Strong gravitational lensing and absorption cross section for black holes

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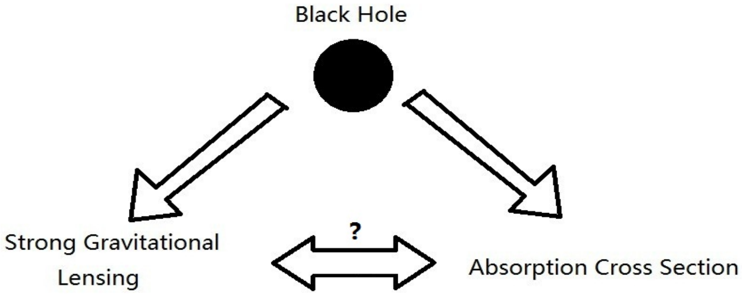
Black Hole
Strong Gravitational Lensing
?
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Introduction

- **What is gravitational lensing?**

The phenomena that the photons deviate from their straight path in a gravitational field.

- **What is gravitational lens?**

The object which causes a detectable deflection. (*such as the sun, galaxy or black hole.*)

Classes of gravitational lensing:

1. **Strong gravitational lensing**

   large deflection angle

2. **Weak gravitational lensing**

   small deflection angle

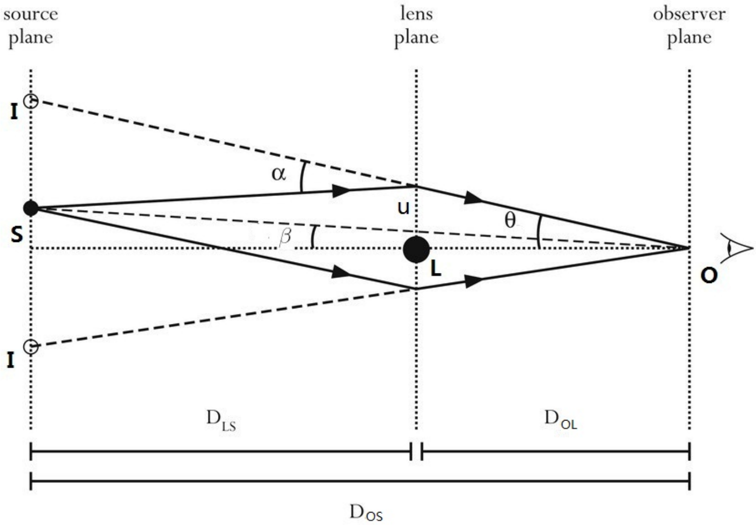
3. **Microlensing**

   change in time

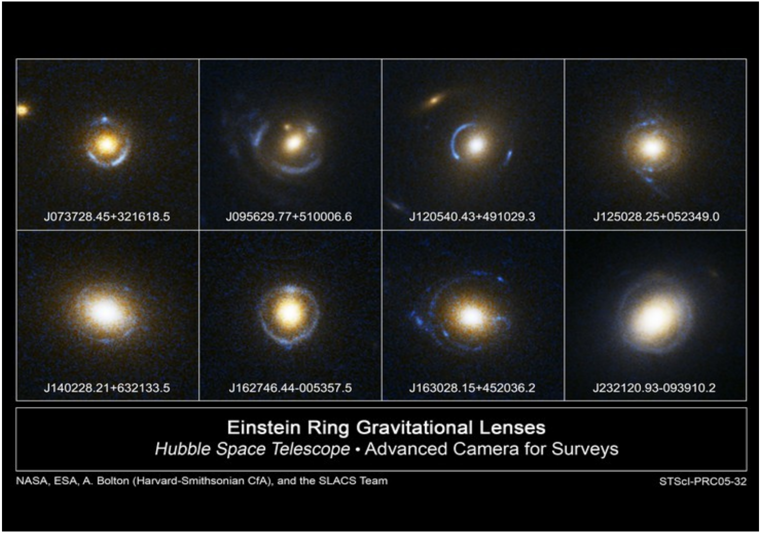
Why gravitational lensing?

1. **provide information about the distant stars.**
2. **test the exotic objects** (*like cosmic strings, gravitomagnetic mass*) **in the universe**.
3. **estimate the values of the cosmological parameters.**
4. **provide a profound verification of alternative theories of gravity in their strong field regime.**
5. **......**

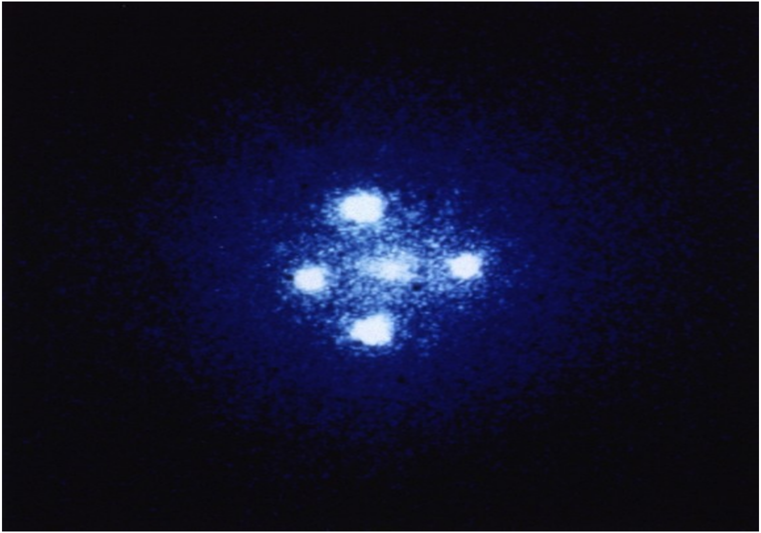
Strong gravitational lensing



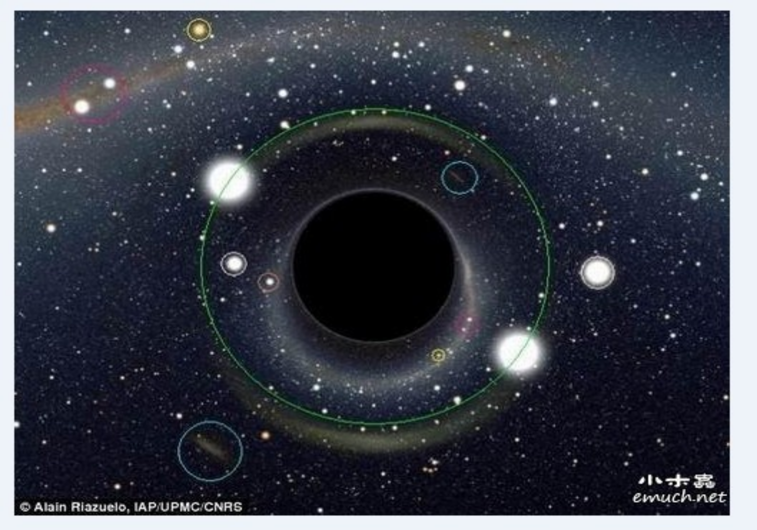
Einstein ring:



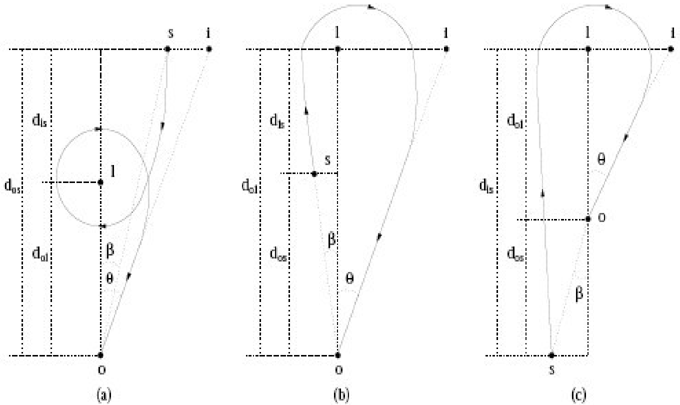
Einstein cross:



Black hole lensing:

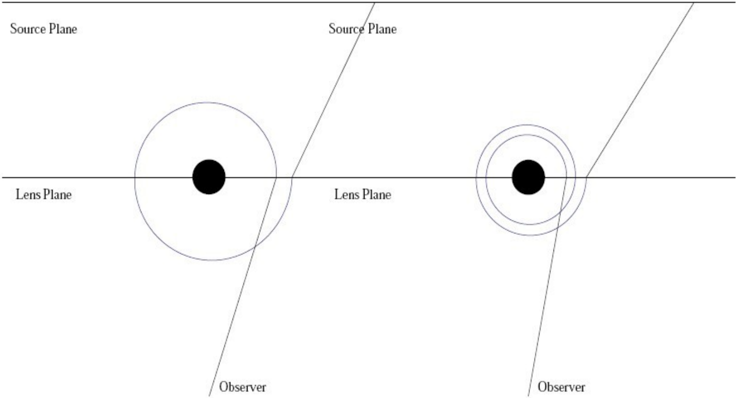


Black hole lensing:



(a) standard lensing, and (b), (c) to retrolensing.

Black hole lensing:



Spherically symmetric spacetime

Consider a spherically symmetric spacetime

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1)$$

where

$$A(r \to \infty) = 1 - \frac{2M}{r}, \qquad (2)$$

$$B(r \to \infty) = 1 + \frac{2M}{r}, \qquad (3)$$

$$C(r \to \infty) = r^2. \qquad (4)$$

Lensing described by this metric correctly matches the weak gravitational field far from the lensing object.

Equations of motion for photon on the equatorial plane $(\theta = \pi/2)$

$$p_t = -A\dot{t} = -E = \text{Const}, \tag{5}$$

$$p_\phi = C\dot{\phi} = L = \text{Const}, \tag{6}$$

$$u^\mu u_\mu = 0 = \text{Const}. \tag{7}$$

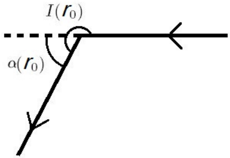
$$\Downarrow$$

$$\dot{t} = \frac{E}{A}, \tag{8}$$

$$\dot{r}^2 = \frac{E^2}{AB} - \frac{L^2}{BC}, \tag{9}$$

$$\dot{\phi} = \frac{L}{C}. \tag{10}$$

Consider the case that a photon incomes from infinity with the impact parameter u, reaches a minimum distance r_0, then returns to infinity.



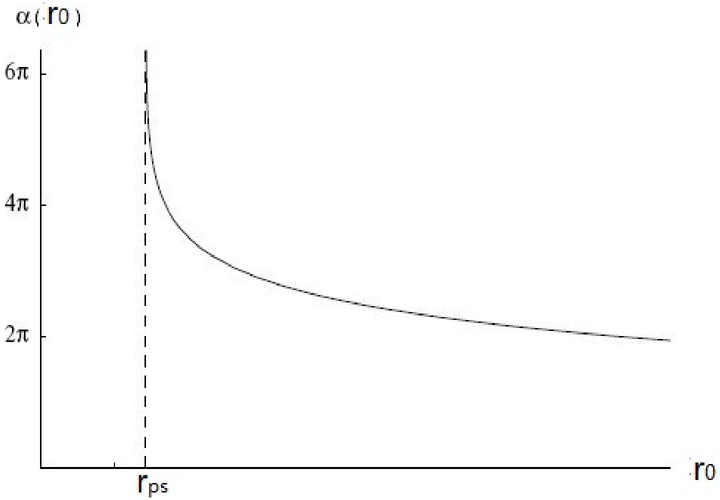
The deflection angle

$$\alpha(r_0) = I(r_0) - \pi, \quad (11)$$

with the total azimuthal angle

$$I(r_0) = \int d\phi \;=\; 2\int_{r0}^{\infty}\left[\frac{d\phi}{d\tau}\left(\frac{dr}{d\tau}\right)^{-1}\right]dr \;=\; 2\int_{r_0}^{\infty}\frac{\sqrt{B}}{\sqrt{C}\sqrt{\frac{CA_0}{C_0A}-1}}. \quad (12)$$

Behavior of the deflection angle



Photon sphere equation

The photon sphere with radius r_{ps} is an unstable circular of light, which satisfies the three conditions

$$V_{eff}\Big|_{r=r_{\text{ps}}} = 0, \tag{13}$$

$$V'_{eff}\Big|_{r=r_{\text{ps}}} = 0, \tag{14}$$

$$V''_{eff}\Big|_{r=r_{\text{ps}}} < 0, \tag{15}$$

with effective potential $V_{eff} = -\dot{r}^2 = \frac{L^2}{BC} - \frac{E^2}{AB}$.

- (13) gives the minimum impact parameter $u_{\text{ps}} = \sqrt{\frac{C_{\text{ps}}}{A_{\text{ps}}}}$.
- (14) leads to the photon sphere equation

$$A_{\text{ps}} C'_{\text{ps}} - A'_{\text{ps}} C_{\text{ps}} = 0. \tag{16}$$

- (15) is the unstable condition.

The deflection angle formula

Define two new variables

$$y = A(r), \tag{17}$$

$$z = \frac{y - y_0}{1 - y_0}. \tag{18}$$

The total azimuthal angle (12) can be expressed as

$$I(r_0) = \int_0^1 R(z, r_0) f(z, r_0) dz, \tag{19}$$

where

$$R(z, r_0) = \frac{2\sqrt{By}}{CA'} (1 - y_0) \sqrt{C_0}, \tag{20}$$

$$f(z, r_0) = \frac{1}{\sqrt{y_0 - [(1 - y_0) z + y_0] \frac{C_0}{C}}}. \tag{21}$$

Expanding the argument of the square root in $f(z, r_0)$ to the second order in z ($f(z, r_0)$ is singular at $z = 0$):

$$
\begin{aligned}
f \quad (z, r_0) &\sim f_0(z, r_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}}, \qquad (22) \\
\alpha &= \frac{1 - y_0}{C_0 A_0'} \left(C_0' A_0 - C_0 A_0' \right), \\
\beta &= \frac{(1 - y_0)^2}{2 C_0^2 A_0'^{\,3}} \left[2 C_0 C_0' A_0'^{\,2} + \left(C_0 C_0'' - 2 C_0'^{\,2} \right) y_0 A_0' - C_0 C_0' y_0 A_0'' \right].
\end{aligned}
$$

1. $\alpha \neq 0 \Rightarrow f(z, r_0)$ **is finite.**
2. $\alpha = 0 \Rightarrow f(z, r_0)$ **is unlimited.**

Note: $\alpha = 0 \Rightarrow$ **photon sphere equation (16).**

Splitting the total azimuthal angle (19) into two pieces

$$I(r_0) = I_D(r_0) + I_R(r_0), \tag{23}$$

with divergent part $I_D(r_0)$ **and regular part** $I_R(r_0)$

$$I_D(r_0) = \int_0^1 R(0, r_{\mathsf{ps}}) f_0(z, r_0) dz, \tag{24}$$

$$I_R(r_0) = \int_0^1 g(z, r_0) dz, \tag{25}$$

where

$$g(z, r_0) = R(z, r_0) f(z, r_0) - R(0, r_{\mathsf{ps}}) f_0(z, r_0). \tag{26}$$

Divergent part $I_D(r_0)$ can be solved exactly,

$$I_D(r_0) = R(0, r_{\text{ps}}) \frac{2}{\sqrt{\beta}} \log \frac{\sqrt{\beta} + \sqrt{\alpha + \beta}}{\sqrt{\alpha}}. \tag{27}$$

Expanding α and β to order $\mathcal{O}(r_0 - r_{\text{ps}})$, it can be written as

$$I_D(r_0) = -a \log \left(\frac{r_0}{r_{\text{ps}}} - 1 \right) + b_D + O(r_0 - r_{\text{ps}}), \tag{28}$$

$$a = \frac{R(0, r_{\text{ps}})}{\sqrt{\beta_{\text{ps}}}}, \tag{29}$$

$$b_D = \frac{R(0, r_{\text{ps}})}{\sqrt{\beta_{\text{ps}}}} \log \frac{2(1 - y_{\text{ps}})}{A'_{\text{ps}} r_{\text{ps}}}, \tag{30}$$

$$\beta_{\text{ps}} = \beta|_{r=r_{\text{ps}}}. \tag{31}$$

Regular part $I_R(r_0)$ in powers of $(r_0 - r_{\mathsf{ps}})$:

$$I_R(r_0) = \sum_{n=0}^{\infty} \frac{1}{n!}(r_0 - r_{\mathsf{ps}})^n \int_0^1 \left.\frac{\partial^n g}{\partial r_0^n}\right|_{r_0=r_{\mathsf{ps}}} dz, \tag{32}$$

up to order $\mathcal{O}(r_0 - r_{\mathsf{ps}})$, it is

$$I_R(r_0) = \int_0^1 g(z, r_{\mathsf{ps}})dz + O(r_0 - r_{\mathsf{ps}}), \tag{33}$$

and

$$b_R = I_R(r_{\mathsf{ps}}), \tag{34}$$

which can be easily evaluated numerically.

Combing the two parts,

$$\alpha(r_0) = -a\log\left(\frac{r_0}{r_{\text{ps}}}-1\right)+b+\mathcal{O}\left(r_0-r_{\text{ps}}\right), \quad (35)$$

$$b = -\pi+b_D+b_R. \quad (36)$$

From $\alpha(r_0)$ to $\alpha(\theta)$, the deflection angle $\alpha(\theta)$[1]

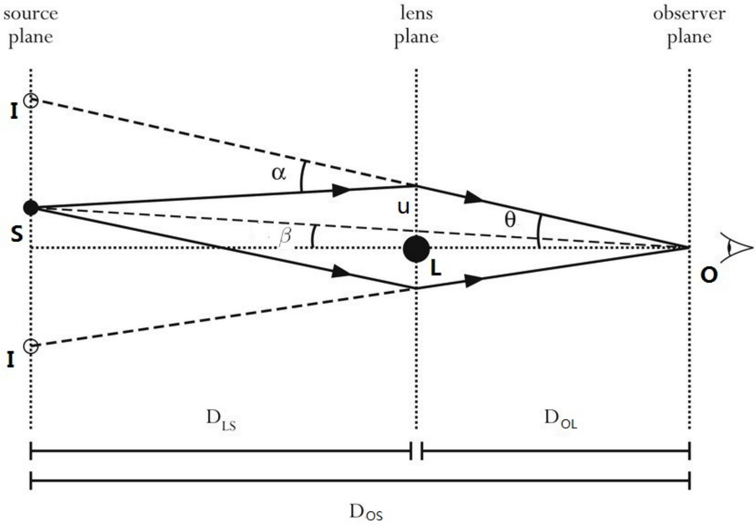
$$\alpha(\theta) = -\bar{a}\log\left(\frac{\theta D_{OL}}{u_{\text{ps}}}-1\right)+\bar{b}, \quad (37)$$

with the strong deflection limit coefficients

$$\bar{a} = \frac{R(0,r_{\text{ps}})}{2\sqrt{\beta_{\text{ps}}}}, \quad (38)$$

$$\bar{b} = -\pi+b_R+\bar{a}\log\frac{2\beta_{\text{ps}}}{y_{\text{ps}}}. \quad (39)$$

[1] **Bozza, PRD 66, 103001 (2002)**


source plane
lens plane
observer plane
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D_OS

Lens equation: [2]

$$\tan\beta = \tan\theta - \frac{D_{\mathsf{LS}}}{D_{\mathsf{OS}}}\big[\tan(\alpha-\theta)+\tan\theta\big]. \tag{40}$$

Highly aligned case ($\beta, \theta \sim 0$)

$$\beta = \theta - \frac{D_{\mathsf{LS}}}{D_{\mathsf{OS}}}\Delta\alpha_n, \tag{41}$$

with $\Delta\alpha_n = \alpha - 2n\pi$ and n denotes the number of loops done by the photon.

[2] **Virbhadra, Ellis, PRD 62, 084003 (2000)**

Angular positions

$$\theta_n = \theta_n^0 + \frac{u_{\mathsf{ps}} e_n \left(\beta - \theta_n^0\right) D_{OS}}{\overline{a} D_{LS} D_{OL}}, \tag{42}$$

Magnification

$$\mu_n = e_n \frac{u_{\mathsf{ps}}^2 \left(1 + e_n\right) D_{OS}}{\overline{a} \beta D_{OL}^2 D_{LS}}, \tag{43}$$

with

$$\theta_n^0 = \frac{u_{\mathsf{ps}}}{D_{OL}} \left(1 + e_n\right), \quad e_n = e^{\frac{\overline{b} - 2n\pi}{\overline{a}}}. \tag{44}$$

1. **For non-zero β the first image is the brightest one among the relativistic images and its brightness decreases quickly with the increasing of the distance D_{OL}.**
2. **It will be no longer valid when $\beta = 0$.**

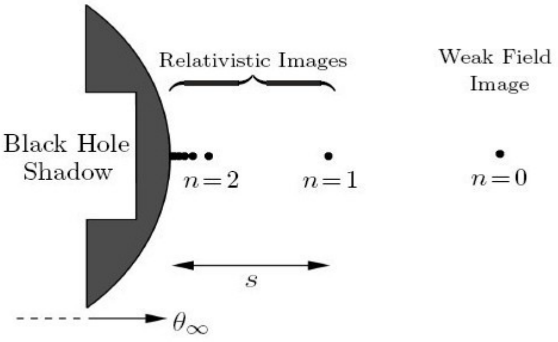
Observable quantities

Observable quantities:

1. The angular position θ_∞
2. The angular separation s
3. The ratio of the flux $\tilde{r}$

Relation between strong field limit coefficients and observable quantities:[3]

1. $\theta_\infty = \frac{u_{\text{ps}}}{D_{\text{OL}}}$
2. $s = \theta_1 - \theta_\infty = \theta_\infty e^{\frac{\bar{b}-2\pi}{\bar{a}}}$
3. $\tilde{r} = \mu_1 / \sum_{n=2}^{\infty} \mu_n = e^{2\pi/\bar{a}}$



[3]Bozza, PRD 66, 103001 (2002)

Kerr black hole pierced by a cosmic string [4]

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta^2 + \beta^2\left(r^2 + a^2 + \frac{2Mra^2\sin^2\vartheta}{\rho^2}\right)\sin^2\vartheta d\phi^2 - \beta\frac{4Mra\sin^2\vartheta}{\rho^2}dtd\phi, \quad (45)$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad (46)$$

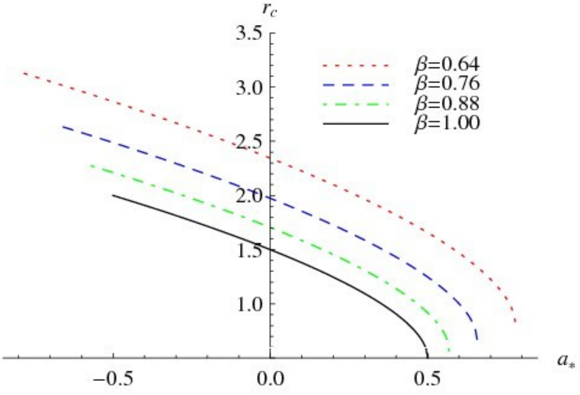
$$\rho^2 = r^2 + a^2\cos^2\vartheta. \quad (47)$$

[4] Wei, Liu arXiv:1107.3023[hep-th]

The photon circle equation in the equatorial plane:

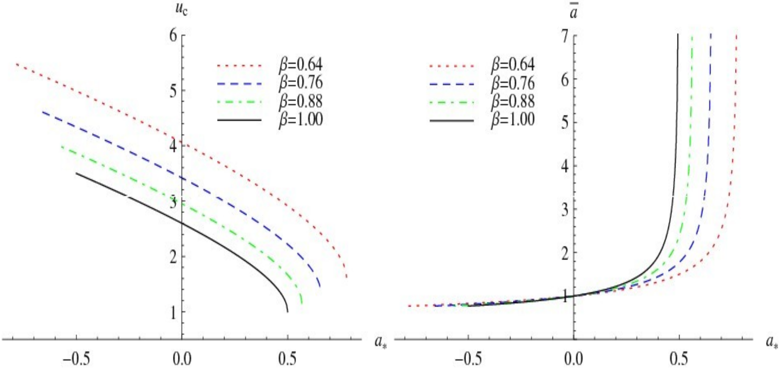
$$r_c(2\beta r_c - 3)^2 - 8\beta a_*^2 = 0, \tag{48}$$

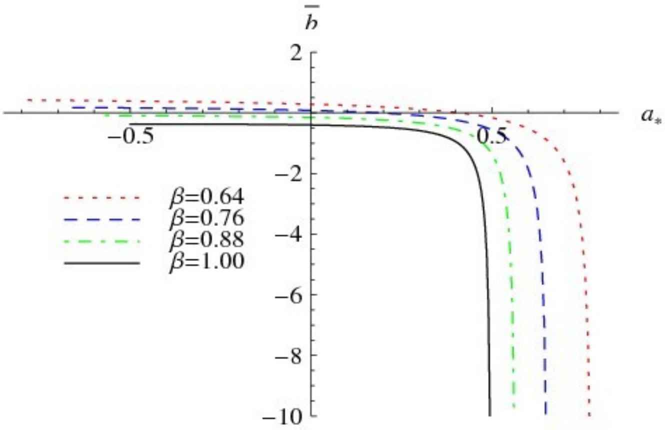
with $a_* = a/2M_{\text{phys}} = a/2M\beta$**.**



The deflection angle

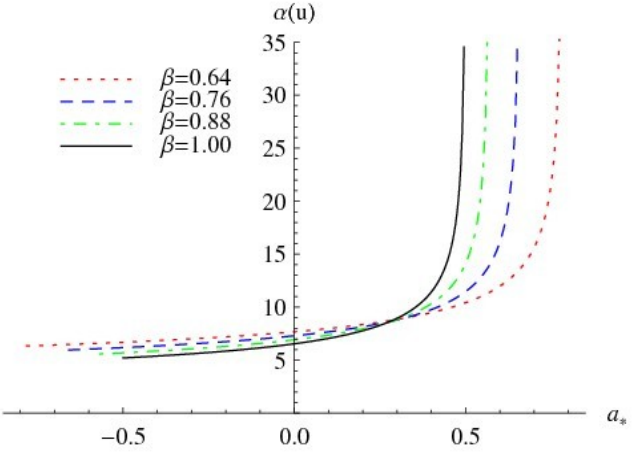
$$\alpha(u) = -\overline{a}\log\left(\frac{u}{u_{\mathsf{c}}} - 1\right) + \overline{b} \qquad (49)$$





$\bar{b}$
2
a_*
−0.5
0.5
−2
−4
−6
−8
−10
β=0.64
β=0.76
β=0.88
β=1.00

Behavior of the deflection angle with fixed $u = u_c + 0.0025$



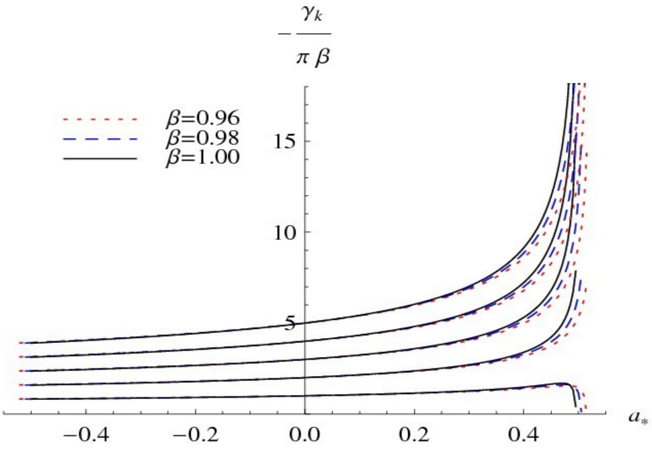
Magnification:

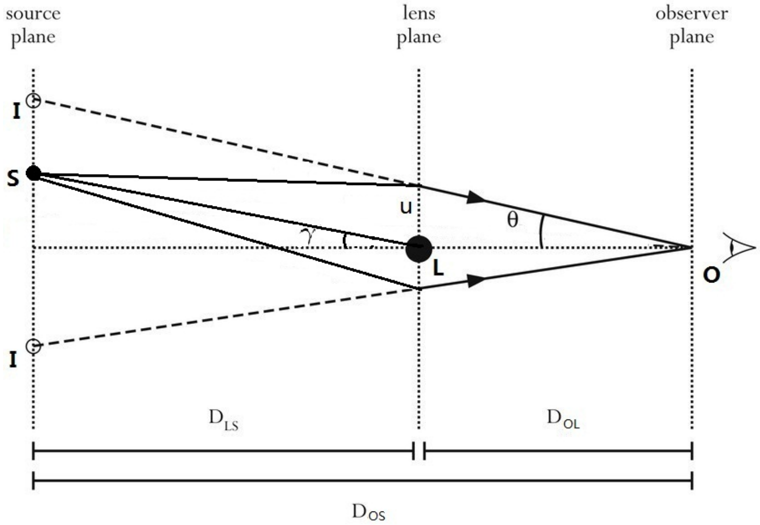
$$\mu = \frac{d^2\mathcal{A}_I}{d^2\mathcal{A}_S} = \frac{(D_{\mathsf{OL}} + D_{\mathsf{LS}})^2}{D_{\mathsf{OL}} D_{\mathsf{LS}}} \frac{\bar{u} u_{\mathsf{c}} e_\gamma}{\bar{a} K(\gamma)}, \tag{50}$$

with $e_\gamma = e^{(\bar{b}+\gamma)/\bar{a}}$**.**

Angular positions of the caustic points:

$$K(\gamma) = -\bar{u}(D_{\mathsf{LS}} + D_{\mathsf{OL}})C_n + D_{\mathsf{OL}} D_{\mathsf{LS}} S_n = 0. \tag{51}$$




source plane
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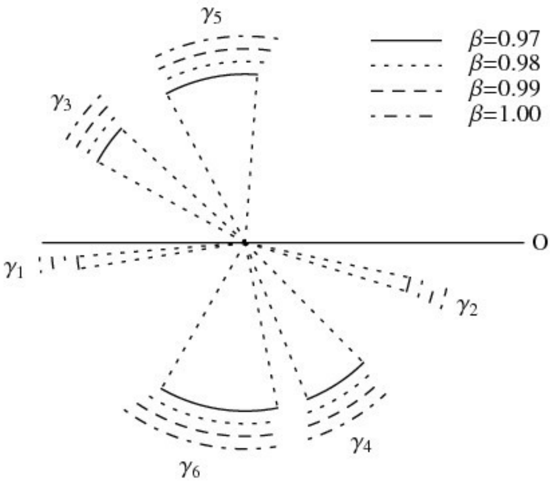
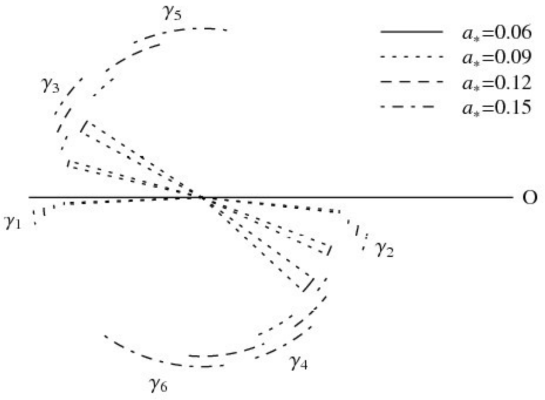
Caustic structure:

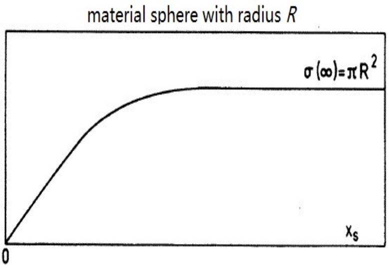


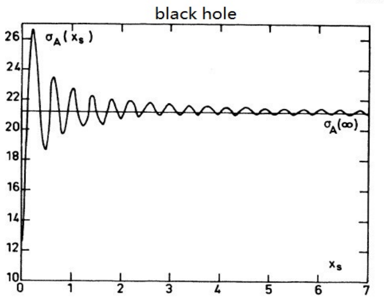
Caustic structure:



Absorption cross section

A measure for the probability of an absorption process.





Black hole

- **Low energy:** $\sim A_{\mathsf{h}}$

- **High energy: oscillates around the geometrical cross section of its photon sphere**

High-energy absorption cross section

- **metric (**$f(r \to +\infty) \to 1$**)**

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{(D-2)}, \tag{52}$$

- **equatorial hyperplane** $(\theta_i = \pi/2)$
- **in the eikonal approximation**

High-energy absorption cross section

$$\sigma_{\text{abs}}(\omega) = \sigma_{\text{lim}} + \sigma_{\text{osc}}, \tag{53}$$

$$\begin{aligned}
\sigma_{\text{lim}} &= \sigma_{\text{geo}} = \frac{\pi^{\frac{D-2}{2}} b_{\text{ps}}^{D-2}}{\Gamma(D/2)}, \\
\sigma_{\text{osc}}(\omega) &= (-1)^{D-3} 4(D-2)\pi\eta_{\text{ps}} e^{-\pi\eta_{\text{ps}}} \frac{\sin(\omega T_{\text{ps}})}{\omega T_{\text{ps}}} \sigma_{\text{geo}},
\end{aligned}$$

with

$$b_{\text{ps}} = \frac{r_{\text{ps}}}{\sqrt{f_{\text{ps}}}}, \quad \eta_{\text{ps}} = \sqrt{f_{\text{ps}} - \frac{1}{2} r_{\text{ps}}^2 f''_{\text{ps}}}. \tag{54}$$

- **Orbital period:** $T_{\text{ps}} = 2\pi b_{\text{ps}}$
- **Geometrical cross section:** σ_{geo}

Under metric (52), the strong deflection limit coefficients

$$u_{\mathsf{ps}} = L = \frac{r_{\mathsf{ps}}}{\sqrt{f_{\mathsf{ps}}}}, \tag{55}$$

$$\bar{a} = \sqrt{\frac{2}{2f_{\mathsf{ps}} - r_{\mathsf{ps}}^2 f_{\mathsf{ps}}''}}, \tag{56}$$

$$\bar{b} = -\pi + b_{\mathsf{R}} + \bar{a}\ln\left[\frac{1}{2f_{\mathsf{ps}}^3}\left(\frac{1-f_{\mathsf{ps}}}{\bar{a}}\right)^2\right]. \tag{57}$$

Relation between high-energy absorption cross section and gravitational lensing

$$b_{\mathsf{ps}} = \frac{u_{\mathsf{ps}}}{c} \tag{58}$$

$$\eta_{\mathsf{ps}} = \frac{1}{\bar{a}} \tag{59}$$

or, in terms of the observables

$$\eta_{\mathsf{ps}} = \frac{1}{2\pi} \ln \tilde{r}, \tag{60}$$

$$b_{\mathsf{ps}} = \frac{D_{\mathsf{OL}} \theta_\infty}{c} = 2\pi \Delta T_{2,1} \tag{61}$$

$\Delta T_{2,1}$**: time delay between the first relativistic image and the second one.**

New expression: [5]

$$\sigma_{\mathsf{abs}}(\omega) = (2\pi\Delta T_{2,1})^{D-2}\frac{2\pi K\ln\tilde{r}}{\sqrt{\tilde{r}}}\cdot\frac{\sin(4\pi^2\omega\Delta T_{2,1})}{4\pi^2\omega\Delta T_{2,1}} + \frac{2^{D-2}\pi^{(3D-6)/2}(\Delta T_{2,1})^{D-2}}{\Gamma(\frac{D}{2})}. \quad (62)$$

Total energy per unit time and energy interval $d\omega$ is

$$\frac{d^2E(\omega)}{d\omega dt} = \left((4\pi)^{\frac{D-1}{2}}\Gamma\left(\frac{D-1}{2}\right)\right)^{-1}\frac{2\sigma_{\mathsf{abs}}(\omega)}{e^{\frac{\omega}{T_{\mathsf{H}}}}-1}\omega^{D-1}, \quad (63)$$

[5]**Wei, Liu et.al., PRD(R) 84, 041501 (2011)**

Thanks for your attention!