The AdS space origin of hidden conformal symmetry in non-extremal black holes

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1. Introduction

- **Extremal black holes**: the near-horizon geometry is usually AdS space
- **AdS/CFT**: explicit realisation of holography; miricroscopic interpretation of black hole entropy in the dual CFT
- **Near-extremal black holes**: viewed as linear excitation of extremal black holes, CFT at small temperature
- **Generic non-extremal black holes**: non-linear excitation and strong backreaction; the near-horizon geometry is usually taken as Rindler space, but not AdS space any more

1. Introduction

- AdS space and conformal symmetry: AdS$_{p+2}$/CFT$_{p+1}$
  - AdS_{p+2} space embedded in $(p+3)$-dimensional flat space:

$$X_0^2 + X_{p+2}^2 - X_i X^i = R^2, \quad (1)$$

    with the metric:

$$ds^2 = -dX_0^2 + dX_{p+2}^2 + dX_i dX^i. \quad (2)$$

    Isometry: $SO(2, p+1)$.
  - On the other hand, $SO(2, p+1)$ is also the conformal group in $(p+1)$-dimensional Minkowski spacetime:
    Poincare transformation + scale transformation $[x^\mu \to \lambda x^\mu]$
    + special conformal transformation
    $[x^\mu \to (x^\mu + a^\mu x^2)/(1 + 2a \cdot x + a^2 x^2)]$.
    If we do not involve the Lorentz transformations ($M_{\mu\nu}$), the generators for translation (P_μ), scaling transformation (D) and special conformal transformation (K_μ) obey

$$[D, K_\mu] = iK_\mu, \quad [D, P_\mu] = -iP_\mu, \quad [P_\mu, K_\nu] = -2i\eta_{\mu\nu} D \quad (3)$$

    where $\mu = 1, \cdots, p$.

Introduction

Coordinate transformations:

- Three kinds of AdS spaces (e.g., [Gibbons 11])
  (I) $X^0/X^{p+2} = \mathrm{const}$:

$$ds^2 = -(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2 d\Omega_{p,1}^2, \qquad (4)$$

  where $d\Omega_{p,1}$: sphere S^p.

  - AdS space in global coordinates.
  - Redefinition $r = \sinh\rho$:

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_{p,1}^2. \qquad (5)$$

  - Globally static because of no Killing horizon.

Introduction

(II) $X^0/(X^{p+2} + X^{p+1}) = \text{const}$:

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\Omega_{p,0}^2, \tag{6}$$

where $d\Omega_{p,0}$: flat E^p.

- AdS space in Poincare coordinates.
- Globally static with Killing horizon at $r = 0$.
- Usually the near-horizon geometry of extremal black holes.
  The AdS/CFT correspondence is well established in this case.

Introduction

(III) $X^0/X^{p+1} = \text{const}$:

$$ds^2 = -(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + r^2 d\Omega_{p,-1}^2, \tag{7}$$

where $d\Omega_{p,-1}$: hyperbolic space H^p.

- Redefinition $r = \cosh\rho$:

$$ds^2 = -\sinh^2\rho dt^2 + d\rho^2 + \cosh^2\rho d\Omega_{p,-1}^2. \tag{8}$$

- Not globally static because of non-degenerate Killing horizon at $r = 1$.
- $\rho \to 0$ and so $\sinh\rho \simeq \rho$: Rindler space with temperature $T_H = 1/(2\pi)$.
- Usually the near-horizon geometry of near-extremal black holes.
- We shall focus on this space and show that it could be the near-horizon geometry of generic non-extremal black holes, accounting for the "hidden conformal symmetries" observed in them.

2. Hidden conformal symmetries in non-extremal black holes

- The symmetry and (thermo)dynamics of black holes are usually revealed by studying probe fields (like scalars) propagating in their spacetime
- Conformal symmetry is known to exist in wave equations of fields propagating near extremal and near-extremal black holes because their near-horizon geometry is AdS space
- However, conformal symmetry is also found in field equations near generic non-extremal black holes.
- The near-horizon geometry of generic non-extremal black holes is usually viewed as Rindler space. So this conformal symmetry seemingly has no geometric origin and is called "hidden".

Hidden conformal symmetries

DDF model of quantum mechanics near horizons:
First introduced by [de Alfaro, Fubini & Furlan 76] and later found to be a description of dynamics near horizons of extremal RN black holes [Claus, et al 98] and generic non-extremal black holes [Govindarajan 00].
For a general black hole solution

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{9}$$

the Klein-Gordon equation for a massless scalar $\phi(r)$ (neglecting the dependence on other coordinates) is

$$-\frac{1}{F}\frac{d}{dr}\left(F\frac{d}{dr}\right)\phi(r) = 0. \tag{10}$$

Hidden conformal symmetries

Redefinition: $\sqrt{r^2F}\phi = \psi$, the equation becomes

$$H\psi = -\frac{d^2}{dr^2}\psi + \left[\frac{(r^2F)''}{2F} - \left(\frac{(r^2F)'}{2F}\right)^2\right]\psi = 0. \qquad (11)$$

For non-extremal black holes: $F = (r - r_-)(r - r_+)/r^2$, taking the near-horizon limit yields

$$H\psi = \left(-\frac{d^2}{dx^2} - \frac{1}{4x^2}\right)\psi = 0, \qquad (12)$$

where $x = r - r_+$.
Define the dilatation and conformal boost

$$D = \frac{i}{4}\left(x\frac{d}{dx} + \frac{d}{dx}x\right), \qquad K = \frac{1}{4}x^2. \qquad (13)$$

Hidden conformal symmetries

We have the $SL(2, R)$ algebra:

$$[D, H] = -iH, \quad [D, K] = iK, \quad [H, K] = 2iD.$$

The solution from the eigenvalue function $H\psi = E\psi$ is:

$$\psi_n(x) = \sqrt{2E_n x} K_0(E_n x), \tag{14}$$

$$E_n = \exp[\frac{\pi}{2}(1 - 8n) \cot \frac{z}{2}]. \tag{15}$$

where n is integer, K_0 is the modified Bessel function and z is a variable.
Counting states...

Hidden conformal symmetries

New hidden conformal symmetries

- Kerr black holes [Castro, Malnony & Strominger 10]
  The Kerr metric for rotating black holes

$$ds^2 = -\frac{\triangle}{\chi^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\chi^2}((r^2+a^2)d\phi - adt)^2 + \frac{\chi^2}{\triangle}dr^2 + \chi^2 d\theta^2,$$

where

$$\triangle = (r - r_+)(r - r_-), \quad r_\pm = M \pm \sqrt{M^2 - a^2}$$

$$\chi^2 = r^2 + a^2\cos^2\theta, \quad a = \frac{J}{M}.$$

$r_\pm$ are the radii of the inner and outer horizons, respectively, and J is the angular momentum of the black hole.

Hidden conformal symmetries

Consider the Klein-Gordon (KG) equation of a massless scalar:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0. \tag{16}$$

Separate the function Φ in terms of symmetries:

$$\Phi = e^{-i\omega t + im\phi}R(r)S(\theta). \tag{17}$$

The KG equation in Kerr spacetime:

$$(\nabla^2_{S^2} + \omega^2 a^2 \cos^2\theta)S(\theta) = -K_l S(\theta), \tag{18}$$

$$\Big[\partial_r \triangle \partial_r + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)} + (r+2M)^2\omega^2\Big] R(r) = K_l R(r).$$

Hidden conformal symmetries

If we adopt the low-frequency condition

$$\omega M \ll 1, \tag{19}$$

and the "near-region" condition

$$r \ll \frac{1}{\omega}, \tag{20}$$

the KG equation reduces to

$$-J^2 S(\theta) = \nabla^2_{S^2} S(\theta) = -l(l+1)S(\theta), \tag{21}$$

$$\left[\partial_r \triangle \partial_r + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)}\right] R(r) = K_l R(r),$$

with $K_l = l(l+1)$.
Solution: hypergeometric functions bearing the $SL(2,R)_L \times SL(2,R)_R$ symmetry.

Hidden conformal symmetries

The $SL(2,R)_L$ generators:

$$H_1 = ie^{-2\pi T_R\phi}\left(\triangle^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r-M}{\triangle^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr-a^2}{\triangle^{1/2}}\partial_t\right), \quad (22)$$

$$H_0 = \frac{i}{2\pi T_R}\partial_\phi + 2iM\frac{T_L}{T_R}\partial_t, \quad (23)$$

$$H_1 = ie^{2\pi T_R\phi}\left(-\triangle^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r-M}{\triangle^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr-a^2}{\triangle^{1/2}}\partial_t\right), \quad (24)$$

and the $SL(2,R)_R$ generators

$$\bar{H}_1 = ie^{-2\pi T_L\phi+\frac{t}{2M}}\left(\triangle^{1/2}\partial_r - \frac{a}{\triangle^{1/2}}\partial_\phi - 2M\frac{r}{\triangle^{1/2}}\partial_t\right), \quad (25)$$

$$\bar{H}_0 = -2iM\partial_t, \quad (26)$$

$$\bar{H}_1 = ie^{2\pi T_L\phi-\frac{t}{2M}}\left(-\triangle^{1/2}\partial_r - \frac{a}{\triangle^{1/2}}\partial_\phi - 2M\frac{r}{\triangle^{1/2}}\partial_t\right). \quad (27)$$

Hidden conformal symmetries

The left and right temperatures are defined as

$$T_R = \frac{r_+ - r_-}{4\pi a}, \qquad T_L = \frac{r_+ + r_-}{4\pi a}.$$

They are related to the Hawking temperature T_H via

$$\frac{1}{T_H} = \frac{1}{T_L} + \frac{1}{T_R}. \tag{28}$$

The generators obey the $SL(2,R)_L \times SL(2,R)_R$ symmetry:

$$[H_0, H_{\pm 1}] = \mp i H_{\pm 1}, \quad [H_1, H_{-1}] = 2iH_0.$$

$$[H_n, \bar{H}_m] = 0. \tag{29}$$

$$[\bar{H}_0, \bar{H}_{\pm 1}] = \mp i \bar{H}_{\pm 1}, \quad [\bar{H}_1, \bar{H}_{-1}] = 2i\bar{H}_0.$$

The $SL(2,R)$ Casimir:

$$H^2 = \bar{H}^2 = -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1). \tag{30}$$

Thus, the radial KG equation can be re-expressed as

$$H^2 R(r) = \bar{H}^2 R(r) = l(l+1) R(r). \tag{31}$$

Hidden conformal symmetries

- From the hypergeometric solution of $R(r)$, we can calculate the scattering amplitude and cross section. They really take the same form as in 2D CFT at finite temperature.
- Key fact supporting this symmetry:
  The CFT dual of extremal $(M = a = J/M)$ and near-extremal Kerr black holes (Kerr/CFT) can reproduce the Bekenstein-Hawking entropy. But the result is true for generic values of M and J. So there should be a general conformal symmetry for generic Kerr black holes.

Hidden conformal symmetries

- Schwarzschild black holes [Bertini, Cacciatori & Klemm 11]

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{32}$$

where $r_0 = 2M$ and M is the mass of the black hole.

  - Redefining $r = r_0(1 + \rho^2/4)$ and $\hat{t} = t/(2r_0)$, we get the Rindler space

$$ds^2 = r_0^2(-\rho^2 d\hat{t}^2 + d\rho^2 + d\Omega_2^2). \quad (\rho \ll 1) \tag{33}$$

    It does not accommodate conformal symmetry.
  - However, conformal symmetry appears to exist in the KG wave equation! So this symmetry is hidden.

Hidden conformal symmetries

For a massless scalar $\Phi = e^{-i\omega t+im\phi}R(\rho)S(\theta)$, the KG equation:

$$\nabla^2_{S^2}S(\theta) = -l(l+1)S(\theta), \tag{34}$$

$$\left[\partial_r \triangle \partial_r + \frac{\omega^2 r^4}{\triangle} - l(l+1)\right]R(r) = 0, \tag{35}$$

where $\triangle = r^2 - 2Mr$.
Taking the $\omega r \ll 1$ and $M\omega \ll 1$ limits:

$$\left[\partial_r \triangle \partial_r + \frac{16\omega^2 M^4}{\triangle}\right]R(r) = l(l+1)R(r), \tag{36}$$

Hidden conformal symmetries

This equation has the $SO(2,1)$ or $SL(2,R)$ symmetry, whose generators are

$$H_1 = ie^{\frac{t}{4M}} \left(\triangle^{1/2} \partial_r - 4M(r-M) \triangle^{-1/2} \partial_t \right), \tag{37}$$

$$H_0 = -4iM\partial_t, \tag{38}$$

$$H_{-1} = -ie^{-\frac{t}{4M}} \left(\triangle^{1/2} \partial_r + 4M(r-M) \triangle^{-1/2} \partial_t \right), \tag{39}$$

They satisfy

$$[H_0, H_{\pm 1}] = \mp i H_{\pm 1}, \quad [H_1, H_{-1}] = 2iH_0.$$

Thus, the KG equation can be re-expressed as the $SL(2,R)$ Casimir:

$$H^2 R(r) = [-H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1)] R(r) = l(l+1) R(r). \tag{40}$$

3. The geometrical origin of hidden conformal symmetry

We suggest that the hidden conformal symmetry should arise from the AdS space of type (III).

- Schwarzschild black holes

  Claim: The reason that there is "hidden conformal symmetry" in Schwarzschild spacetime is that the near-horizon geometry has been equivalently taken as the following AdS_2 space of type (III)

$$ds^2 = r_0^2(-\sinh^2 \rho d\hat{t}^2 + d\rho^2 + d\Omega_2^2), \quad (\rho \ll 1) \qquad (41)$$

  instead of the Rindler space (33):

$$ds^2 = r_0^2(-\rho^2 d\hat{t}^2 + d\rho^2 + d\Omega_2^2). \quad (\rho \ll 1) \qquad (42)$$

  [But this AdS_2 space does not provide apparent explanation to the conformal symmetry in the DDF model.]

The geometrical origin

Why this geometry?

Let's see this from the RN metric for charged black holes

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2 d\Omega_2^2. \tag{43}$$

Redefinitions:

$$r = r_+(1+\lambda^2), \quad \lambda^2 = \delta^2 \sinh\frac{\rho}{2}, \quad \delta^2 = \frac{r_+ - r_-}{r_+}. \tag{44}$$

The RN metric is exactly rewritten as

$$ds^2 = r_+^2\left[-\frac{\sinh^2\rho}{(1+\lambda^2)^2}d\hat{t}^2 + (1+\lambda^2)^2 d\rho^2 + (1+\lambda^2)^2 d\Omega_2^2\right],$$

where $\hat{t} = \delta^2 t/(2r_+)$. Taking the near-horizon limit

$$\lambda = \delta \sinh(\rho/2) \ll 1, \tag{45}$$

we get $AdS_2 \times S^2$.

- $\delta \to 0$: the extremal limit and ρ can be very large
- $\delta \sim 1$: the off-extremal case ($\delta = 1$: Schwarzschild) and ρ can

The geometrical origin

Adopting this AdS_2 space, the KG equation is

$$\left[\frac{1}{\sinh\rho}\partial_\rho \sinh\rho\partial_\rho + \frac{r_+^4\omega^2}{\epsilon^2\sinh^2\rho} - l(l+1)\right] R(\rho) = 0. \tag{46}$$

It has the $SL(2,R)$ symmetry, with the generators:

$$\begin{aligned} H_1(\rho,\hat{t}) &= ie^{\hat{t}}(\partial_\rho - \coth\rho\partial_{\hat{t}}), \\ H_{-1}(\rho,\hat{t}) &= -ie^{-\hat{t}}(\partial_\rho + \coth\rho\partial_{\hat{t}}), \\ H_0(\rho,\hat{t}) &= -i\partial_{\hat{t}}. \end{aligned} \tag{47}$$

These vectors are exactly the same as the ones given in Eqs. (37-39), via the same coordinate redefinitions:
$r = r_+(1+\delta^2\sinh\frac{\rho}{2})$ and $\hat{t} = (\delta^2/2r_+)t$, given in Eq. (44).

The geometrical origin

- Kerr black holes

  Denote:

$$r_0 = \frac{1}{2}(r_+ + r_-), \quad \epsilon = \frac{1}{2}(r_+ - r_-), \tag{48}$$

  Redefine the coordinates:

$$U = r - r_0, \quad \bar{t} = \frac{t}{2r_+^2}, \quad \hat{\phi} = \phi - r_0 \bar{t}. \tag{49}$$

  In the new coordinates, the near-horizon geometry of non-extremal Kerr (16) is

$$ds^2 = r_+^2(1+\cos^2\theta)\left[-(U^2-\epsilon^2)d\bar{t}^2 + \frac{dU^2}{U^2-\epsilon^2} + d\theta^2\right] + \frac{4r_+^2\sin^2\theta}{1+\cos^2\theta}\left(d\hat{\phi} + U d\bar{t}\right)^2.$$

  This is the warped AdS_3 space obtained in the near-extremal case in [Castro & Larsen 09]. We assume that it still exist in generic non-extremal cases, replacing the Rindler space.

The geometrical origin

Further redefining: $\cosh\alpha = U/\epsilon$ and $\hat{t} = \epsilon\bar{t}$, the warped AdS_3 space of type (III) is

$$ds^2 = r_+^2(1+\cos^2\theta)\Big[-\sinh^2\alpha d\hat{t}^2 + d\alpha^2 + d\theta^2 + \frac{4\sin^2\theta}{(1+\cos^2\theta)^2}(d\hat{\phi} + \cosh\alpha d\hat{t})^2\Big].$$

In this geometry, the KG equation becomes

$$\left[\frac{1}{\sinh(2\rho)}\partial_\rho \sinh(2\rho)\partial_\rho + \frac{(2Mr_+\omega - am)^2}{\epsilon^2\sinh^2\rho} - \frac{(2Mr_-\omega - am)^2}{\epsilon^2\cosh^2\rho}\right]R(\rho) = 4l(l+1)R(\rho),$$

where $\rho = \alpha/2$.

The geometrical origin

This is the Laplacian in the following AdS_3 space:

$$ds^2_{AdS_3} = R^2(-\sinh^2\rho d\tau^2 + d\rho^2 + \cosh^2\rho d\sigma^2), \qquad (50)$$

$$\nabla^2_{AdS_3} = \frac{1}{\sinh(2\rho)}\partial_\rho \sinh(2\rho)\partial_\rho - \frac{\partial_\tau^2}{\sinh^2\rho} + \frac{\partial_\sigma^2}{\cosh^2\rho}. \qquad (51)$$

with the coordinate relations between (t, ϕ) and (τ, σ):

$$\tau = \frac{1}{4M}t + \pi(T_R - T_L)\phi, \quad \sigma = -\frac{1}{4M}t + \pi(T_R + T_L)\phi, \qquad (52)$$

The temperatures

$$T_L = \frac{r_0}{2\pi a}, \quad T_R = \frac{\epsilon}{2\pi a}. \qquad (53)$$

The geometrical origin

The generators obtained from the AdS_3 space are

$$H_1 = ie^{2\pi T_R \phi} \left\{ \partial_\alpha - \frac{1}{\sinh\alpha} \left[2M \left(1 + \cosh\alpha \frac{T_L}{T_R} \right) \partial_t + \frac{\cosh\alpha}{2\pi T_R} \partial_\phi \right] \right\}$$

$$H_{-1} = -ie^{-2\pi T_R \phi} \left\{ \partial_\alpha + \frac{1}{\sinh\alpha} \left[2M \left(1 + \cosh\alpha \frac{T_L}{T_R} \right) \partial_t + \frac{\cosh\alpha}{2\pi T_R} \partial_\phi \right] \right\}$$

$$H_0 = -i \left(2M \frac{T_L}{T_R} \partial_t + \frac{1}{2\pi T_R} \partial_\phi \right)$$

and

$$\bar{H}_1 = ie^{\frac{1}{2M} t - 2\pi T_L \phi} \left\{ \partial_\alpha - \frac{1}{\sinh\alpha} \left[2M \left(\cosh\alpha + \frac{T_L}{T_R} \right) \partial_t + \frac{1}{2\pi T_R} \partial_\phi \right.\right.$$

$$\bar{H}_{-1} = -ie^{-\frac{1}{2M} t + 2\pi T_L \phi} \left\{ \partial_\alpha + \frac{1}{\sinh\alpha} \left[2M \left(\cosh\alpha + \frac{T_L}{T_R} \right) \partial_t + \frac{1}{2\pi T_R} \partial_\phi \right.\right.$$

$$\bar{H}_0 = -2i$$

where $\alpha = 2\rho$. They are the same as those given in Eqs. (22-27) after coordinate redefinitions.

4. Conclusions

- Hidden conformal symmetries in non-extremal black holes do have geometrical origin: the AdS space of type (III)
- This implies that the near-horizon geometries of generic non-extremal black holes might be the AdS space of type (III), but not simply the Rindler space
- If so, the AdS/CFT originally established in extremal case can be extended to generic non-extremal black holes (but the AdS spaces in extremal and non-extremal cases are different)

Thank you!