The Beauty of String Perturbation Theory

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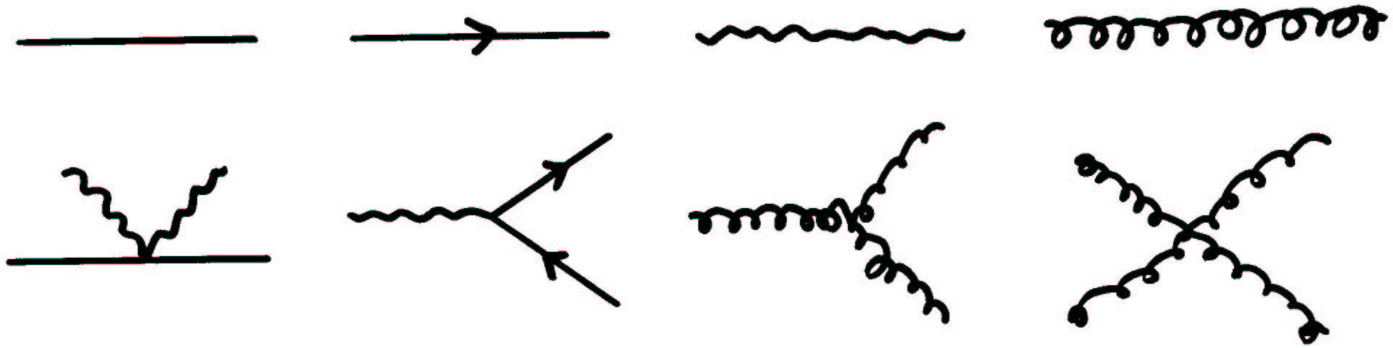
1. Introduction: Particles vs. Strings

Quantum theory and special theory of relativity $\Rightarrow$ Quantum Field Theory (QFT): the fundamental theory of elementary particles (the Standard Model).

Basic assumption: point particles with local interactions:



The problems are

- **The interactions are arbitrary**
- **The loop corrections are mostly divergent**

Partial solution

- **The gauge principle (“symmetry dictates interaction”)**
- **Renormalization**

This leads to the $SU(3)_c \otimes SU(2)_I \otimes U(1)_Y$ **Standard Model** which is the most precise theory of elementary particle physics.

However it left a lot of questions unanswered.

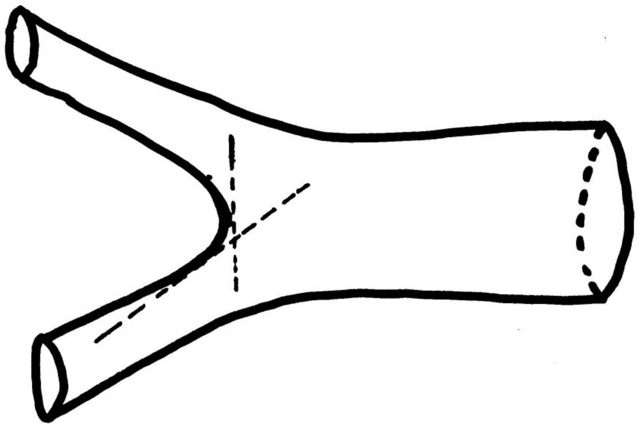
- The family problem (why 3?)
- Why the specific particle spectrum in each family?
- Why the 19 free parameters take the observed values?

• . . .

• **Gravity is not renormalizable. No quantum gravity**

The string jump

point particles $\Rightarrow$ 1-dimensional string with "local" interaction:



Why strings?

- May explain the structure of the Standard Model
- Natural step in unification: GUT, higher dimensions (Kaluza-Klein) and Supersymmetry
- Natural step to consider the extension of the point particle idea: unification of all particles
- Gravity is included and a finite theory of quantum gravity

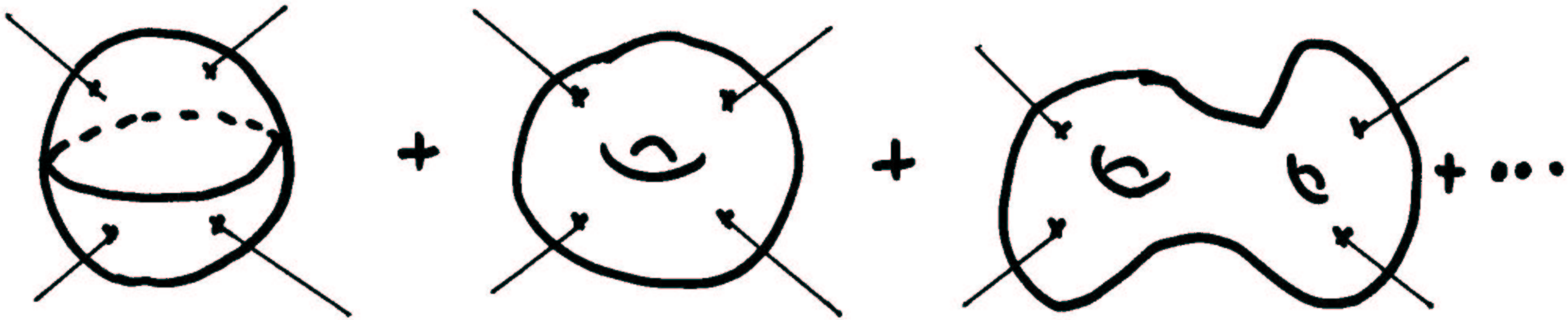
2. String Perturbation Theory

Amplitudes can be easily constructed. We have

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\text{top.}} \int \frac{\mathcal{D}X\mathcal{D}g}{\text{Vol.}(\text{Diff} \times \text{Weyl})} \mathrm{e}^{-S_X - \lambda\chi}$$

$$\times \prod_{i=1}^{n} \int \mathrm{d}^2\sigma_i (\det g(\sigma_i))^{1/2} V_{j_i}(k_i, \sigma_i)$$

$V_j(k,\sigma)$ **is a vertex operator describing a specific particle.**



3. Superstring Theory: The 3 Different Formalisms

1) Green-Schwarz formalism:

- **quantization only in light-cone gauge**
- **space-time super-symmetric string theory**
- **computation of tree amplitude is quite easy**
- **It has never been used for computing multi-particle and higher-loop amplitudes (dependence on insertion points)**

2) Ramond-Neveu-Schwarz (RNS) formalism

- **Spacetime supersymmetric only after GSO proj.**
- **Higher loops: summation over spin structure and modular invariance.**
- **Applied to multi-particle, higher-loop (2-loop, see below) and topological string theory amplitudes.**

3) Berkovits' pure spinor formalism

- **Lorentz covariant and manifestly spacetime supersymmetric (no summation over spin structures).**
- **All integer dimensional free fields on (ordinary) Riemann surface.**
- **Shortcoming: pure spinor constraints and very complicated composite $\tilde{b}$ fields.**

4. String Amplitudes: Some Explicit Results

The n $(NS, \widetilde{NS})$ particle amplitudes:

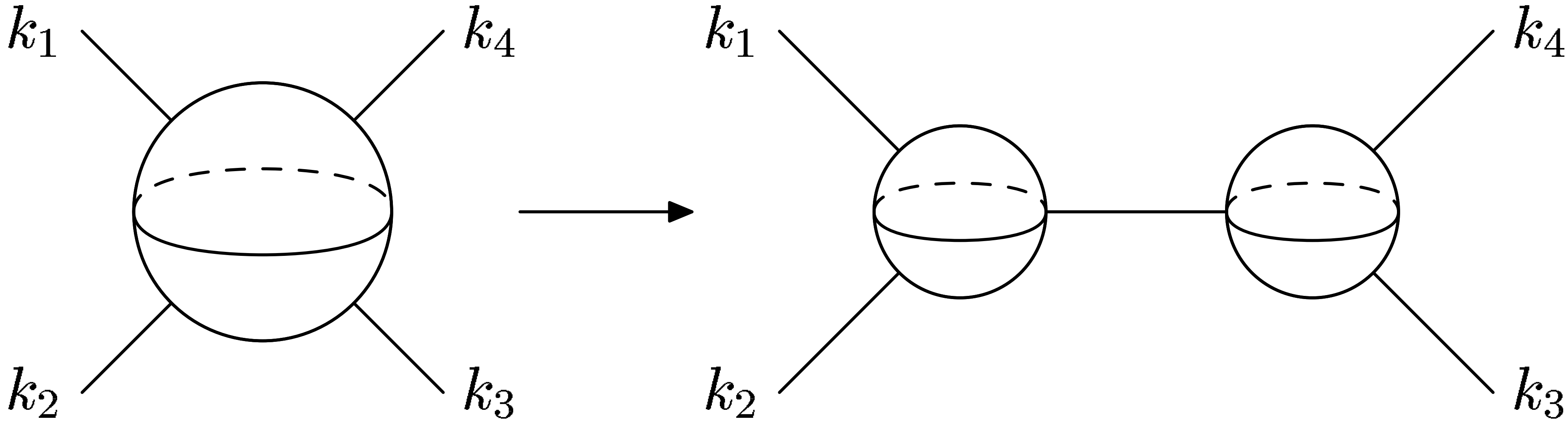
$$
\begin{aligned}
i\mathcal{A}_n(k_i, \epsilon_i) \; = \; & \int \prod_{i=4}^{n} \mathrm{d}^2 z_i \langle [c\mathcal{V}_B^{(-1)}](z_1, k_1, \epsilon_1)[c\mathcal{V}_B^{(0)}](z_2, k_2, \epsilon_2) \\
& \times [c\mathcal{V}_B^{(-1)}](z_3, k_3, \epsilon_3) \prod_{i=4}^{n} [c\mathcal{V}_B^{(0)}](z_i, k_i, \epsilon_i) \\
& \times (\textbf{right-moving part}) \rangle,
\end{aligned}
$$

To compute the 4-particle we do need the right-moving part to get the full amplitude.

Explicitly:

$$
\begin{aligned}
i\mathcal{A}_3(k_i,\epsilon_i,\tilde{\epsilon}_i) &= 4\,\pi i\, g_c\; K_3(k_i,\epsilon_i)\tilde{K}_3(k_i,\tilde{\epsilon}_i) \\
i\mathcal{A}_4(k_i,\epsilon_i,\tilde{\epsilon}_i) &= c \times \frac{-i\kappa^2(\alpha')^3}{4} K(k_i,\epsilon_i)\tilde{K}(k_i,\tilde{\epsilon}_i) \\
&\quad \times \frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}
\end{aligned}
$$

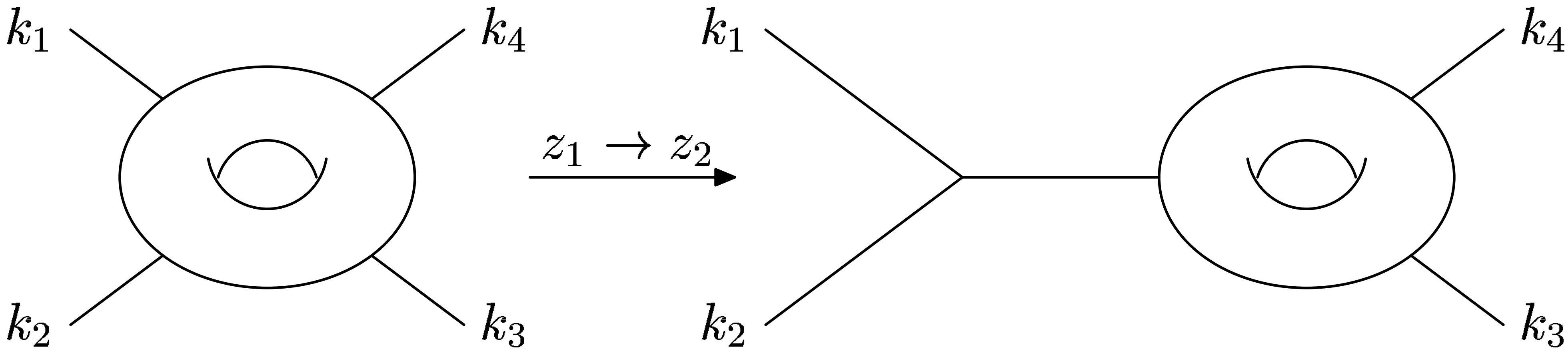
Factorization gives $c = 1$. See Polchinski's book (vol. 2). $\alpha' = 2$. $K_3(k_i,\epsilon_i)$ and $K(k_i,\epsilon_i)$ are kinematic factors and s, t, u are Mandelstam variables: $s = -(k_1+k_2)^2, \cdots$.


k_1
k_4
k_2
k_3
k_1
k_4
k_2
k_3

One-loop amplitudes: the massless 4-particle case

$$\mathcal{A}_4^{1-\text{loop}} = g_4^{1-\text{loop}} K(k_i, \epsilon_i) \int_F \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\tau)^2} \int \prod_{i=1}^{4} \frac{\mathrm{d}^2 z_i}{\mathrm{Im}\tau} \times \prod_{r<s} \left| \frac{\Theta_1(z_{rs}|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\mathrm{Im}\tau} (\mathrm{Im} z_{rs})^2 \right) \right|^{\alpha' k_r \cdot k_s}$$

You may fix one z_i to an arbitrary point.



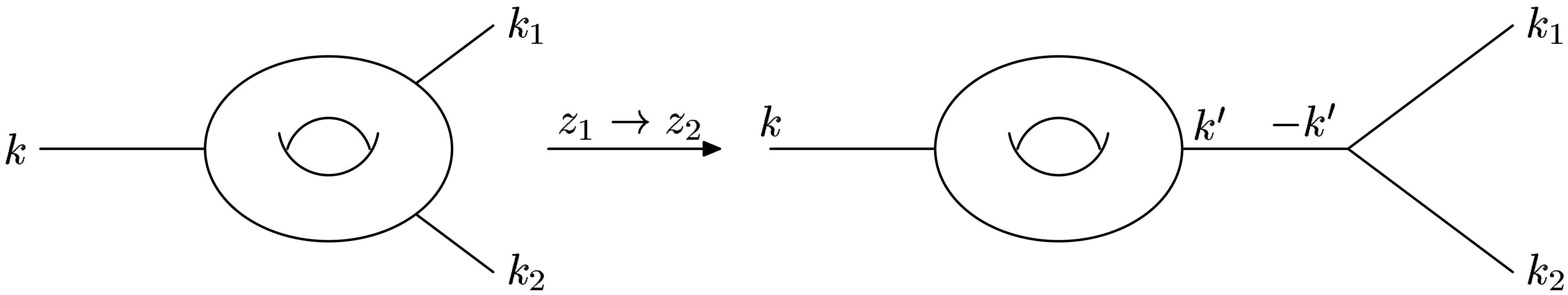
k_1
k_4
k_2
k_3
$z_1 \to z_2$
k_1
k_4
k_2
k_3

One-loop amplitudes: the 3-particle case—2 massless and 1 massive

$$\mathcal{A}^{1-\text{loop}}_{MBB}(k_1, k_2, k) = g_M^{1-\text{loop}} K_M \tilde{K}_M \int_F \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\tau)^5} \int \prod_{i=1}^{2} \mathrm{d}^2 z_i \times \prod_{r<s}^{3} \left| \frac{\Theta_1(z_{rs}|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\mathrm{Im}\tau} (\mathrm{Im} z_{rs})^2 \right) \right|^{\alpha' k_r \cdot k_s}$$

$$g_4^{1-\text{loop}} = \frac{g_M \, g_M^{1-\text{loop}}}{4\pi/\alpha'}$$

by using factorization or explicit computation.



k
k_1
k_2
$z_1 \to z_2$
k
k'
$-k'$
k_1
k_2

One-loop amplitudes: the 2-particle case—2 massive tensors

$$
\begin{aligned}
\mathcal{A}_{MM}^{1-\mathrm{loop}}(k,k') &= g_{MM}^{1-\mathrm{loop}} K_{MM}\tilde{K}_{MM} \int_F \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\tau)^5} \int \mathrm{d}^2 z \\
&\quad \times \left| \frac{\Theta_1(z|\tau)}{\partial\Theta_1(0|\tau)} \exp\left(\frac{-\pi}{\mathrm{Im}\tau}(\mathrm{Im}z)^2\right)\right|^{\alpha' k\cdot k'} \\
g_3^{1-\mathrm{loop}} &= \frac{g_M\, g_{MM}^{1-\mathrm{loop}}}{4\pi\alpha'} \\
K_{MM} &= -6\alpha_{\mu_1\nu_1\rho_1}(k)\alpha^{\mu_1\nu_1\rho_1}(k') + \sigma_{\mu\nu}(k)\sigma^{\mu\nu}(k')
\end{aligned}
$$

Modular invariant only for $k\cdot k' = -k^2 = \frac{4}{\alpha'}$**.**

The 2-loop 4-particle amplitude:

$$\mathcal{A}_{II} \sim \int \frac{1}{T^5} \frac{\prod_{i=1}^{6} \mathrm{d}^2 a_i}{\mathrm{d}V_{pr} |\prod_{i<j} a_{ij}|^2} \prod_{i=1}^{4} \frac{\mathrm{d}^2 z_i}{|y(z_i)|^2} \prod_{i<j} \mathrm{e}^{-k_i \cdot k_j \langle X(z_i) X(z_j) \rangle}$$

$$\times \left| s(z_1 z_2 + z_3 z_4) + t(z_1 z_4 + z_2 z_3) + u(z_1 z_3 + z_2 z_4) \right|^2$$

$$\mathrm{d}V_{pr} = \frac{\mathrm{d}^2 a_i \mathrm{d}^2 a_j \mathrm{d}^2 a_k}{|a_{ij} a_{ik} a_{jk}|^2}, \quad T = \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2 |z_1 - z_2|^2}{|y(z_1) y(z_2)|^2},$$

$$\langle X(z_i) X(z_j) \rangle \equiv G(z_i, z_j) = -\ln |E(z_i, z_j)|^2$$

$$+2\pi (\mathrm{Im}\, \Omega)^{-1}_{IJ} \left(\mathrm{Im} \int_{z_i}^{z_j} \omega_I\right) \left(\mathrm{Im} \int_{z_i}^{z_j} \omega_J\right)$$

A better but equivalent form derived by D' Hoker and Phong (hep-th/0501197):

$$
\begin{aligned}
\mathbf{A}_{II}(\epsilon_i, k_i) &= \frac{K\bar{K}}{2^{12}\pi^4} \int \frac{|\prod_{I\le J} d\Omega_{IJ}|^2}{(\det \mathrm{Im}\,\Omega)^5} \\
&\quad \times \int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp\Big(-\sum_{i<j} k_i \cdot k_j\, G(z_i, z_j)\Big) \\
\mathcal{Y}_S &= +(k_1 - k_2)\cdot(k_3 - k_4)\,\Delta(z_1, z_2)\Delta(z_3, z_4) + \cdots \\
&\propto s(z_1 z_2 + z_3 z_4) + t(z_1 z_4 + z_2 z_3) + u(z_1 z_3 + z_2 z_4) \\
\Delta(z, w) &\equiv \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)
\end{aligned}
$$

5. Factorization and Unitarity

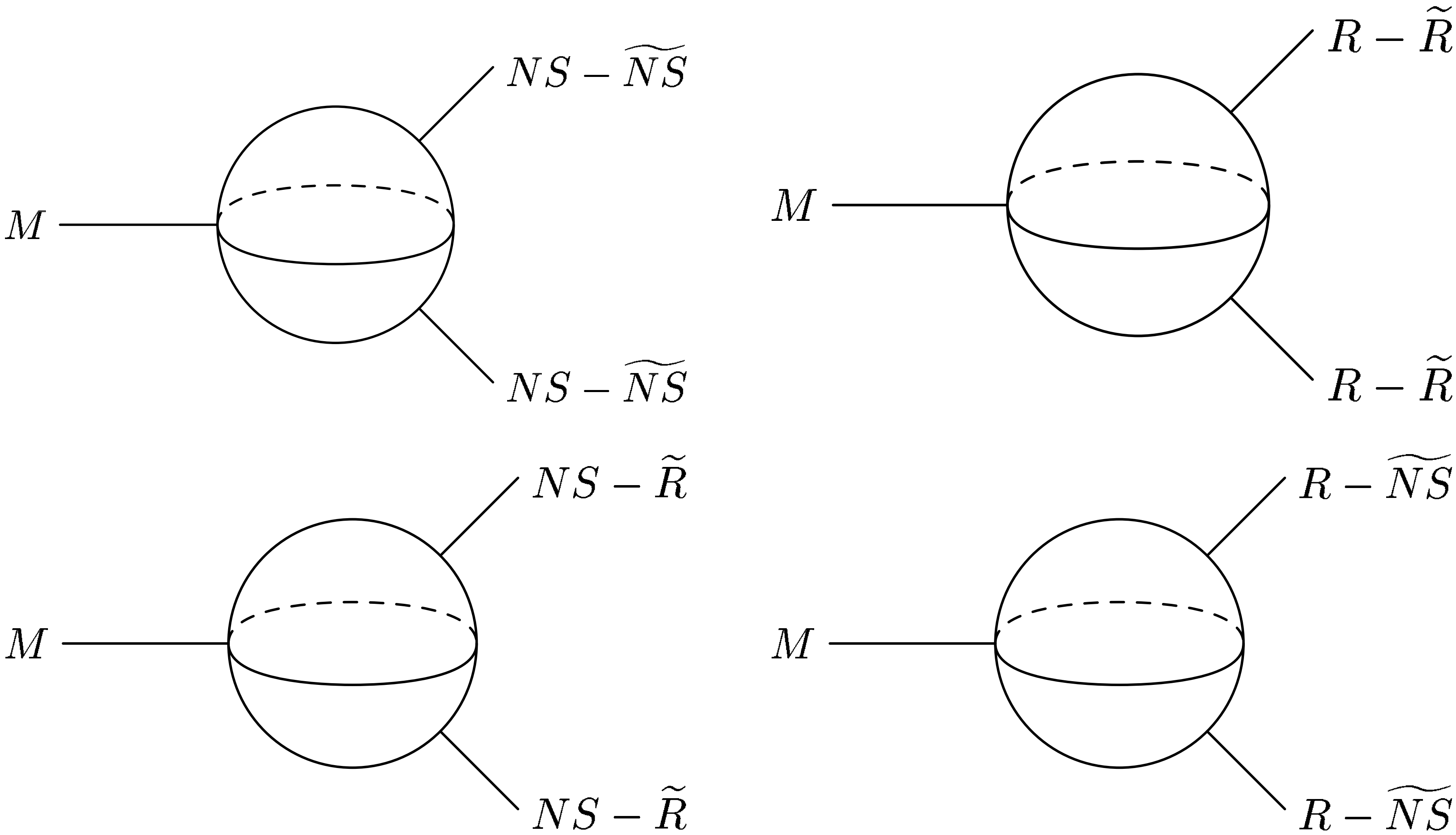
The unitarity relation at 1-loop:

$$
\begin{aligned}
\mathcal{A}(s+i\epsilon) \; - \; & \mathcal{A}(s-i\epsilon) \\
= \; & \frac{i}{2!}\int \frac{\mathrm{d}^{\mathrm{D}}k_1}{(2\pi)^D}\, 2\pi\delta(k_1^2) \int \frac{\mathrm{d}^{\mathrm{D}}k_2}{(2\pi)^D}\, 2\pi\delta(k_2^2) \\
& \times (2\pi)^D \delta^D(k_1+k_2+k)\, |A^{\mathrm{tree}}(k_1;k_2;k)|^2
\end{aligned}
$$

The factor of 2 is due to the propagation of intermediate identical particles.

In $D = 10$ superstring theory, the difficult part is to compute $|A^{\text{tree}}(k_1; k_2; k)|^2$. By summing over all possible intermediate states, we have:

$$\sum_{\text{all intermediate states}} |A^{\text{tree}}_{M**}|^2(k_1; k_2; k) = (g_M)^2\, K_{MM}\tilde{K}_{MM}$$


$NS - \widetilde{NS}$
M
$NS - \widetilde{NS}$
$R - \widetilde{R}$
M
$R - \widetilde{R}$
$NS - \widetilde{R}$
M
$NS - \widetilde{R}$
$R - \widetilde{NS}$
M
$R - \widetilde{NS}$

Here we used the following:

$$\int \frac{\mathrm{d}^{\mathrm{D}} k_2}{(2\pi)^D} k_1^\mu \delta(k_1^2) \delta((k-k_1)^2)$$
$$= \frac{1}{2} k^\mu \int \frac{\mathrm{d}^{\mathrm{D}} k_1}{(2\pi)^D} k_1^\mu \delta(k_1^2) \delta((k-k_1)^2)$$

$$\int \frac{\mathrm{d}^{\mathrm{D}} k_2}{(2\pi)^D} k_1^\mu k_2^\nu \delta(k_1^2) \delta((k-k_1)^2)$$
$$= \frac{-1}{4(D-1)} \left(k^2 \eta^{\mu\nu} - D\, k^\mu k^\nu\right)$$

$$\times \int \frac{\mathrm{d}^{\mathrm{D}} k_1}{(2\pi)^D} k_1^{\mu} \delta(k_1^2) \delta((k-k_1)^2),$$

etc. By using this result we have:

$$g_{MM}^{1-\text{loop}} \frac{(\alpha')^{D/2}}{(2\pi)^{\alpha' k \cdot k'}} \frac{(2\pi)^D}{(\pi\alpha')^2} = \frac{(g_M)^2}{2}, \qquad g_M^{1-\text{loop}} = \frac{g_c^2}{2\pi^2(\alpha')^5},$$

$$g_3^{1-\text{loop}} = \frac{g_c^3}{\pi^2(\alpha')^5}, \qquad g_4^{1-\text{loop}} = \frac{2g_c^3}{\pi^2(\alpha')^5}$$

Factorization and Unitarity are also true at 2 loops and can be used to determine the 2-loop overall

coefficient:

$$C_{II} = \frac{2^4\, g_c^6}{(\alpha')^7 \pi}.$$

In period matrix language:

$$\mathcal{A}_{II} \;=\; C_{II}\,\frac{1}{(2\pi)^6\, 2^5}\, K(k_i, \epsilon_i) \int \frac{|\mathrm{d}^3\tau|^2}{(\det \mathrm{Im})^5}$$
$$\times \int \prod_{i=1}^{4} \mathrm{d}^2 z_i |3\mathcal{Y}_s|^2 \prod_{i<j} \exp\{-k_i \cdot k_j \langle X(z_i) X(z_j)\rangle\}$$

and the overall coefficient is

$$\hat{C}_{II} = C_{II}\,\frac{1}{(2\pi)^6\,2^5} = \frac{g_c^6}{(2\pi\alpha')^7},$$

which agrees with D'Hoker, Gutperle and Phong (hep-th/0501197) by taking into account the different convention for $\mathrm{d}^2 z$ (we use $\mathrm{d}^2 z = \mathrm{d}x\mathrm{d}y$ for $z = x + iy$).

The above result also agrees with S-duality.

6. The Frontier: Superstring Multiloop Amplitudes?

In Berkovits' pure spinor formalism:

- **Basic variables:** $X^\mu(z,\bar z)$, $\theta^\alpha(z)$ **and** $p_\alpha(z)$ **(conjugate to** $\theta(z)$**). (For NRS:** $X^\mu(z,\bar z)$**,** $\psi^\mu(z)$**,** $b(z)$, $c(z)$ **and** $\beta(z)$, $\gamma(z)$**.)**

- **ALso introducing bosonic pure spinor ghost variable** λ^α **(and their conjugates** w_α**):**

$$\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta = 0$$

The BRST operator

$$Q = \oint \lambda^\alpha d_\alpha$$

is used to impose the fermionic constraints:

$$d_\alpha = p_\alpha - \frac{1}{2}(\gamma^\mu \theta)_\alpha \left[\partial X_\mu + \frac{1}{4} \partial\theta\gamma_\mu\theta \right] = 0$$

To insure Lorentz covariance, w_α only appears in:

$$J = w_\alpha \lambda^\alpha, \qquad N_{mn} = \frac{1}{2} w_\alpha (\gamma_{mn})^\alpha{}_\beta \lambda^\beta$$

Many checks have be done. The physical vertex operators and (manifestly supersymmetric covariant) tree amplitudes are shown to be in agreement with other formalisms. A quantizable σ-model action for the superstring in curved backgrounds (with Ramond-Ramond flux). Infinite set of nonlocal (classically and quantum) conserved charges. $\Rightarrow$ Exactly solvable theory for superstring theory on $AdS_5 \times S^5$. AdS/CFT correspondence, ...

Loop amplitudes (Berkovits, hep-th/0406055)

- vanishing of the $m = 0$ N-particle amplitudes $N \leq 3$
- 1-loop 4-particle amplitude
- vanishing of the multi-loop 4-particle leading contribution (absence of the R^4 term)
- the complete multi-loop 4-particle amplitude

But we have doubts about these results. 0 even for 4-particle amplitude?

Key points

- **Picture changing operators:**
  - **"picture-lowering" operator:** $Y_C = C_\alpha \theta^\alpha \delta(C_\beta \lambda^\beta)$
  - **"picture-raising" operator:**

$$Z_B = \frac{1}{2} B_{mn} \lambda \gamma^{mn} d\, \delta(B^{pq} N_{pq}), \qquad Z_J = \lambda^\alpha d_\alpha \delta(J)$$

- **A construction of the "b-ghost fields":**

$$\{Q, b(z)\} = T(z)$$

$$\{Q, \tilde{b}_B(z,w)\} = T(z) Z_B(w)$$

Schematicaly, Berkovits got:

$$\begin{aligned} b_B &= B(dd\Pi + dN\partial\theta + NN + N\Pi\Pi)\delta(BN) \\ &\quad + BB(dddd + ddN\Pi + NN\Pi\Pi + NNd\partial\theta)\partial\delta(BN) \\ &\quad + BBB(ddddN + ddNN\Pi)\partial^2\delta(BN)\cdots \\ &\quad + BBBB(ddddNN)\partial^3\delta(BN) \end{aligned}$$

$$\tilde{b}_B(z,w) = b_B(z) + T(z)\int_z^w \mathrm{d}u\, B_{pq}\partial N^{pq}(u)\delta(BN(u))$$

- **Integration over θ^α and p_α requires a $16+16g$ zero modes to give a non-vanishing result.**

$$\mathcal{A} = \int d^2\tau_1 ... d^2\tau_{3g-3} \langle \mid \prod_{P=1}^{3g-3} \int d^2 u_P \mu_P(u_P) \widetilde{b}_{B_P}(u_P, z_P)$$

$$\times \prod_{P=3g-1}^{10g} Z_{B_P}(z_P)$$

$$\times \prod_{R=1}^{g} Z_J(v_R) \prod_{I=1}^{11} Y_{C_I}(y_I) \mid^2 \prod_{T=1}^{N} \int d^2 t_T U_T(t_T) \rangle$$

$$U = \mathrm{e}^{ik\cdot X}(\partial\theta^{\alpha}A_{\alpha}(\theta)+\Pi^{m}A_{m}(\theta)+d_{\alpha}W^{\alpha}(\theta)+\frac{1}{2}N^{mn}\mathcal{F}_{mn}(\theta))$$

- **Counting of the d_{α} zero modes:**
  - **The massless vertex operator can give at most 1 zero modes of p_{α}.**
  - **Each picture-raising operator Z_B gives at most 1 zero modes of p_{α}.**
  - **Each "b-ghost fields" can give at most 4 zero modes of p_{α}, but there are other restrictions (the "engineering dimension" and conformal**

dimension).

Two possible methods to compute the g-loop 4-particle amplitude:

1) Explicit calculation at $g = 2, 3$ to see a pattern;

2) Guess a formula and fix it by factorization in the dividing degeneration limit (incorrect in hep-th/0503001).

We hope that there do exist a nice formula for the massless 4-particle amplitude and its 3-particle and

2-particle factorization limits.

The crispness and precision of standard string theory should not stand just around $g = 2$.