Exact solutions in the DBI effective theory and the dynamics of D-branes

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Contents

Introduction

- Two kinds of D-branes in Type II superstring theories:

  —**Stable** or **BPS branes**: supersymmetric, RR charged

  —**Unstable** or **non-BPS branes**: non-susy, uncharged

- The low energy dynamics of D-branes can be captured by the Dirac-Born-Infeld (DBI) type effective action

  —For a non-BPS D-brane, there is a tachyon field T in the DBI action

  —For a BPS D-brane, there are only stable fields in the DBI action

- The DBI effective action for a non-BPS D-brane (in the absence of other worldvolume fields) is given by

$$S = -\int d^{p+1}x\mathcal{C}(T)\sqrt{1+\eta^{\mu\nu}\partial_\mu T\partial_\nu T}, \tag{1}$$

where $\mathcal{C}(T)$ is the runaway tachyon potential which has the maximum value $\mathcal{C}_m$ at $T = 0$ and the minimum value 0 as $|T| \to \infty$.

— Non-BPS D-branes can decay into BPS ones of fewer dimensions —— the **tachyon condensation** process [a series of work by *Sen*], which can be described by this DBI action

— Applications of this action to understand inflation, dark matter, the producation and evolution of cosmic strings [*Gibbons02,Choudhury02,Frolov02, Kofman02,Sarangi02,Dvali03*] has been investigated. It is shown that the detailed models are problematic [*Kofman02*]

- Singularities

The quation of motion from the above tachyon field action is:

$$(1 + \partial T \cdot \partial T)\left(\Box T - \frac{\mathcal{C}'}{\mathcal{C}}\right) = \partial_\mu T \partial_\nu T \partial^\mu \partial^\nu T, \qquad (2)$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $\mathcal{C}' = \partial\mathcal{C}(T)/\partial T$.

Both analytical and numerical calculations have shown that the time evolution of the tachyon field in this equation develops into two featured regions:

$-$**kinks** and **anti-kinks** ($T = 0$), corresponding to the produced stable D-branes and anti-D-branes at the end of tachyon condensation

$-$**extrema** ($|T| \to \infty$), the vacuum

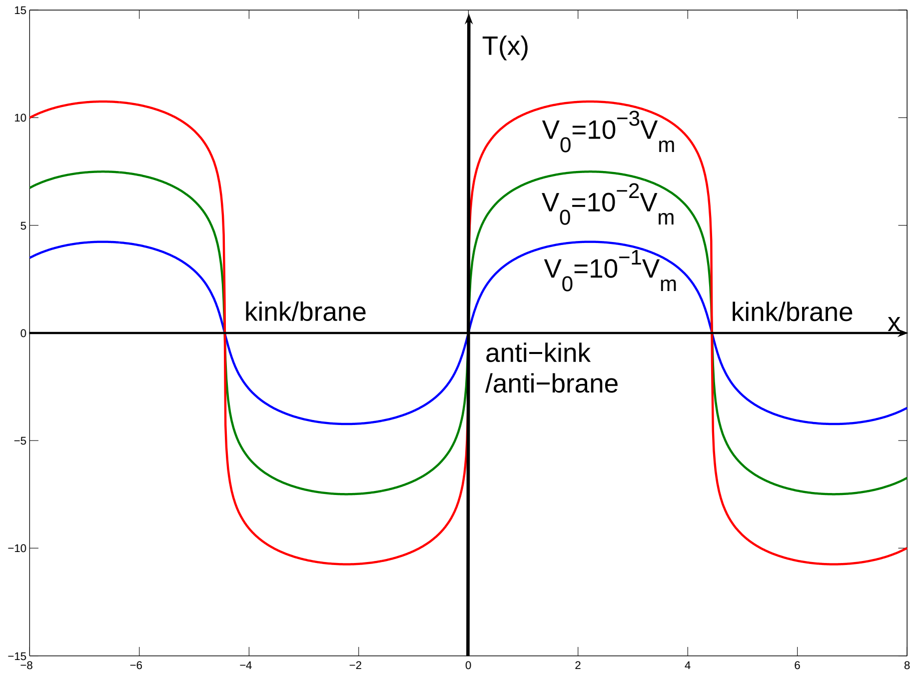

Figure 1: The static kink solution in the $p = 1$.

Further studies show that there develope singularities in these two regions:

—Around **(anti-)kinks** [*Cline03*]: the field gradients grows to infinity in finite time

—Around **extrema** [*Felder et al 02,04,Hindmarsh&Li09*]: caustic development, i.e., the second order derivatives of the tachyon field blow up in finite time

The existence of the singularities makes it hard to get complete numerical simulations based on this theory in the spacetime-dependent case. So it is hard to apply this theory.

- In this work, we try to explore exact analytical solutions in the DBI effective theory.

Exact solutions in the DBI effective theory

For completeness, we generalise the DBI effective action of a real tachyon field T to the action of a general real scalar field X:

$$S = -\int d^{p+1}x\mathcal{C}(X)\sqrt{1 + \eta^{\mu\nu}\partial_\mu X \partial_\nu X}, \tag{3}$$

where $\mathcal{C}(X)$ is the potential.

—For a **tachyon**, we denote $X = T$. It describes an **unstable** D-brane

—For a **massless scalar**, we denote $X = Y$. It can describe the fluctuations of a **stable** D-brane in a transverse direction

—For a **massive scalar**, we denote $X = \Phi$. It may describe confined fluctuations (probably!) of a **stable** D-brane ——- less interesting

- **Equation of motion**

$$(1 + \partial X \cdot \partial X)\left(\Box X - \frac{\mathcal{C}'}{\mathcal{C}}\right) = \partial_\mu X \partial_\nu X \partial^\mu \partial^\nu X, \qquad (4)$$

where $\mathcal{C}' = \partial\mathcal{C}(X)/\partial X$.

- **The energy-momentum tensor**

$$T_{\mu\nu}(X) = \mathcal{C}(X)\left[\frac{\partial_\mu X \partial_\nu X}{\sqrt{1 + \partial X \cdot \partial X}} - \eta_{\mu\nu}\sqrt{1 + \partial X \cdot \partial X}\right], \qquad (5)$$

- **Exact solution**

  Now we try to get exact solutions in the above DBI effective theory using the field redefinition technique.

  Adopt the following field redefinition relation:

$$X = f(\phi) \tag{6}$$

The equation of motion becomes

$$\frac{1}{f'^2}\left[\Box\phi + \left(\frac{f''}{f'} - \frac{\mathcal{C}'}{\mathcal{C}} f'\right)(\partial\phi\cdot\partial\phi) - \frac{\mathcal{C}'}{\mathcal{C}}\frac{1}{f'}\right] + [(\partial\phi\cdot\partial\phi)\Box\phi - \partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi] = 0, \tag{7}$$

where $f' = \partial f(\phi)/\partial\phi$ and $\mathcal{C}' = \partial\mathcal{C}(f)/\partial f$.

Consider the case:

$$\frac{f''}{f'} - \frac{\mathcal{C}'}{\mathcal{C}} f' = 0, \quad -\frac{\mathcal{C}'}{\mathcal{C}} \frac{1}{f'} = \alpha\phi. \tag{8}$$

Under these two conditions, the equation of motion (7) is

$$\frac{1}{f'^2}(\Box\phi + \alpha\phi) + [(\partial\phi \cdot \partial\phi)\Box\phi - \partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi] = 0. \tag{9}$$

Obviously, this equation has the following exact solution of **a single momentum mode** k

$$\phi = \phi_+ e^{ik\cdot x} + \phi_- e^{-ik\cdot x}, \quad k^2 = \alpha. \tag{10}$$

- $\alpha = 0$, ϕ is a massless scalar
- $\alpha < 0$, ϕ is a massive scalar
- $\alpha > 0$, ϕ is a tachyon

To derive the solution, we need to determine the mapping relation f and the potential $\mathcal{C}$ under these two conditions, depending on the sign of α:

— **Massless scalar**

$$\alpha = 0: \quad Y = f = \phi, \quad \mathcal{C} = \mathcal{C}_m \tag{11}$$

— **Massive scalar**

$$\alpha = -\gamma^2 < 0: \quad \Phi = f(\phi) = \frac{1}{\gamma}\sin^{-1}(\gamma\phi), \quad \mathcal{C}(\Phi) = \frac{\mathcal{C}_m}{\cos(\gamma\Phi)} \tag{12}$$

— **Tachyon**

$$\alpha = \beta^2 > 0: \quad T = f(\phi) = \frac{1}{\beta}\sinh^{-1}(\beta\phi), \quad \mathcal{C}(T) = \frac{\mathcal{C}_m}{\cosh(\beta T)}{}^{a} \tag{13}$$

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[a] This potential has been derived in string field theory [*Kutasov03*] and is widely adopted in studies of tachyon condensation

After the field redefinition, the DBI action of X becomes:

$$S = -\int d^{p+1}x V(\phi)\sqrt{U(\phi) + \partial_\mu\phi\partial^\mu\phi}. \tag{14}$$

where

$$V(\phi) = |f'|\mathcal{C}(f) = \frac{\mathcal{C}_m}{1+\alpha\phi^2}, \tag{15}$$

$$U(\phi) = \frac{1}{(f')^2} = 1 + \alpha\phi^2. \tag{16}$$

This action for a tachyon field case has been derived in string field theory [*Kutasov03*].

- **More about the exact solution**:

$$\phi = \phi_+ e^{ik\cdot x} + \phi_- e^{-ik\cdot x}, \quad k^2 = \alpha, \tag{17}$$

(a) For a tachyon field $\alpha = \beta^2 > 0$, no real solution has been found in the full spacetime-dependent case (because for a tachyon field of growing mode, k_0 is imaginary but $\vec{k}$ is real). But real solutions can exist in the time-dependent and the space-dependent cases, which are respectively

$$T(t) = \frac{1}{\beta}\sinh^{-1}[T_{sh}\sinh(\beta t) + T_{ch}\cosh(\beta t)], \tag{18}$$

and

$$T(\vec{x}) = \frac{1}{\beta}\sinh^{-1}[T_s \sin(\vec{k}\cdot\vec{x}) + T_c \cos(\vec{k}\cdot\vec{x})], \quad \vec{k}^2 = \beta^2. \text{[a]} \tag{19}$$

[a] In the $p = 1$ case, it is the well known kink-anti-kink solution.

(b) For a massless scalar $\alpha = 0$, besides the above **oscillating mode** solution, there exists an extra **zero mode** solution:

$$\partial_\mu Y = \partial_\mu \phi = a_\mu, \tag{20}$$

with $a^2 \geq -1$. It describes a p-dimensional plane moving with the velocity $\dot{Y} = a_0$.

In contrast to the **mode expansions of strings**, here the exact solution implies: the Dp-brane moves or oscillates in a single mode.

- **The energy-momentum tensor from the exact solution**

  — The energy-momentum tensor from the exact solutions shows that the pressure of **massless and massive scalars** is generically **negative** on stable D-branes.

  For example, for the oscillating mode solution of the massless scalar, the energy-momentum tensor is

$$T_{\mu\nu} = \mathcal{C}_m(\partial_\mu Y \partial_\nu Y - \eta_{\mu\nu}). \tag{21}$$

  — The energy-momentum tensor of **tachyon field** on unstable D-branes cannot be obtained from the exact solution, but has been determined in previous work. The result shows that the tachyon field evolves into a **pressureless** state, probably corresponding to some unknown **tachyon matter**.

Exact solutions in the presence of worldvolume massless fields

In the presence of worldvolume massless fields, the DBI effective action (for a single Dp-brane) is given by [*Garousi00,Sen03,04*]:

$$S = -\int d^{p+1}x\,\mathcal{C}(X)\sqrt{-\det(\eta_{\mu\nu} + \partial_\mu X\partial_\nu X + \partial_\mu Y^I\partial_\nu Y^I + F_{\mu\nu})}, \tag{22}$$

where

- $Y^I(I = p+1, \cdots, D-1)$: a set of massless scalars transverse to the worldvolume of the Dp-brane, describing the fluctuations of the Dp-brane in transverse directions

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: the field strength of the $U(1)$ gauge field A_μ on the Dp-brane

Coupling to transverse massless scalars

Dropping the gauge fields, we can write the action in the explicit form:

$$\mathcal{L} = -\mathcal{C}(X)\left[1 + M + \partial_\mu X \partial^\mu X + \frac{1}{2} H^{IJ}_{\mu\nu} H^{IJ\mu\nu} + \frac{1}{2} H^{I}_{\mu\nu}(X) H^{I\mu\nu}(X)\right]^{\frac{1}{2}}, \tag{23}$$

where

$$M = \sum_I \partial_\mu Y^I \partial^\mu Y^I, \tag{24}$$

$$H^I_{\mu\nu}(X) = \partial_\mu X \partial_\nu Y^I - \partial_\nu X \partial_\mu Y^I, \tag{25}$$

$$H^{IJ}_{\mu\nu} = \partial_\mu Y^I \partial_\nu Y^J - \partial_\nu Y^I \partial_\mu Y^J. \tag{26}$$

After the field redefinitions $X = f(\phi)$ adopted in the previous section, the action is rewritten

$$\mathcal{L} = -V(\phi)\left[\left(1 + M + \frac{1}{2}H^{IJ}_{\mu\nu}H^{IJ\mu\nu}\right)U(\phi) + \partial_\mu\phi\partial^\mu\phi + \frac{1}{2}H^{I}_{\mu\nu}H^{I\mu\nu}\right]^{\frac{1}{2}}, \tag{27}$$

where

$$U(\phi) = \frac{\mathcal{C}_m}{V(\phi)} = 1 + \alpha\phi^2, \tag{28}$$

$$H^{I}_{\mu\nu} = \partial_\mu\phi\partial_\nu Y^I - \partial_\nu\phi\partial_\mu Y^I, \tag{29}$$

$\alpha = 0$: ϕ massless scalar; $\alpha < 0$: ϕ massive scalar; $\alpha > 0$: ϕ tachyon.

- **Equations of motion**

The equations of motion of Y^I and ϕ from this Lagrangian are respectively

$$\partial^\mu \left(\frac{\partial_\mu Y^I + H^{IJ}_{\mu\nu} \partial^\nu Y^J - V(\phi) H^I_{\mu\nu} \partial^\nu \phi}{y} \right) = 0, \tag{30}$$

$$\partial^\mu \left(\frac{\partial_\mu \phi + H^I_{\mu\nu} \partial^\nu Y^I}{y} \right) + \left(1 + M + \frac{1}{2} H^{IJ} \cdot H^{IJ} \right) \frac{\alpha\phi}{y} = 0, \tag{31}$$

where y is the kinetic part of the DBI action: $\mathcal{L} = -V(\phi) y$.

- **Exact solutions in the time-dependent case**

For a **massless scalar** $\alpha = 0$, ϕ has the massless scalar solution.

For a **tachyon** $\alpha = \beta^2$,

$$\phi = \phi_{sh} \sinh(\omega t) + \phi_{ch} \cosh(\omega t), \quad \omega^2 = (1 - a_0^2)\beta^2, \tag{32}$$

For a **massive scalar** $\alpha = -\gamma^2$,

$$\phi = \phi_s \sin(\omega t) + \phi_c \cos(\omega t), \quad \omega^2 = (1 - a_0^2)\gamma^2, \tag{33}$$

where a_0 is the velocity of the whole brane

$$a_0^2 = \sum_I (\dot{Y}^I)^2. \tag{34}$$

Implications:

—D-branes move or not: massless scalar propagate with the same speed.

—Unstable D-branes (like D-particles) decay slower when moving faster;

—Massive scalar oscillates slower on a stable D-brane moving faster;

- **Exact solutions in the spacetime-dependent case**

  We find exact solutions from the equations of motion

$$\partial_\mu Y^I(x^\mu) = a^I_\mu, \quad a^2 = \sum_I a^I_\mu a^{I\mu} \geq -1, \tag{35}$$

$$\phi(x^\mu) = \phi_+ e^{ik\cdot x} + \phi_- e^{-ik\cdot x}, \quad k^2 = (1+a^2)\alpha, \tag{36}$$

with their momenta satisfying the coupling relations

$$\frac{a^I_\mu}{a^I_\nu} = \frac{a^J_\mu}{a^J_\nu} = \frac{k_\mu}{k_\nu}, \tag{37}$$

which equivalently give

$$\frac{a^I_0}{a^J_0} = \frac{a^I_1}{a^J_1} = \cdots \frac{a^I_p}{a^J_p} = \text{const}, \quad \frac{a^I_0}{k_0} = \frac{a^I_1}{k_1} = \cdots = \frac{a^I_p}{k_p} = \text{const}, \tag{38}$$

Coupling to gauge fields

Dropping the transverse massless scalars Y^I, we get the DBI action of X coupling to gauge fields

$$S = -\int d^{p+1}x\mathcal{C}(X)\sqrt{-\det(\eta_{\mu\nu} + \partial_\mu X\partial_\nu X + F_{\mu\nu})}. \tag{39}$$

We take the $p = 3$ brane as an example. In this case, this action can be written explicitly

$$S = -\int d^4x\mathcal{C}(X)\left[1 + \partial_\mu X\partial^\mu X + \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16}(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2 - G_X^2\right]^{\frac{1}{2}}, \tag{40}$$

where

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}, \tag{41}$$

$$G_X^2 = (\widetilde{F}^{\mu\nu}\partial_\nu X)(\widetilde{F}_{\mu\rho}\partial^\rho X). \tag{42}$$

After field redefinition, the action becomes

$$\mathcal{L} = -Vy = -V\left[\left(1 + \frac{1}{2}F\cdot F - \frac{1}{16}(F\cdot\widetilde{F})^2\right)U + \partial_\mu\phi\partial^\mu\phi - G^2\right]^{\frac{1}{2}}, \tag{43}$$

where

$$U = \frac{\mathcal{C}_m}{V} = 1 + \alpha\phi^2, \tag{44}$$

$$G^2 = G^\mu G_\mu = (\widetilde{F}^{\mu\nu}\partial_\nu\phi)(\widetilde{F}_{\mu\rho}\partial^\rho\phi). \tag{45}$$

- **Equations of motion**

$$\partial^\mu \left(\frac{F_{\mu\nu} - \frac{1}{4}(F \cdot \widetilde{F})\widetilde{F}_{\mu\nu}}{y} \right) = V \epsilon_{\mu\nu\rho\lambda} \partial^\mu \left(\frac{G^\rho}{y} \right) \partial^\lambda \phi, \tag{46}$$

and

$$\partial^\mu \left(\frac{\partial_\mu \phi + \widetilde{F}_{\mu\nu} G^\nu}{y} \right) + \left(1 + \frac{1}{2} F \cdot F - \frac{1}{16} (F \cdot \widetilde{F})^2 \right) \frac{\alpha \phi}{y} = 0. \tag{47}$$

- **Exact solutions in the time-dependent case**

For a **tachyon** $\alpha = \beta^2$

$$\phi = \phi_{sh} \sinh(\omega t) + \phi_{ch} \cosh(\omega t), \quad \omega^2 = (1 - r)\,\beta^2, \tag{48}$$

For a **massive scalar** $\alpha = -\gamma^2$

$$\phi = \phi_s \sin(\omega t) + \phi_c \cos(\omega t), \quad \omega^2 = (1 - r)\,\gamma^2, \tag{49}$$

where

$$r = \frac{\mathbf{E} \cdot \mathbf{E}}{1 + \mathbf{B} \cdot \mathbf{B}}. \tag{50}$$

Implications:

—The electric fields tend to slow down while the magnetic fields tend to expedite the decay process of unstable D-branes.

—Similar results apply to the massive scalar.

- **Exact solutions in the presence of constant EM fields**

In the spacetime-dependent case, setting all components of the EM fields **E** and **B** to be constant, we can have the exact solution from the equations of motion

$$\phi = \phi_+ e^{ik\cdot x} + \phi_- e^{-ik\cdot x}, \tag{51}$$

with

$$k^2 = (\widetilde{F}_{\mu\nu} k^\nu)(\widetilde{F}^{\mu\rho} k_\rho) + [1 + \mathbf{B}^2 - \mathbf{E}^2 - (\mathbf{E}\cdot\mathbf{B})^2]\alpha, \tag{52}$$

Implications:

The first term of the expression of k^2 is non-negative. Thus a tachyon ($\alpha > 0$) will be always a tachyon ($k^2 > 0$) in the presence of constant EM fields.

However, the massless and massive scalars ($\alpha \leq 0$) may become tachyonic ($k^2 > 0$) in the presence of the constant EM fields. To avoid this, massless and massive scalars on D-branes must oscillate in some specific modes such that $k^2 \leq 0$.

- **Exact solutions in the presence of EM wave**

  For simplicity, we switch on a single gauge field component A_3 which satisfies the Lorentz gauge condition $\partial^3 A_3 = 0$ (i.e., A_3 is homogeneous along the x^3 direction). Then the non-vanishing components of the field strength are:

$$F_{03} = \partial_0 A_3 = E_3, \quad F_{13} = \partial_1 A_3 = B_2, \quad F_{23} = \partial_2 A_3 = -B_1 . \tag{53}$$

  For convenience, we denote A_3 by A and $m = 0, 1, 2$.

In this case, the solutions of A and ϕ from the above two field equations can be:

$$A = A_+ e^{iq_m x^m} + A_- e^{-iq_m x^m}, \quad q_0^2 = q_1^2 + q_2^2, \tag{54}$$

$$\phi(x^\mu) = \phi_+ e^{ik_\mu x^\mu} + \phi_- e^{-ik_\mu x^\mu}, \quad \frac{k_0}{q_0} = \frac{k_1}{q_1} = \frac{k_2}{q_2}, \quad k_3^2 = \alpha, \tag{55}$$

Features:

– When $\alpha = 0$, the solutions are valid, indicating that the massless scalar ϕ propagates together with the gauge field A;

– When $\alpha = \beta^2$, k_3 is real. To keep ϕ real, k_0, k_1 and k_2 must be all real as well. Thus ϕ oscillates, describing a scalar propagating faster than light ——- problematic;

– When $\alpha = -\gamma^2$, k_3 is imaginary. To keep ϕ real, k_0, k_1 and k_2 must be all imaginary. Thus ϕ grows ——- problematic.

Summary

- It is possible to obtain exact solutions in the DBI effective theory, even in the presence of worldvolume massless fields, which makes it possible to extract exact information in this theory
- From the exact solutions, we found that the pressure of massless and massive scalars on D-branes was generically negative
- By deriving the exact solutions, we can determine the dynamics of D-branes that are moving at different angles, and that contain EM fields or EM waves on them
- Massless fields (like massless scalars and gauge fields) and the couplings between them are well described in the DBI effective theory (as an action at low energy). In any case, we found valid exact solution