defined from energy-momentum tensor T_{ab}. The entropy S is assumed to be the area of the local Rindler horizon (cross section). The temperature is assumed to be the Unruh temperature of an uniformly accelerating observer. By using Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = \frac{1}{2}\theta^2 + \sigma^{ab}\sigma_{ab} - R_{ab}k^a k^b \,, \tag{14}$$

He gets Einstein equation from $\delta Q = T\delta S$.

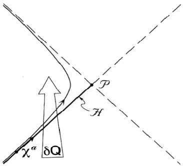


FIG. 1. Spacetime diagram showing the heat flux δQ across the local Rindler horizon $\mathcal{H}$ of a 2-surface element $\mathcal{P}$. Each point in the diagram represents a two dimensional spacelike surface. The hyperbola is a uniformly accelerated worldline, and χ^a is the approximate boost Killing vector on $\mathcal{H}$.

Focusing and cross focusing equations

Raychaudhuri equation is a kind of focusing equation. So, once we hope to study the thermodynamics of the horizons qusi-locally or locally, the focusing and cross focusing equations are important. They link the matter flux and the variation of the horizon:

$$\begin{aligned}
\mathcal{L}_\ell \theta^{(\ell)} &= \kappa_\ell \theta^{(\ell)} - \mathscr{G}_{ab}\ell^a \ell^b - \sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} - \frac{1}{n-2}\theta^{(\ell)}\theta^{(\ell)} \,, \\
\mathcal{L}_n \theta^{(n)} &= -\kappa_n \theta^{(n)} - \mathscr{G}_{ab}n^a n^b - \sigma^{(n)}_{ab}\sigma^{(n)ab} - \frac{1}{n-2}\theta^{(n)}\theta^{(n)} \,, \\
\mathcal{L}_n \theta^{(\ell)} &= \kappa_n \theta^{(\ell)} + \omega_c\omega^c - D_c\omega^c + \mathscr{G}_{ab}\ell^a n^b - \frac{1}{2}R - \theta^{(\ell)}\theta^{(n)} \,, \\
\mathcal{L}_\ell \theta^{(n)} &= -\kappa_\ell \theta^{(n)} + \omega_c\omega^c + D_c\omega^c + \mathscr{G}_{ab}n^a \ell^b - \frac{1}{2}R - \theta^{(\ell)}\theta^{(n)} \,.
\end{aligned} \tag{15}$$

Submanifold

For a spacelike submanifold, from the submanifold theory, one can always decompose the metric of the spacetime into

$$g_{ab} = h_{ab} + q_{ab} \,, \tag{16}$$

The *second fundamental tensor* ${K_{ab}}^c$ is defined as

$${K_{ab}}^c = {q_a}^d {q_b}^e \nabla_d {q_e}^c \,, \tag{17}$$

and it can be defined without introducing any local frame of the spacetime (B.Carter, 1992).

Submanifold

The second fundamental tensor can be decomposed into a traceless part ($C_{ab}{}^c$) and a trace part (K^c), i.e.,

$$K_{ab}{}^c = \frac{1}{n-2} q_{ab} K^c + C_{ab}{}^c \,, \tag{18}$$

$K^c = g^{ab} K_{ab}{}^c$ is called *extrinsic curvature vector* or *mean curvature vector*. For an arbitrary normal vector X, we can define

$$K_{ab}^{(X)} = -K_{ab}{}^c X_c = q_a{}^c q_b{}^d \nabla_c X_d \,,$$

This is the usual second fundamental tensor along X direction, the expansion and the shear tensor are respectively given by

$$\theta^{(X)} = -K^c X_c \,, \qquad \sigma_{ab}^{(X)} = -C_{ab}{}^c X_c \,.$$

Submanifold

After introducing the covariant derivative on the submanifold and normal covariant derivative, we have

Gauss equation:

$$R_{abcd} = K_{ca}{}^{e} K_{bde} - K_{cb}{}^{e} K_{ade} + q_a{}^e q_b{}^f q_c{}^g q_d{}^h \; \mathscr{R}_{efgh} \,, \qquad (19)$$

Ricci equation:

$$\Omega_{abcd} = q_a{}^e q_b{}^f h_c{}^g h_d{}^h \mathscr{R}_{efgh} + K_{aed} K_b{}^e{}_c - K_{bed} K_a{}^e{}_c \,. \qquad (20)$$

Codazzi equation:

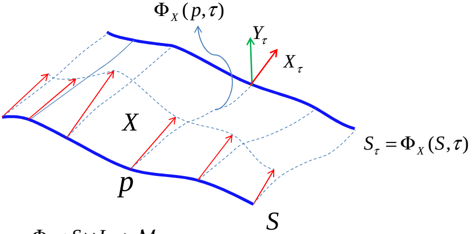
$$\tilde{D}_a K_{bcd} - \tilde{D}_b K_{acd} = -q_a{}^e q_b{}^f q_c{}^h h_d{}^g \mathscr{R}_{efhg} \,. \qquad (21)$$

For an arbitrary normal vector Y, it gives

$$\left(\frac{n-3}{n-2}\right) D_a \theta^{(Y)} - D_b \sigma^{(Y)b}_{\;a} + K_d \tilde{D}_a Y^d - K_a{}^b{}_d \tilde{D}_b Y^d = q_a{}^e q^{bc} Y^d \mathscr{R}_{ebcd} \,. \qquad (22)$$

The deformation defined by Andersson et al

L. Andersson, M. Mars and W. Simon, Phys. Rev. Lett. 95, 111102 (2005); Adv. Theor. Math. Phys. 12, 853 (2008).

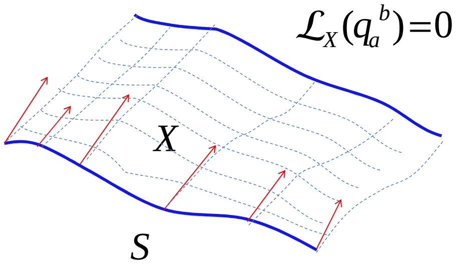


$$\Phi_X : S \times I \to \mathcal{M}$$

$$\delta_X \theta^{(Y)} \equiv \partial_\tau \theta^{(Y_\tau)}\Big|_{\tau=0}$$

Our definition of the deformation

The deformation operator is Lie derivative constrained by $\mathcal{L}_X({q_a}^b) = 0$.



Our definition of the deformation

By this definition of deformation, we have

$$\mathcal{L}_X q_{ab} = q_a{}^c q_b{}^d \mathcal{L}_X q_{cd} = -2K_{ab}{}^c X_c = 2K^{(X)}_{ab} \,. \tag{23}$$

Similarly, one finds the expansion along X can be expressed as

$$\mathcal{L}_X \epsilon_q = \theta^{(X)} \epsilon_q \,, \tag{24}$$

where ϵ_q is the area element of the $(n-2)$-dimension submanifold S.

Deformation equation with an arbitrary codimension

By using our requirement $\mathcal{L}_X({q_a}^b) = 0$, we have

$$\begin{aligned}\mathcal{L}_X K^{(Y)}_{ab} &= {q_a}^c {q_b}^d X^e Y^f \mathscr{R}_{ecdf} + K^{(Y)c}_{a} K^{(X)}_{bc} - Y^c \tilde{D}_a \tilde{D}_b X_c \\ &\quad + K_{acb}\left(Y_d \tilde{D}^c X^d\right) - K_{abc}\left(X^d \nabla_d Y^c\right) . \qquad (25)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_X \theta^{(Y)} &= q^{cd} X^e Y^f \mathscr{R}_{ecdf} - K^{(Y)ab} K^{(X)}_{ab} \\ &\quad - Y^c \tilde{D}_a \tilde{D}^a X_c - K_c\left(X^d \nabla_d Y^c\right) . \qquad (26)\end{aligned}$$

Deform along tangent direction

For a tangent vector, for example, ϕ^a, the Lie derivative of $\theta^{(Y)}$ along ϕ^a is constrained by the Codazzi equations (21) and (22):

$$\left(\frac{n-3}{n-2}\right)\mathcal{L}_\phi \theta^{(Y)} = \phi^a D_b \sigma^{(Y)b}_{\ a} - \left(\frac{n-3}{n-2}\right)\phi^a K_d \tilde{D}_a Y^d + \phi^a C_a{}^b{}_d \tilde{D}_b Y^d + q^{fg}\phi^e Y^h \mathscr{R}_{efgh}\,. \tag{27}$$

Codimension-1

In the case of codimension-1, we can set $h_{ab} = -u_a u_b$, where u^a is an unit timelike normal vector of the hypersurface. So the extrinsic curvature is simply given by $K_{abc} = K_{ab} u_c$. In this case, X is just the evolution vector $X_a = N u_a$ with lapse function N. We can select $Y_a = u_a$ such that $\theta^{(Y)}$ is given by

$$\theta^{(Y)} = K = -K^a u_a,$$

then we have

$$-\frac{1}{N}\mathcal{L}_X K_{ab} = -q_a{}^c q_b{}^d \mathscr{R}_{cd} + R_{ab} + K K_{ab} - 2K_{ac}K_b{}^c - \frac{1}{N} D_a D_b N \tag{28}$$

$$-\frac{1}{N}\mathcal{L}_X K = \mathscr{R}_{ab} u^a u^b + K^{ab} K_{ab} - \frac{1}{N} D^a D_a N\,. \tag{29}$$

These are just the evolution equations of hypersurface in Einstein gravity theory.

Codimension-2

From the Gauss equation (19), we find that eq.(26) becomes

$$\begin{aligned} \mathcal{L}_X \theta^{(Y)} &= -\left(\mathscr{G}_{ab} + K_{cda} K^{cd}{}_b\right) \left[X^a Y^b - h^{ab} \left(X_e Y^e\right)\right] \\ &\quad + \frac{1}{2} \left(R - K_{abc} K^{abc} - K_c K^c\right) \cdot \left(X_e Y^e\right) \\ &\quad - Y^e \tilde{D}_c \tilde{D}^c X_e - K_c \left(X^e \nabla_e Y^c\right) . \end{aligned} \tag{30}$$

Local frames

For codimension-2 case, we can introduce two null vector fields ℓ and n such that the Lorentz part of the metric has a form

$$h_{ab} = -\ell_a n_b - n_a \ell_b = \varepsilon_{IJ} e^I_a e^J_b \,. \tag{31}$$

where I and J take values $\{1, 2\}$, and $e^1 = \ell$, $e^2 = n$. The symbol ε_{IJ} represents a constant matrix given by

$$\varepsilon_{11} = \varepsilon_{22} = 0,\ \varepsilon_{12} = \varepsilon_{21} = -1$$

Local frame

For an arbitrary normal vector $X_a = \alpha \ell_a + \beta n_a$, it's easy to find

$$\tilde{D}_a X_b = (D_a \alpha + \omega_a \alpha)\ell_b + (D_a \beta - \omega_a \beta) n_b \,, \tag{32}$$

where ω_a is defined as

$$\omega_a = -q_a{}^e n_d \nabla_e \ell^d \,. \tag{33}$$

ω_a is the $SO(1,1)$ connection of the $SO(1,1)$ normal bundle.

Local frame

After some calculation, we find

$$
\begin{aligned}
Y^e \tilde{D}_c \tilde{D}^c X_e &= \varepsilon_{IJ}\left(Y^I D^c D_c X^J\right) + 2\varepsilon_{IJ}\left(\omega^c \epsilon_{bd} Y^b e^{Id} D_c X^J\right) \\
&+ D_c \omega^c \left(\epsilon_{bd} Y^b X^d\right) + \omega_c \omega^c (X_e Y^e)\,. \qquad (34)
\end{aligned}
$$

Focusing and cross focusing equations

By setting $Y_a = \ell_a$ and $X_a = A\ell_a - Bn_a$, we have $X_e Y^e = B$. So eqs.(30) and (34) give result

$$
\begin{aligned}
\mathcal{L}_X \theta^{(\ell)} &= \kappa_X \theta^{(\ell)} - D_c D^c B + 2\omega^c D_c B \\
&- B\Big[\omega_c \omega^c - D_c \omega^c + \mathscr{G}_{ab}\ell^a n^b - \frac{1}{2}R - \theta^{(\ell)}\theta^{(n)}\Big] \\
&- A\Big[\mathscr{G}_{ab}\ell^a \ell^b + \sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + \frac{1}{n-2}\theta^{(\ell)}\theta^{(\ell)}\Big]\,. \qquad (35)
\end{aligned}
$$

Here, we have introduced an important quantity—"surface gravity"

$$\kappa_X = -n^c X^e \nabla_e \ell_c\,, \qquad (36)$$

In the case where $A = 1, B = 0$, eq.(35) just the so called focusing equation. In the case where $A = 0, B = -1$, eq.(35) gives the cross focusing equation

Focusing and cross focusing equations

Similarly, by setting $Y_a = n_a$ and $X_a = A\ell_a - Bn_a$, we get

$$\begin{aligned}
&\mathcal{L}_X\theta^{(n)} = -\kappa_X\theta^{(n)} + D_cD^cA + 2\omega^cD_cA \\
&+A\Big[\omega_c\omega^c + D_c\omega^c + \mathscr{G}_{ab}n^a\ell^b - \frac{1}{2}R - \theta^{(\ell)}\theta^{(n)}\Big] \\
&+B\Big[\mathscr{G}_{ab}n^an^b + \sigma^{(n)}_{ab}\sigma^{(n)ab} + \frac{1}{n-2}\theta^{(n)}\theta^{(n)}\Big]\,. \qquad (37)
\end{aligned}$$

Y is dual to X

we have

$$\kappa_X \theta^{(X)} = \mathscr{G}_{ab} X^a Y^b + \sigma^{(X)}_{ab} \sigma^{(Y)ab} + \frac{1}{n-2} \theta^{(X)} \theta^{(Y)} + D_e (A D^e B - B D^c A - 2AB\omega^e) + A \mathcal{L}_X \theta^{(\ell)} + B \mathcal{L}_X \theta^{(n)} \,, \tag{38}$$

and

$$\int \kappa_X \mathcal{L}_X \epsilon_q = \int \epsilon_q \left[\mathscr{G}_{ab} X^a Y^b + \sigma^{(X)}_{ab} \sigma^{(Y)ab} + \frac{1}{n-2} \theta^{(X)} \theta^{(Y)} \right] + \int \epsilon_q \left[A \mathcal{L}_X \theta^{(\ell)} + B \mathcal{L}_X \theta^{(n)} \right] \,, \tag{39}$$

This is a very important equation to study the thermodynamic of quasi-local horizon.

Damour-Navier-Stokes like Equation

From the definition of ω_a in eq.(33), it's not hard to find

$$\mathcal{L}_X\omega_a = K_a{}^b{}_c\tilde{D}_b(\epsilon^{cd}X_d) + D_a\kappa_X - \frac{1}{2}q_a{}^b X^d \epsilon^{ce}\mathscr{R}_{dbce}\,. \qquad (40)$$

From the generalized Codazzi equation (22), we have

$$\mathcal{L}_X\omega_a = D_a\kappa_X + \left(\frac{n-3}{n-2}\right) D_a\theta^{(Y)} - D_c\sigma^{(Y)c}_a + K_c\tilde{D}_aY^c + q_a{}^bY^c\mathscr{G}_{bc}\,, \qquad (41)$$

where $Y_a = \epsilon_{ab}X^b$.

In the case where X is self-dual or anti-self-dual, i.e., $X = \pm Y$, by considering the Einstein equation, this equation is a kind of *Damour-Navier-Stokes equation*.

Damour-Navier-Stokes like Equation

Let ϕ^a be a tangent vector which satisfies $\mathcal{L}_X \phi^a = 0$ and $D_a \phi^a = 0$, then we get

$$\begin{aligned} \mathcal{L}_X \int \epsilon_q \left(\phi^a \omega_a\right) &= \int \epsilon_q \Bigg\{ \frac{1}{2} \left(D^a \phi^b + D^b \phi^a \right) \sigma^{(Y)}_{ab} + \phi^a Y^b \mathscr{G}_{ab} \\ &\quad + A \phi^a D_a \theta^{(\ell)} + B \phi^a D_a \theta^{(n)} \Bigg\} . \end{aligned} \tag{42}$$

The angular momentum can be defined as

$$J_\phi = \int \epsilon_q (\phi^a \omega_a) .$$

So, from Damour-Navier-Stokes equation, we can get the deformation equation of angular momentum.

Trapping horizon

- The codimension-2 spacelike surface with $\theta^{(\ell)}\theta^{(n)} = 0$ is called *marginal trapped surface*.
- The surface with $\theta^{(\ell)}\theta^{(n)} > 0$ is called *trapped*, and $\theta^{(\ell)}\theta^{(n)} < 0$ is called *untrapped*.
- A *trapped* (*untrapped*) *region* is the union of all trapped (untrapped) surfaces.

We can give similar definitions by using the extrinsic curvature vector K^a from the relation

$$K^c K_c = -2\theta^{(\ell)}\theta^{(n)} .$$

Trapping horizon

- A marginal trapped surface is called *future* if $\theta^{(\ell)} = 0$, $\theta^{(n)} < 0$.
  - (i). if $\mathcal{L}_n \theta^{(\ell)} < 0$, we call the future marginal trapped surface is *outer*.
  - (ii). if $\mathcal{L}_n \theta^{(\ell)} > 0$, the future marginal trapped surface is called *inner*.
- The *past* marginal trapped surface is defined by $\theta^{(n)} = 0$, $\theta^{(\ell)} > 0$.
  - (i). The past marginal trapped surface with $\mathcal{L}_\ell \theta^{(n)} > 0$ is called outer.
  - (ii). The past marginal trapped surface with $\mathcal{L}_\ell \theta^{(n)} < 0$ is called inner.

The so called *trapping horizon* is the closure of a hypersurface foliated by the marginal trapped surfaces[Hayward,1994].

The classification of the trapping horizon inherits from the classification of the marginal trapped surfaces.

Evolution vector

The trapping horizon is foliated by marginal trapped surfaces S_τ. Here τ is called the foliation parameter of the tapping horizon. Assume X is the so called "evolution" vector, i.e., the vector which is tangent to $\mathcal{H}$ and normal to S_τ and satisfies $\mathcal{L}_X \tau = 1$.

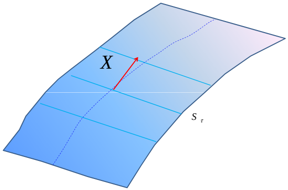


Figure: The evolution vector on trapping horizon

Methods to study horizon thermodynamics

There are two ways to study the thermodynamics of quasilocal horizons:

- The method with quasi-local energy
  This method heavily depend on some quasilocal energy inside a given codimension-2 surface. By using Einstein equation and the quasilocal energy, one may directly gets some first law like equation or balance equation.
  deformation of energy $\Rightarrow$ matter flux $+$ gravitational radiation

Methods to study horizon thermodynamics

- The method without quasi-local energy
  - (i). Focusing (cross focusing) equations + energy flux⇒Clausius like equation
  - (ii). Damour-Stokes equation ⇒ vaiation of angular momentum of the horizon.
  - (iii). First law of thermodynamics is still valid on the quasilocal horizon

  The first law like equation will give the energy of the horizon. We need not introduce some quasilocal energy in advance. Contrarily, the energy of the horizon can be regarded as a byproduct of the theory

The method with quasi-local energy: spherically symmetric case

For generally spherically symmetric spacetime

$$g = \beta_{\mu\nu}(y)dy^{\mu}dy^{\nu} + r(y)^2\gamma_{ij}(z)dz^i dz^j \,, \tag{43}$$

We have generalized Misner-Sharp energy:

$$\mathscr{E} = \frac{(n-2)\Omega_{n-2}}{16\pi G}r^{n-3}\left(1 - \nabla_a r \nabla^a r\right) \,, \tag{44}$$

By defining

$$\psi_a = \mathscr{T}_{ab}\nabla^b r + w\nabla_a r \,, \qquad w = -\frac{1}{2}h^{ab}\mathscr{T}_{ab} \,, \tag{45}$$

we get

$$\mathcal{L}_X\mathscr{E} = \mathscr{A}\psi_a X^a + w\mathcal{L}_X\mathscr{V} \,, \tag{46}$$

The method with quasi-local energy: spherically symmetric case

By selecting X to be the evolution vector on the trapping horizon , on the trapping horizon, we have

$$\mathscr{A}\psi_a X^a = \left(\frac{\kappa}{2\pi}\right)\mathcal{L}_X S\,, \tag{47}$$

The surface gravity is defined as

$$\frac{\kappa}{2\pi} = \frac{4G}{n-2}\left[\left(\frac{n-3}{\Omega_{n-2}}\right)\frac{\mathscr{E}}{r^{n-2}} - wr\right]\,. \tag{48}$$

The evolution of $\mathscr{E}$ on the trapping horizon becomes

$$\mathcal{L}_X\mathscr{E} = \left(\frac{\kappa}{2\pi}\right)\mathcal{L}_X S + w\mathcal{L}_X\mathscr{V}\,. \tag{49}$$

This is a first law like equation.

The method with quasi-local energy: general cases

For general case, we can define a generalized energy

$$\mathscr{E} = \frac{\left(\int \epsilon_q\right)^{\frac{n-3}{n-2}}}{16\pi G \left(\Omega_{n-2}\right)^{\frac{1}{n-2}} (n-3)} \left\{ \frac{\int \epsilon_q R}{\left(\int \epsilon_q\right)^{\frac{n-4}{n-2}}} - \left(\frac{n-3}{n-2}\right) \frac{\int \epsilon_q K_c K^c}{\left(\int \epsilon_q\right)^{\frac{n-4}{n-2}}} \right\}. \tag{50}$$

- for $n = 4$, this energy reduces to usual four dimension Hawking energy (mass).
- In spherically symmetric case, this energy reduces to Misner-Sharp energy ($n \geq 4$).

The method with quasi-local energy: general cases

The deformation of the energy is given by

$$\mathcal{L}_X \mathscr{E} = \left(\frac{n-3}{n-2}\right)\left(\frac{\mathscr{E}}{\mathscr{A}}\right)\mathcal{L}_X \mathscr{A} + \mathscr{A}^{\frac{n-3}{n-2}}\mathcal{L}_X\left(\frac{\mathscr{E}}{\mathscr{A}^{\frac{n-3}{n-2}}} - \mathscr{K}\right) \quad (51)$$

if $\mathcal{L}_X \mathscr{K} = 0$, where

$$\mathscr{K} = \frac{1}{16\pi G\left(\Omega_{n-2}\right)^{\frac{1}{n-2}}(n-3)}\left(\frac{\int \epsilon_q R}{\left(\int \epsilon_q\right)^{\frac{n-4}{n-2}}}\right), \quad (52)$$

and on the horizon, by selecting X to be the evolution vector of the horizon, we have

$$\mathcal{L}_X \mathscr{E} = \left(\frac{n-3}{n-2}\right)\left(\frac{\mathscr{E}}{\mathscr{A}}\right)\mathcal{L}_X \mathscr{A}\,. \quad (53)$$

The method with quasi-local energy: general cases

Since we have

$$\mathcal{L}_X \mathscr{K} \sim \left(\frac{n-4}{n-2}\right) \int \epsilon_q \left\{ R \left(\frac{\mathcal{L}_X \mathscr{A}}{\mathscr{A}} + K^e X_e \right) \right\} , \qquad (54)$$

- Obviously, the right hand of above equation is identically vanished in the four dimension because that $\int \epsilon_q R / (\int \epsilon_q)^{\frac{n-4}{n-2}}$ is just the Euler number of some two dimensional closed manifold.
- If the codimension-2 surface is assumed to be a closed Einstein manifold (R is a constant).
- Selecting a special deformation vector X such that $K^a X_a$ is a constant on the codimension-2 surface, then we have $\mathcal{L}_X \mathscr{A} / \mathscr{A} + K^e X_e = 0$.

The method with quasi-local energy: general cases

More detailed, the deformation of the energy is given by

$$\mathcal{L}_X \mathscr{E} = \int \epsilon_q \left\{ \left(\frac{\mathcal{E}}{n-2} \right) \left(\frac{\mathcal{L}_X \mathscr{A}}{\mathscr{A}} + K_e X^e \right) \right\}$$

$$+ \frac{1}{8\pi G} \left(\frac{L}{n-2} \right) \int \epsilon_q \Bigg\{ - K^e \tilde{D}_c \tilde{D}^c X_e$$

$$- \left(\mathscr{G}_{ab} + C_{cda} C^{cd}{}_b \right) \left[K^a X^b - \frac{1}{2} h^{ab} \left(K_e X^e \right) \right]$$

$$+ \frac{1}{2} \left(\mathscr{G}_{ab} h^{ab} \right) \cdot \left(K_e X^e \right) \Bigg\}, \qquad (55)$$

where $L = \mathscr{A}^{\frac{1}{n-2}} / \left(\Omega_{n-2} \right)^{\frac{1}{n-2}}$,

The method with quasi-local energy: general cases

and

$$\mathcal{E} = \frac{L}{16\pi G(n-3)}\left[R - \left(\frac{n-3}{n-2}\right)K_c K^c\right] . \tag{56}$$

Here, $\mathcal{E}$ is a quantity like an energy density.

- In the general spherically symmetric cases, it reduces to the one given in eq.(46).
- If $n = 4$, the energy $\mathscr{E}$ is the Hawking mass, and eq.(55) reduces to the one given by Bray et.al[Bray,2006]. Of course, we can also consider the cases with cosmological constant as in [Bray,2006].
- It's interest to study the monotonicity of this energy $\mathscr{E}$ as in [Bray,2006,Mizuno,2009].

The method with quasi-local energy: general cases

If we introduce null frames, generally, the evolution of the energy on the trapping horizon is given by

$$\mathcal{L}_X \mathscr{E} = \int \epsilon_q \left[\alpha \left(\mathscr{T}_{ab}\ell^a \ell^b + \frac{1}{8\pi G} \sigma^{(\ell)}_{ab} \sigma^{(\ell)ab} \right) + \beta \left(\mathscr{T}_{ab}\ell^a n^b + \frac{\zeta_a \zeta^a}{8\pi G} \right) \right]. \quad (57)$$

where α and β are determined by the components of X, and ζ has close relation to the SO(1,1) connection ω_a.

- The contribution of the usual matter fields — $\mathscr{T}_{ab}\ell^a\ell^a$ and $\mathscr{T}_{ab}\ell^a n^a$;
- The contribution of the gravitational radiation — $\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab}$ and $\zeta_a\zeta^a$.

Method without quasi-local energy: equilibrium state

Equilibrium state: Null trapping horizon.

The evolution vector X is null. From focusing and cross focussing equations, we find: On the null future trapping horizon, we have

$$\sigma_{ab}^{(\ell)} = 0\,, \qquad \mathscr{G}_{ab}\ell^a\ell^b = 0\,, \tag{58}$$

and on the null past trapping horizon, we have

$$\sigma_{ab}^{(n)} = 0\,, \qquad \mathscr{G}_{ab}n^a n^b = 0\,. \tag{59}$$

$\mathscr{G}_{ab}\ell^a\ell^b = 0$ and $\mathscr{G}_{ab}n^a n^b = 0$ just imply that there are no matter flux across the codimension-2 surface. $\sigma_{ab}^{(\ell)} = 0$ and $\sigma_{ab}^{(n)} = 0$ means that there are no gravitational radiation across the codimension-2 surface.

Method without quasi-local energy: equilibrium state

While, from Damour-Navier-Stokes equation, we find

$$\mathcal{L}_X \omega_a - D_a \kappa_X = 0 \,. \tag{60}$$

if one requires that ω_a does not evolve, i.e., $\mathcal{L}_X \omega_a = 0$, then, from above equation, one gets $D_a \kappa_X = 0$ on the codimension-2 surface.

Furthermore, if $\mathcal{L}_X \kappa_X = 0$ is required, then κ_X is a constant on the null trapping horizon.

In these null cases, we have

$$X^a \nabla_a X^b = \pm \kappa_X X^b \,, \tag{61}$$

Method without quasi-local energy: equilibrium state

Conclusively, on these null trapping horizons, there are no gravitational radiation and matter flux, and κ_X's are constants. These properties correspond to the equilibrium state of the thermodynamics of the horizon. Further, now eqs.(39) and (42) just mean

$$\left(\frac{\kappa_X}{2\pi}\right)\mathcal{L}_X S = 0\,, \qquad \mathcal{L}_X J_\phi = 0\,, \tag{62}$$

where $S \sim \int \epsilon_q$ and $J_\phi \sim \int \epsilon_q\,(\phi^a\omega_a)$ can be explained as the entropy and the angular momentum associated with the null trapping horizons.

Method without quasi-local energy: Near equilibrium state

The near equilibrium means that X is almost a null vector.

$$X^a = \ell^a - Cn^a \,, \tag{63}$$

For the future trapping horizon, Booth *et.el.*(2003) give three *slowly expanding conditions* :

(F-i). The so called evolving parameter $\epsilon \ll 1$ with

$$\frac{\epsilon^2}{L^2} = \max\left[|C|\left(\|\sigma^{(n)}\|^2 + (8\pi G)\mathscr{T}_{ab}n^a n^b + \frac{1}{n-2}\theta^{(n)}\theta^{(n)}\right)\right]; \tag{64}$$

(F-ii). The Ricci scalar, the $SO(1,1)$ normal connection and the energy-momentum tensor satisfy

$$|R|\,, \quad \|\omega_a\|^2 \quad \text{and} \quad (8\pi G)\mathscr{T}_{ab}\ell^a n^b \preceq \frac{1}{L^2}\,;$$

Method without quasi-local energy: Near equilibrium state

(F-i). The derivatives of horizon fields are at most the same order in ϵ as the (maximum of the) original fields. For example,

$$\|D_a C\| \preceq \frac{C_m}{L}\,, \qquad \|D_a D_b C\| \preceq \frac{C_m}{L^2}\,.$$

Here, $\|\cdot\|$ is the norm of (tangent) tensor fields on the codimension-2 Riemannian manifold, while $|\cdot|$ is the absolute value of some scalar. The quantity L is some length scale of the codimension-2 surface. For example, the radius of the closed $(n-2)$ manifold: $L = (\mathscr{A}/\Omega_{n-2})^{\frac{1}{n-2}}$ which has been defined just bellow eq.(55). C_m is the maximum value of $|C|$ on the codimension-2 surface. The relation $E \preceq F$ means $E \leq k_0 F$ for some constant k_0 of order one.

Method without quasi-local energy: Near equilibrium state

The slowly evolving parameter ϵ defined in the condition (F-i) is independent of the relabeling of the foliation and the rescaling of the null frame.

Remembering in the case of future null trapping horizon, to ensure that some physical quantities do not evolve, we have required the condition $\mathcal{L}_X \omega_a = 0$ and $\mathcal{L}_X \kappa_X = 0$. These just mean that ω_a and κ_X do not evolve respect to the evolution vector X^a. Similarly, here there are also *slowly evolving conditions* :

(F-i'). $\|\mathcal{L}_X \omega_a\|$ and $|\mathcal{L}_X \kappa_X| \preceq \epsilon/L^2$;

(F-ii'). $|\mathcal{L}_X \theta^{(n)}| \preceq \epsilon/L^2$.

Method without quasi-local energy: Near equilibrium state

For the past trapping horizon, we can gives similar conditions to describe the slowly expanding properties:

(P-i). The evolving parameter $\epsilon \ll 1$ with

$$\frac{\epsilon^2}{L^2} = \max\left[|C|\left(\|\sigma^{(\ell)}\|^2 + (8\pi G)\mathscr{T}_{ab}\ell^a\ell^b + \frac{1}{n-2}\theta^{(\ell)}\theta^{(\ell)}\right)\right];$$

(P-ii). The Ricci scalar, the $SO(1,1)$ normal connection and the energy-momentum tensor satisfy

$$|R|, \quad \|\omega_a\|^2 \quad \text{and} \quad (8\pi G)\mathscr{T}_{ab}\ell^a n^b \preceq \frac{1}{L^2};$$

(P-iii). The derivatives of horizon fields are at most the same order in ϵ as the (maximum of the) original fields. For example,

$$\|D_a C\| \preceq \frac{C_m}{L}, \qquad \|D_a D_b C\| \preceq \frac{C_m}{L^2}.$$

Method without quasi-local energy: Near equilibrium state

The slowly evolving conditions are given:

(P-i'). $\|\mathcal{L}_X \omega_a\|$ and $|\mathcal{L}_X \kappa_X| \preceq \epsilon / L^2$;

(P-ii'). $|\mathcal{L}_X \theta^{(\ell)}| \preceq \epsilon / L^2$.

With these conditions, one can find that κ_X is nearly a constant on the past trapping horizon. So it can also be expanded as

$$\kappa_X = \kappa_o + \mathscr{O}(\epsilon)$$

.

Method without quasi-local energy: Near equilibrium state

Clausiu like equations:
For the future slowly evolving trapping horizon

$$\left(\frac{\kappa_o}{8\pi G}\right)\mathcal{L}_X\mathscr{A} = \int \epsilon_q\left[\mathscr{T}_{ab}\ell^a\ell^b + \sigma^{(\ell)}_{ab}\sigma^{(\ell)ab}\right], \qquad (65)$$

Similarly, for the past slowly evolving horizon, we have

$$-\left(\frac{\kappa_o}{8\pi G}\right)\mathcal{L}_X\mathscr{A} = \int \epsilon_q\left[\mathscr{T}_{ab}n^a n^b + \sigma^{(n)}_{ab}\sigma^{(n)ab}\right]. \qquad (66)$$

Geometry of FRW universe

The metric of the FLRW universe $(\mathcal{M}, g)$ is

$$g = -dt^2 + \frac{a^2}{1-kr^2}dr^2 + a^2 r^2 d\Omega_{n-2}^2 \,, \tag{67}$$

by introducing two null vectors ℓ and n

$$\ell_a dx^a = \sqrt{\frac{1}{2}}\left(-dt + \frac{a}{\sqrt{1-kr^2}}dr\right) \,, \tag{68}$$

$$n_a dx^a = \sqrt{\frac{1}{2}}\left(-dt - \frac{a}{\sqrt{1-kr^2}}dr\right) \,. \tag{69}$$

So we have $h_{ab} = -\ell_a n_b - n_a \ell_b$, while q_{ab} is just the metric for the sphere part, i.e.,

$$q_{ab} dx^a dx^b = a^2 r^2 d\Omega_{n-2}^2.$$

Geometry of FRW universe

The expansions of the sphere along these two null directions are given by

$$\theta^{(\ell)} = q^{ab}\nabla_a \ell_b = \sqrt{2}\left(H + \sqrt{\frac{1}{\tilde{r}^2} - \frac{k}{a^2}}\right), \qquad (70)$$

$$\theta^{(n)} = q^{ab}\nabla_a n_b = \sqrt{2}\left(H - \sqrt{\frac{1}{\tilde{r}^2} - \frac{k}{a^2}}\right). \qquad (71)$$

Here $\tilde{r}$ is defined as $\tilde{r} = ar$.

Geometry of FRW universe

It's also easy to find

$$\begin{aligned}
\mathcal{L}_\ell \theta^{(\ell)} &= \dot{H} - \frac{1}{\tilde{r}^2} - H\sqrt{\frac{1}{\tilde{r}^2} - \frac{k}{a^2}}\,, \\
\mathcal{L}_n \theta^{(\ell)} &= \dot{H} + \frac{1}{\tilde{r}^2} - H\sqrt{\frac{1}{\tilde{r}^2} - \frac{k}{a^2}}\,, \\
\mathcal{L}_\ell \theta^{(n)} &= \dot{H} + \frac{1}{\tilde{r}^2} + H\sqrt{\frac{1}{\tilde{r}^2} - \frac{k}{a^2}}\,, \\
\mathcal{L}_n \theta^{(n)} &= \dot{H} - \frac{1}{\tilde{r}^2} + H\sqrt{\frac{1}{\tilde{r}^2} - \frac{k}{a^2}}\,.
\end{aligned} \tag{72}$$

From these equations, we can find trapping horizons and realize the classification of the horizons.

Null trapping horizons

Null trapping horizons exist only when $k = 0$. Further, only inner horizon exists in the future case, and only outer horizon exists in the past case. In the following discussions, we always set $k = 0$. On the null trapping horizons (future and past), the Hubble parameter H is always a constant.

We only consider the past outer case.

Slowly past evolving trapping horizons in FRW universe

The evolution vector X can be expressed as

$$X^a = \alpha \ell^a - n^a \,, \tag{73}$$

where

$$\alpha = \frac{\dot{H}}{\dot{H} + 2H^2} \,. \tag{74}$$

From the definition, the evolving parameter ϵ in the condition (P-i) becomes (we only consider the four dimension case, and choose L to be the radius $\tilde{r} = 1/|H|$ for $k = 0$.)

$$\frac{\epsilon^2}{\tilde{r}^2} = |\alpha| \left(\mathscr{G}_{ab} \ell^a \ell^b + \frac{1}{2} \theta^{(\ell)} \theta^{(\ell)} \right) \,. \tag{75}$$

Straightforward calculation shows: on the trapping horizons, ϵ's are given by

$$\epsilon^2 = |\alpha| \left(4 - \frac{\dot{H}}{H^2}\right) . \tag{76}$$

By defining

$$s = -\frac{\dot{H}}{H^2} > 0 \,, \tag{77}$$

then, from the expression of α in eq.(74), we have

$$\alpha = -\frac{s}{2-s} \,. \tag{78}$$

The evolution parameter ϵ now has a simple form

$$\epsilon^2 = s \left(\frac{4+s}{2-s}\right) . \tag{79}$$

So, the requirement of the evolving parameter $\epsilon \ll 1$ automatically implies that $s = -\dot{H}/H^2$ is very small.

It's not hard to find

$$\mathcal{L}_X \kappa_X = \frac{2H^2 s}{(2-s)^3}\left[2 - s + s^2 + \left(\frac{\ddot{H}}{\dot{H}H}\right)\right]. \tag{80}$$

So the slowly evolving condition of κ_X requires that $|\ddot{H}/H^3|$ is also a small quantity.

Thermodynamics on slowly evolving trapping horizon in FRW

For the past horizon, from eq.(66), we have

$$-\frac{\kappa_o}{8\pi G}\mathcal{L}_X\mathscr{A} = \int \epsilon_q \mathscr{T}_{ab} n^a n^b \,. \qquad (81)$$

Up to second order of ϵ (or the first order of s).

The temperature of the system can be expanded as

$$T = \frac{\kappa_X}{2\pi} \sim \frac{H}{2\pi}\left(1 - \frac{s}{2}\right) + \mathscr{O}(\epsilon^4)\,.$$

Temperature from the formalism with quasi-local energy

The surface gravity κ in eq.(48) becomes

$$\frac{\kappa}{2\pi} = -\frac{H}{2\pi}\left(1 - \frac{s}{2}\right) , \tag{82}$$

where s is defined in eq.(77). So the temperature of the past outer trapping horizon is

$$T = \frac{|\kappa|}{2\pi} = \frac{H}{2\pi}\left(1 - \frac{s}{2}\right)$$

The Clausius relation is

$$A\psi_a X^a = \frac{\kappa}{2\pi}\mathcal{L}_X S \, . \tag{83}$$

Questions and Discussions

- How to define a slowly evolving quasi-local horizon without using a local frame?
- The detailed relation between slowly evolving quasi-local horizon in FRW universe and slow-roll inflation is still not clear.
- How to establish a first law (on the general quasi-local horizon) which has a similar form like the one for stationary black hole?

Thanks for your attention!