Introduction to Loop Quantum Gravity

Yongge Ma

Department of Physics, Beijing Normal University

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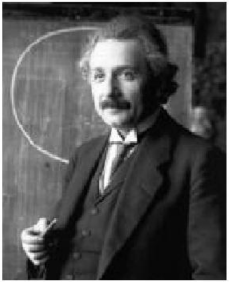
mayg@bnu.edu.cn

Outline

1. **Introduction: Motivations and Ideas**
2. **Classical Connection Dynamics of GR**
3. **Fundamental Structure of LQG**
4. **Applications of LQG**
5. **Summary and Outlook**

Motivations of quantum gravity

From the beginning of last century to now, two fundamental theories of physics, QM and GR, have destroyed the coherent pictures of the physical world.



- *Classical Gravity - Quantum Matter Inconsistency*

$$R_{\alpha\beta}[g] - \frac{1}{2}R[g]g_{\alpha\beta} = \kappa T_{\alpha\beta}[g].$$

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  In 1960s Penrose and Hawking proved that singularities are inevitable in general spacetimes with certain reasonable conditions on energy and causality by the well-known singularity theorem.
  Thus general relativity can not be valid unrestrictedly.

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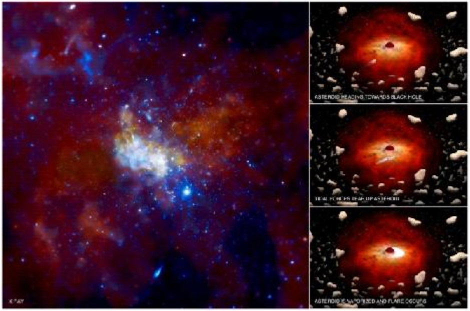
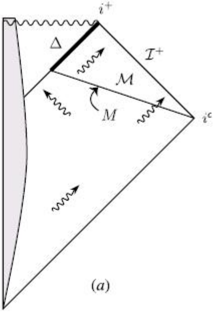
- *Infinity in Quantum Field Theory*
  It is expected that some quantum gravity theory, playing a fundamental role at Planck scale, would provide a natural cut-off to cure the UV singularity in quantum field theory.

- *The interpretation of black hole thermodynamics*

$$S_{BH} = \frac{k_B c^3 Ar_{BH}}{4G\hbar}.$$

This equation brings together the three pillars of fundamental physics.





The basic ideas of LQG

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  The separation of the gravitational field from background spacetime is in strident contradiction with the very lesson of GR.
- ⋆ *The viewpoint of background independence:*
  GR's revolution: particle and fields are neither immersed in external space nor moving in external time, but live on one another.
  The quanta of the field cannot live in background spacetime. They should build spacetime themselves.

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- ⋆ The choice of the algebra of field functions to be quantized:
  Not the positive and negative components of the field modes as in conventional QFT; but the holonomies of the gravitational connection and the electric flux.
- ⋆ The physical meaning of holonomies:
  Faraday - lines of force: the relevant variables do not refer to what happens at a point, but rather refer to the relation between different points connected by a line.

$$A(c) = \mathcal{P} \exp\big(-\int_0^1 [A_a^i \dot{c}^a \tau_i]\ dt \big).$$

The Lagrangian formulation

- The generalized Palatini action in which we are interested is given by [Holst 1996]:

$$S_G[e_K^\beta, \omega_\alpha{}^{IJ}] = \frac{1}{2\kappa}\int_M d^4x(e)e_I^\alpha e_J^\beta(\Omega_{\alpha\beta}{}^{IJ} + \frac{1}{2\beta}\epsilon^{IJ}{}_{KL}\Omega_{\alpha\beta}{}^{KL}), \quad (1)$$

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  where β is the Barbero-Immirzi parameter.
- The variation of S_G with respect to the connection $\omega_\alpha{}^{IJ}$ leads to the conclusion that the connection $\omega_\alpha{}^{IJ}$ is the unique torsion-free Levi-Civita spin connection compatible with the tetrad e^α_I.
- As a result, the second term in the action S_G is vanishing. So the generalized Palatini action returns to the Palatini action, which will certainly give the Einstein field equation.

Classical Connection Dynamics of GR

- After the 3+1 decomposition and the Legendre transformation, the generalized Palatini action can be expressed as:

$$S_G = \int_{\mathbf{R}} dt \int_{\Sigma} d^3x [\widetilde{P}^a_i \mathcal{L}_t A^i_a - \mathcal{H}_{tot}(A^i_a, \widetilde{P}^b_j, \Lambda^i, N, N^c)],$$

where the configuration and conjugate momentum are defined respectively by:

$$\begin{aligned} A^i_a &:= \Gamma^i_a + \beta K^i_a, \\ \widetilde{P}^a_i &:= \frac{1}{2\kappa\beta}\widetilde{\eta}^{abc}\epsilon_{ijk}e^j_b e^k_c = \frac{1}{\kappa\beta}\sqrt{\det q}\, e^a_i. \end{aligned}$$

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- The Hamiltonian density $\mathcal{H}_{tot}$ is a linear combination of constraints:

$$\mathcal{H}_{tot} = \Lambda^i G_i + N^a V_a + NS,$$

where $\Lambda^i \equiv -\frac{1}{2} \epsilon^i{}_{jk} \omega_t{}^{jk}$.

The Hamiltonian formulation

- The three constraints are expressed as:

$$
\begin{aligned}
G_i &= D_a\widetilde{P}_i^a := \partial_a\widetilde{P}_i^a + \epsilon_{ij}{}^{k}A_a^j\widetilde{P}_k^a, \\
V_a &= \widetilde{P}_i^b F_{ab}^i - \frac{1+\beta^2}{\beta}K_a^i G_i, \\
S &= \frac{\kappa\beta^2}{2\sqrt{\det q}}\widetilde{P}_i^a\widetilde{P}_j^b[\epsilon^{ij}{}_{k}F_{ab}^k - 2(1+\beta^2)K_{[a}^i K_{b]}^j] \\
&+ \kappa(1+\beta^2)\partial_a\Big(\frac{\widetilde{P}_i^a}{\sqrt{\det q}}\Big)G^i,
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here the configuration variable A_a^i performs as a $su(2)$-valued connection on Σ, and F_{ab}^i is the $su(2)$-valued curvature 2-form of A_a^i. The constraints are all of first class.

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S &= \frac{\kappa\beta^2}{2\sqrt{\det q}} \widetilde{P}^a_i \widetilde{P}^b_j [\epsilon^{ij}{}_k F^k_{ab} - 2(1+\beta^2) K^i_{[a} K^j_{b]}] \\
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  here the configuration variable A^i_a performs as a $su(2)$-valued connection on Σ, and F^i_{ab} is the $su(2)$-valued curvature 2-form of A^i_a. The constraints are all of first class.

- Thus general relativity is cast as a dynamical theory of connections with a compact structure group.

A new understanding of the Barbero-Immirzi parameter

- An alternative action for Poincare gauge theory of gravity:

$$S = \int d^4x \left(e\, e^\mu_I e^\nu_J \Omega_{\mu\nu}{}^{IJ}(\omega) - \frac{1}{8\beta} \left[\epsilon^{\mu\nu\rho\sigma} T^I{}_{\mu\nu} T_{I\rho\sigma} \right] \right)$$

where $T^I{}_{\mu\nu}$ is the torsion of the connection ω.

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  where $T^I{}_{\mu\nu}$ is the torsion of the connection ω.
- In the vacuum case, this action gives exactly the same $SU(2)$-connection dynamics of GR as the generalized Palatini action [Yang, Banerjee, YM, 2013].
- Thus the Barbero-Immirzi parameter acquires its meaning as the relative contribution of torsion in comparison with curvature in the action.

Kinematical Structure of LQG

- Classical configurations:
  $\mathcal{A} = \{A|_{\Sigma},\ smooth,\ suitable\ boundary\ condition\}$.

Build the infinite dimensional integration theory from the finite one.

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- Quantum configuration space and Hilbert space: $\mathcal{H}_{kin} = L^2(\overline{\mathcal{A}},\ d\mu^0)$.
  [Ashtekar and Isham, 1992; Ashtekar and Lewandowski, 1995]
- Uniqueness Theorem [LOST, 2005]:
  There is a unique gauge and diffeomorphism invariant representation of the holonomy-flux $*-$algebra on $\mathcal{H}_{kin}$.

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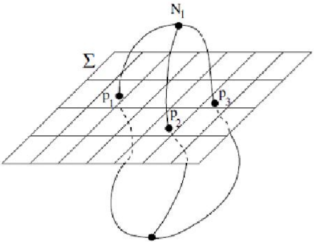

Geometric operators

- Area operator [Rovelli and Smolin, 1995; Ashtekar and Lewandowski, 1997]
  Given a closed 2-surface or a surface S with boundary, its area can be well defined as a self-adjoint operator $\hat{A}_S$ on $\mathcal{H}_{kin}$:

$$\hat{A}_S\psi_\gamma = 4\pi\beta\ell_p^2 \sum_{v\in V(\gamma\cap S)} \sqrt{(\hat{J}^{(S,v)}_{i(u)} - \hat{J}^{(S,v)}_{i(d)})(\hat{J}^{(S,v)}_{j(u)} - \hat{J}^{(S,v)}_{j(d)})\delta^{ij}}\ \psi_\gamma .$$

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One can find some finite linear combinations of spin network basis in $\mathcal{H}_{kin}$ which diagonalize $\hat{A}_S$ with eigenvalues given by finite sums,

$$a_S = 4\pi\beta\ell_p^2 \sum_I \sqrt{2j^{(u)}(j^{(u)}+1) + 2j^{(d)}(j^{(d)}+1) - j^{(u+d)}(j^{(u+d)}+1)},$$

where $j^{(u)}, j^{(d)}$ and $j^{(u+d)}$ are arbitrary half-integers subject to the standard condition

$$j^{(u+d)} \in \{|j^{(u)} - j^{(d)}|, |j^{(u)} - j^{(d)}| + 1, ..., j^{(u)} + j^{(d)}\}. \quad (3)$$

- Thus the spectrum of the area operator is fundamentally pure discrete, while its continuum approximation becomes excellent for large eigenvalues.

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- Volume operator [Ashtekar and Lewandowski, 1995, 1997; Rovelli and Smolin, 1995]
  The volume of a compact region R can also be well defined as a self-adjoint operator with discrete spectrum.
- Other geometric operators:
  Length operator [Thiemann 1998; YM, Soo, Yang, 2010].
  $\hat{Q}$ operator [YM and Y. Ling, 2000].
  Quasi-local energy operator [Yang and YM, 2009].

Implementation of Quantum Constraints

- The Gaussian constraint operator can be defined in $\mathcal{H}_{kin}$.
  The kernel of the operator is the internal gauge invariant Hilbert space: $\mathcal{H}^G = \oplus_{\alpha,\mathbf{j}} \mathcal{H}'_{\alpha,\mathbf{j},\mathbf{l}=0} \oplus \mathbf{C}$.

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- Hamiltonian constraint operators can be well defined in $\mathcal{H}_{kin}$ or $\mathcal{H}^G$ [Thiemann 1998].
  Thus there is no UV divergence in the background independent quantum theory of gravity with diffeomorphism invariance.
- The quantization technique for the Hamiltonian constraint can be generalized to quantize the Hamiltonian of matter fields coupled to gravity [Thiemann 1998].

Black hole entropy in LQG

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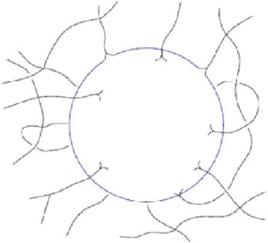
Black hole entropy in LQG

- If one considers the spacetimes which contain an isolated horizon as an internal boundary, the action principle and the Hamiltonian description are well defined.
- By imposing the horizon boundary condition quantum mechanically and employing the spectrum of the area operator in bulk Hilbert space, one can obtain the (black hole) horizon entropy, whose leading term is indeed proportional to the horizon area [Ashtekar el, 2000, 2005; Engle el, 2009].



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- The idea that one should view holonomies rather than connections as basic variables for the quantization of gravity is successfully carried out in the symmetry-reduced models, known as Loop Quantum Cosmology.
- One freezes all but a finite number of degrees of freedom by imposing symmetries.
  The simplified framework provides a simple arena to test ideas and constructions.
- Symmetries: homogeneity and (or) isotropy.
- Example: Spatially flat FRW universe
  - Spatial 3-manifold: $\mathbb{R}^3$
  - Isometry: Euclidean group

The Kinematical Setting of LQC

- One has to introduce an elementary cell $\mathcal{V}$ and restricts all integrations to this cell.
- Fix a fiducial flat metric ${}^{o}q_{ab}$ and denote by V_o the volume of $\mathcal{V}$ in this geometry.
  The gravitational phase space variables —the connections and the density weighted triads — can be expressed as
  $A_a^i = c\, V_o^{-(1/3)}\ {}^{o}\omega_a^i$ and $\widetilde{P}_i^a = p\, V_o^{-(2/3)} \sqrt{{}^{o}q}\ {}^{o}e_i^a$,
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  where $({}^o\omega_a^i, {}^o e_i^a)$ are a set of orthonormal co-triads and triads compatible with ${}^o q_{ab}$ and adapted to $\mathcal{V}$.
- To pass to the quantum theory, one constructs a kinematical Hilbert space $\mathcal{H}_{\rm kin}^{\rm grav} = L^2(\mathbb{R}_{\rm Bohr}, \mathrm{d}\mu_{\rm Bohr})$, where $\mathbb{R}_{\rm Bohr}$ is the Bohr compactification of the real line and $\mathrm{d}\mu_{\rm Bohr}$ is the Haar measure on it.
- There exists no operator corresponding to c, while holonomy operators are well defined.

The Improved Scheme

- It is convenient to introduce new conjugate variables by a canonical transformation:

$$b := \frac{\sqrt{\Delta}}{2}\frac{c}{\sqrt{|p|}}, \qquad \nu := \frac{4}{3\sqrt{\Delta}} sgn(p)|p|^{\frac{3}{2}},$$

where Δ ($\sim 4\sqrt{3}\pi\gamma\, l_p^2$) is the smallest non-zero eigenvalue of area operator in full LQG.

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  where Δ $(\sim 4\sqrt{3}\pi\gamma\, l_p^2)$ is the smallest non-zero eigenvalue of area operator in full LQG.
- In the kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}^{\mathrm{grav}}$, eigenstates of $\hat{\nu}$, which are labeled by real numbers v, constitute an orthonormal basis as: $\langle v_1 | v_2 \rangle = \delta_{v_1, v_2}$.
- The fundamental operators act on $|v\rangle$ as: $\hat{\nu}\,|v\rangle = (8\pi\gamma l_p^2/3) v |v\rangle$ and $\widehat{e^{ib}}\,|v\rangle = |v+1\rangle$.

Alternative Dynamics for LQC

- APS Dynamics
  - The gravitational part of the APS Hamiltonian operator was given in the v representation by [Ashtekar, Pawlowski, Singh, 2006]:

$$\hat{C}_{\mathrm{grav}}\,|v\rangle = f_+(v)|v+4\rangle + f_o(v)|v\rangle + f_-(v)|v-4\rangle. \quad (4)$$

  - To identify a dynamical matter field as an internal clock, one takes a massless scalar field ϕ with Hamiltonian $C_\phi = |p|^{-\frac{3}{2}}\, p_\phi^2/2$, where p_ϕ denotes the momentum of ϕ.

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- Alternative Dynamics
  - LQC Gravitational Hamiltonian operator with Lorentz and Euclidean terms [Yang, Ding, YM, 2009]:

$$\begin{aligned}\hat{H}^F_{\rm grav}|v\rangle = \quad & F'_+(v)|v+8\rangle + f'_+(v)|v+4\rangle + \left(F'_o(v) + f'_o(v)\right)|v\rangle \\ & + f'_-(v)|v-4\rangle + F'_-(v)|v-8\rangle.\end{aligned} \tag{5}$$

  - The new proposed Hamiltonian constraint operator $\hat{H}^F_{\rm grav}$ contains more terms with step of different size comparing to the original APS Hamiltonian operator.

Big Bang Singularity Resolution

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Quantum Nature of the Big Bang

Abhay Ashtekar,* Tomasz Pawlowski,† and Parampreet Singh‡
Institute for Gravitational Physics and Geometry, Physics Department, The Pennsylvania State University, University Park, Pennsylvania 16802, USA
(Received 22 February 2006; published 12 April 2006)

Some long-standing issues concerning the quantum nature of the big bang are resolved in the context of homogeneous isotropic models with a scalar field. Specifically, the known results on the resolution of the big-bang singularity in loop quantum cosmology are significantly extended as follows: (i) the scalar field is shown to serve as an internal clock, thereby providing a detailed realization of the "emergent time" idea; (ii) the physical Hilbert space, Dirac observables, and semiclassical states are constructed rigorously; (iii) the Hamiltonian constraint is solved numerically to show that the big bang is replaced by a big bounce. Thanks to the nonperturbative, background independent methods, unlike in other approaches the quantum evolution is deterministic across the deep Planck regime.

DOI: 10.1103/PhysRevLett.96.141301 PACS numbers: 98.80.Qc, 04.60.Kz, 04.60.Pp

Thanks to the influx of observational data, recent years have witnessed enormous advances in our understanding of the early Universe. To interpret the present data, it is sufficient to work in a regime in which space-time can be taken to be a smooth continuum as in general relativity, lution does not break down at the singularity, as pointed out, e.g., in [7], they did not shed light on what happened before. By constructing the missing conceptual and mathematical infrastructure, we show that the Universe has a classical pre-big-bang branch, joined *deterministically* to

Quantum Bounce

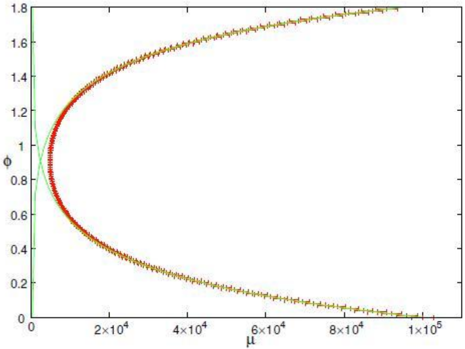

FIG. 2 (color online). The expectation values of Dirac observables $\hat{\mu}|_\phi$ are plotted (in multiples of μ_o), together with their dispersions. They exhibit a quantum bounce which joins the contracting and expanding classical trajectories marked by fainter lines.

Effective Scenarios of LQC

- Effective Hamiltonian and Friedmann Equation
  - We can further obtain an effective Hamiltonian of $\hat{H}_F = \hat{H}^F_{\rm grav} + \hat{H}_\phi$ with the relevant quantum corrections of order $\epsilon^2, 1/v^2\epsilon^2, \hbar^2/\sigma^2 p_\phi^2$ as

$$\begin{aligned}\mathcal{H}^F_{\rm eff} &= -\frac{3^2\sqrt{6}}{2^3}\frac{\hbar^{1/2}}{\gamma^{3/2}\,\kappa^{1/2}}L\,|v|\Big(\sin^2(2b)\left(1-(1+\gamma^2)\sin^2(2b)\right)+2\epsilon^2\Big)\\ &\quad+\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}\frac{|v|}{L}\rho\left(1+\frac{1}{2|v|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2p_\phi^2}\right), \qquad (6)\end{aligned}$$

where $\rho = \frac{1}{2}\left(\frac{6}{\kappa\gamma\hbar}\right)^3\left(\frac{L}{|v|}\right)^2 p_\phi^2$ is the density of the matter field.

Effective Scenarios of LQC

- Effective Hamiltonian and Friedmann Equation
  - We can further obtain an effective Hamiltonian of $\hat{H}_F = \hat{H}^F_{\rm grav} + \hat{H}_\phi$ with the relevant quantum corrections of order $\epsilon^2, 1/v^2\epsilon^2, \hbar^2/\sigma^2 p_\phi^2$ as

$$\mathcal{H}^F_{\rm eff} = -\frac{3^2\sqrt{6}}{2^3}\frac{\hbar^{1/2}}{\gamma^{3/2}\,\kappa^{1/2}} L\,|v|\Big(\sin^2(2b)\left(1-(1+\gamma^2)\sin^2(2b)\right)+2\epsilon^2\Big) + \left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}\frac{|v|}{L}\rho\left(1+\frac{1}{2|v|^2\epsilon^2}+\frac{\hbar^2}{2\sigma^2 p_\phi^2}\right), \qquad (6)$$

    where $\rho = \frac{1}{2}\left(\frac{6}{\kappa\gamma\hbar}\right)^3\left(\frac{L}{|v|}\right)^2 p_\phi^2$ is the density of the matter field.
  - The modified Friedmann equation can then be derived under certain condition as:

$$H_F^2 = \frac{\kappa}{3}\frac{\rho_c}{4(1+\gamma^2)^2}\left(1-\sqrt{1-\chi_F}\right)\left(1+2\gamma^2+\sqrt{1-\chi_F}\right)(1-\chi_F),$$

    where

$$\chi_F = 4(1+\gamma^2)\left(\frac{\rho}{\rho_c}(1+\frac{\hbar^2}{2\sigma^2 p_\phi^2}) - 2\epsilon^2\right). \qquad (7)$$

Effective Scenarios of LQC

PRL **102,** 051301 (2009) PHYSICAL REVIEW LETTERS week ending 6 FEBRUARY 2009

Effective Scenario of Loop Quantum Cosmology

You Ding,* Yongge Ma,† and Jinsong Yang‡
Department of Physics, Beijing Normal University, Beijing 100875, China
(Received 7 August 2008; published 5 February 2009)

Semiclassical states in isotropic loop quantum cosmology are employed to show that the improved dynamics has the correct classical limit. The effective Hamiltonian for the quantum cosmological model with a massless scalar field is thus obtained, which incorporates also the next to leading order quantum corrections. The possibility that the higher order correction terms may lead to significant departure from the leading order effective scenario is revealed. If the semiclassicality of the model is maintained in the large scale limit, there are great possibilities for a $k = 0$ Friedmann expanding universe to undergo a collapse in the future due to the quantum gravity effect. Thus the quantum bounce and collapse may contribute a cyclic universe in the new scenario.

DOI: 10.1103/PhysRevLett.102.051301 PACS numbers: 04.60.Pp, 04.60.Kz, 98.80.Qc

The theoretical search for a quantum theory of gravity has been rather active. The expectation that the singularities predicted by classical general relativity would be resolved bysome quantum gravity theory has been confirmed by the recent study of certain isotropic models in

theory, to predict phenomena which are dramatically different from those of the classical theory is also a hallmark to identify a quantum theory.

The framework that we are considering is the so-called improved dynamics of LQC [3]. In the kinematical setting,

Cyclic Universe

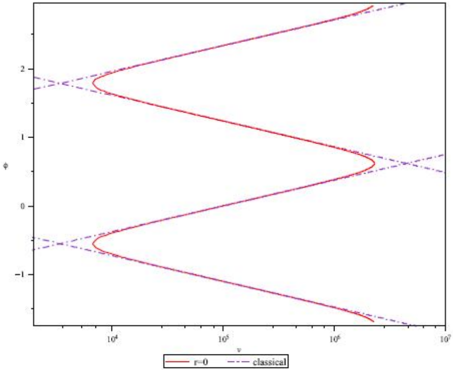

FIG. 2: The cyclic model is compared with expanding and contracting classical trajectories. In this simulation, the parameters were: $G = \hbar = 1$, $p_\phi = 10\,000$, $\epsilon = 0.001$, $\sigma = 0.01$ with initial data $v_o = 100\,000$.

LQC Extension of the Inflationary Scenario

Selected for a Viewpoint in *Physics*

PRL **109,** 251301 (2012) PHYSICAL REVIEW LETTERS week ending 21 DECEMBER 2012

Quantum Gravity Extension of the Inflationary Scenario

Ivan Agullo, Abhay Ashtekar, and William Nelson

Institute for Gravitation and the Cosmos & Physics Department, Penn State, University Park, Pennsylvania 16802, USA

(Received 7 September 2012; published 17 December 2012)

Since the standard inflationary paradigm is based on quantum field theory on classical space-times, it excludes the Planck era. Using techniques from loop quantum gravity, the paradigm is extended to a self-consistent theory from the Planck scale to the onset of slow roll inflation, covering some 11 orders of magnitude in energy density and curvature. This preinflationary dynamics also opens a small window for novel effects, e.g., a source for non-Gaussianities, which could extend the reach of cosmological observations to the deep Planck regime of the early Universe.

DOI: 10.1103/PhysRevLett.109.251301 PACS numbers: 98.80.Qc, 04.60.Pp, 98.80.Cq

The inflationary paradigm has had remarkable success in accounting for the inhomogeneities in the cosmic microwave background (CMB) that serve as seeds for the large scale structure of the Universe. However, it has certain conceptual limitations from particle physics as well as quantum gravity perspectives. For example, (i) the physical that comes from the 7 year WMAP data [5,6]. Throughout we use natural Planck units.

The truncated phase space.—We have $\Gamma_{\mathrm{Trun}} = \Gamma_0 \times \Gamma_1$ where Γ_0 is the four-dimensional phase space of homogeneous fields, and Γ_1, of the first order, purely inhomogeneous perturbations thereon. Γ_0 is conveniently

Radio of LQC scalar power spectrum to that of standard inflation

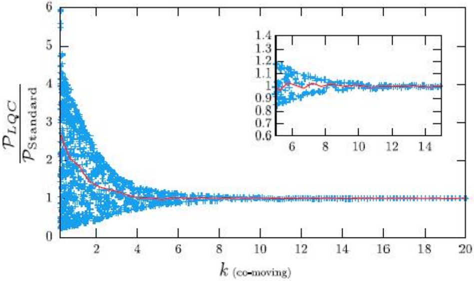

FIG. 1 (color online). Ratio of our LQG power spectrum for scalar perturbations to the standard inflationary power spectrum. The (blue) crosses denote the data points. For small k, the ratio oscillates rapidly with k. The solid (red) curve shows averages over bins of width $\Delta k = 0.5\ell_{\text{Pl}}^{-1}$. The inset shows a blow-up of the interesting region around $k = 9$.

Challenges from Cosmology and Astrophysics

- Dark Universe: There are great challenges to GR coming from astronomic observations.
  - Cosmology - Dark energy problem: Our universe is currently undergoing a period of accelerated expansion.
  - Dark Matter: There are indirect evidences suggesting that the bulk of the matter of the universe is invisible or dark.

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- A large variety of models of $f(\mathcal{R})$ modified gravity have been proposed to account for the present cosmic acceleration.
  Some models of $f(\mathcal{R})$ gravity may even account for the dark matter problem.
  [For a review on $f(\mathcal{R})$ theories, see Sotiriou and Faraoni 2010.]

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  [For a review on $f(\mathcal{R})$ theories, see Sotiriou and Faraoni 2010.]
- The action of $f(\mathcal{R})$ theories reads:

$$S[g] = \frac{1}{2}\int d^4x\sqrt{-g}f(\mathcal{R}), \tag{8}$$

where f is a general function of the scalar curvature $\mathcal{R}$, and we set $8\pi G = 1$.

Extension of LQG to $f(R)$ Theories

PRL **106,** 171301 (2011) PHYSICAL REVIEW LETTERS week ending 29 APRIL 2011

Extension of Loop Quantum Gravity to $f(R)$ Theories

Xiangdong Zhang* and Yongge Ma†
Department of Physics, Beijing Normal University, Beijing 100875, China
(Received 10 January 2011; published 28 April 2011)

The four-dimensional metric $f(\mathcal{R})$ theories of gravity are cast into connection-dynamical formalism with real $su(2)$ connections as configuration variables. Through this formalism, the classical metric $f(\mathcal{R})$ theories are quantized by extending the loop quantization scheme of general relativity. Our results imply that the nonperturbative quantization procedure of loop quantum gravity is valid not only for general relativity but also for a rather general class of four-dimensional metric theories of gravity.

DOI: 10.1103/PhysRevLett.106.171301 PACS numbers: 04.60.Pp, 04.20.Fy, 04.50.Kd

In the recent 25 years, loop quantum gravity (LQG), a background-independent approach to quantize general relativity (GR), has been widely investigated [1–4]. It is remarkable that, as a nonrenormalizable theory, GR can be nonperturbatively quantized by the loop quantization procedure [5]. This background-independent quantization relies on the key observation that classical GR can be constructed by the concept of holonomy, since its definition does not depend on an extra background. In this Letter, we derive the connection-dynamical formulation of $f(\mathcal{R})$ gravity by canonical transformations from its geometrical dynamics. The latter was realized by introducing a nonminimally coupled scalar field to replace the original $f(\mathcal{R})$ action and then doing Hamiltonian analysis.

Alternative Expression of $f(\mathcal{R})$ Theories

- By introducing an independent variable s and a Lagrange multiplier ϕ, an action equivalent to (8) of $f(\mathcal{R})$ theories is proposed as

$$S[g,\phi,s]=\frac{1}{2}\int d^4x\sqrt{-g}(f(s)-\phi(s-\mathcal{R})). \tag{9}$$

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$$S[g,\phi,s] = \frac{1}{2}\int d^4x\sqrt{-g}(f(s) - \phi(s - \mathcal{R})). \tag{9}$$

- The variation of (9) with respect to s yields $\phi = df(s)/ds \equiv f'(s)$.
- Assuming s could be resolved from the above equation, action (9) is reduced to

$$S[g,\phi] = \frac{1}{2}\int d^4x\sqrt{-g}(\phi\mathcal{R} - \xi(\phi)) \equiv \int d^4x\mathcal{L} \tag{10}$$

  where $\xi(\phi) \equiv \phi s - f(s)$.
- The virtue of (10) is that it admits a treatable Hamiltonian analysis [Deruelle el, 2009; Zhang, YM, 2011].

Loop Quantum Scalar-Tensor Theory and Cosmology

- Nonperturbatve loop quantization of scalar-tensor theories (STT)
  - The Hamiltonian connection dynamics of STT of gravity has been obtained [Zhang, YM, 2011].
    The variational principle for this connection dynamical formalism is also found [Zhou, Guo, Han, YM, 2013].
  - Due to the $su(2)$-connection dynamical formalism, the 4-dimensional STT have been quantized by extending LQG scheme.

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  - Due to the $su(2)$-connection dynamical formalism, the 4-dimensional STT have been quantized by extending LQG scheme.
- Loop quantum Brans-Dicke cosmology
  [Zhang, Artymowski, YM, 2013]
  - The scalar may play the role of emergent (internal) time.
    The notion of time emerges from gravity itself rather than an outside matter field.
  - The effective Hamiltonian can also be obtained and confirm the correct classical limit.
    The classical big bang singularity is again replaced by a quantum bounce.

Success of LQG

- ⋆ It is remarkable that, as a non-renormalizable theory, GR can be non-perturbatively quantized by the loop quantization procedure. The idea of background independence is successful realized in the construction of LQG.

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- ⋆ The success of canonical LQG relies on the key observation that classical GR can be cast into the connection-dynamical formalism with a compact structure group.
- ⋆ Geometrical operators in LQG, such as area and volume, have discrete spectrum.
- ⋆ There is no UV divergence in LQG with standard matter fields. Thus the divergence of QFT could be cured by LQG without renormalization.

Advances

- ⋆ The idea and technique of LQG have been successfully carried out in the symmetry-reduced models, known as Loop Quantum Cosmology.
    - Alternative Hamiltonian operators for LQC have been proposed, which have correct classical limit.
    - The big bang singularity of classical GR can be resolved by a quantum bounce of LQC.
    - There is an amazing possibility that quantum gravity manifests itself in the large scale cosmology
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  - There is an amazing possibility that quantum gravity manifests itself in the large scale cosmology
  - The inflationary scenario can be extend to Planck scale with a small window of novel effects.
- Other Advances:
  - LQG is applicable to higher dimensional GR [Bodendorfer, Thiemann, Thurn, 2011], as well as higher dimensional scalar-tensor theories of gravity [Han, Ma, Zhang, 2013].
  - The path integral formulation of LQG: Spin foam models.
  - Inflation in loop quantum cosmology [Bojowald 2002; Ashtekar and Sloan, 2009, 2011].

Outlook

- Applicable Scope:
  - ⋆ The non-perturbative loop quantization procedure can also be extended to $f(\mathcal{R})$ theories and more general scalar-tensor theories.
  - ⋆ Our Conservative Observation: Metric theories of gravity (with well-defined geometrical dynamics) in arbitrary dimensions.

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  - ⋆ Our Conservative Observation: Metric theories of gravity (with well-defined geometrical dynamics) in arbitrary dimensions.
- Open Problems:
  - The classical limit of LQG: Semiclassical analysis.
  - Contact with low energy phys, QFT on curved spacetime.
  - Singularity resolution in full LQG.
  - Hawking effect and the information issue from first principles.
  - Contact between the full LQG and the symmetric models.
  - Contact between canonical and spin foam programs.
  - Applications of loop quantum $f(\mathcal{R})$ theories and Brans-Dicke theory to cosmology and black holes are desirable [Guo, Han, Zhang, YM...].
  - Contact with string (or M-) theory, noncommutative geometry...

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!Thanks!