The exact solution of one-dimensional quantum many-body systems

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   - Eigenvalues
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Introduction to the exactly solvable models

低维系统

- ◆量子点
- ◆自旋链
- ◆梯型材料
- ◆层状氧化物
- ◆…

新奇量子现象

- ●有机导体
- ●高温超导
- ●整数量子霍尔效应
- ●分数量子霍尔效应
- ●……

传统理论框架

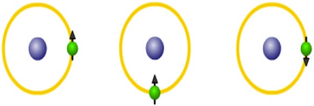
普适的量子多体理论框架

诺贝尔物理奖 1982 、1985 、1987、1998……

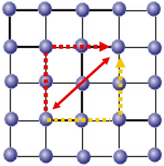
如何理解和刻画这类系统成为凝聚态物理研究中最基本、异常困难而又亟待解决的重要问题之一。

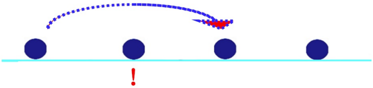
Research object: one-dimensional systems.

Correlation effect: many-body effect induced by the short-range Coulomb repulsion between electrons.



The correlation effects in low dimensional systems are more prominent.





Many interesting phenomena such as fractional elementary excitations, various phases of quantum liquids, nonlinear effects, collective modes, critical behaviors are induced by the strong correlation.

Methods for many-body systems:

Numerical: Exact diagonalization, DMRG, Monte Carlo, Tensor network, Machine learning, DFT, MD

Analytical: Mean field, Perturbation, Exact solution

- Due to the strong correlation, many traditional methods such as mean field and Perturbation are invalid.

- No universal quantum many body theory. Exact solution is a good method.

Actual physical problems $\Rightarrow$ Exactly solvable models $\Rightarrow$ Quantitative results

$\Uparrow$ $\Downarrow$

Physical mechanism $\Leftarrow$ Experiments $\Leftarrow$ Universal class theory

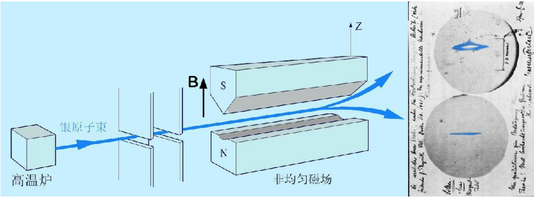
• Exact solution can provide the benchmark for many new phenomena and physical concepts, and check the correction of numerical methods and numerical results.

Examples: 2D Ising model (thermodynamic phase transition), 1D Hubbard model (Mott insulator), Heisenberg model (spinon, fractional charge), Hydrogen atom (quantum mechanics).

• It is an important branch of condensed matter physics, statistical physics, theoretical and mathematical physics.

Exact solvable model: Quantum spin chain

1921, Stern Grach



- Spin singlet and triplet states, exchanging interactions

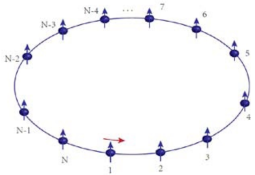
$$\vec{\sigma}_1 \cdot \vec{\sigma}_2$$

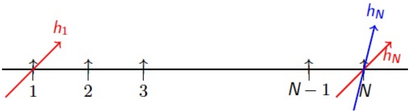
- Heisenberg model

$$H = J \sum_{j=1} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$$

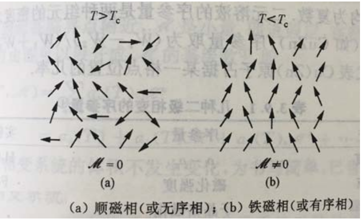
quantum magnetism, anisotropy, quantum phase transition, spinon, Bethe ansatz

- Boundary conditions





- Ising model



- $J_1 - J_2$ model

$$H = \sum_{j=1} J_1 \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + J_2 \vec{\sigma}_j \cdot \vec{\sigma}_{j+2}$$

- Dzyloshinsky-Moriya interaction

$$H = \sum_{j=1} \hat{D} \cdot (\vec{\sigma}_j \times \vec{\sigma}_{j+1})$$

- Chiral three spins interaction

$$H = \sum_{j=1} \vec{\sigma}_j \cdot (\vec{\sigma}_{j+1} \times \vec{\sigma}_{j+2})$$

- Gaudin model

$$H_i = \sum_{j=1} \frac{\vec{\sigma}_j \cdot \vec{\sigma}_j}{i - j}$$

- Haldane-Shastry model

$$H = \sum_{i,j=1} \frac{\vec{\sigma}_j \cdot \vec{\sigma}_j}{(i - j)^2}$$

- New model: integrable quantum spin chain with competing interactions

$$H = H_{bulk} + H_L + H_R$$

Bulk and periodic

$$H_{bulk} = \sum_{j=1}^{2N-1} \left\{ J_1 \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + J_2 \vec{\sigma}_j \cdot \vec{\sigma}_{j+2} + J_3 (-1)^j \vec{\sigma}_{j+1} \cdot (\vec{\sigma}_j \times \vec{\sigma}_{j+2}) \right\}$$

Open boundary

$$H_L = \frac{1-4a^2}{p^2 - a^2} [p\sigma_1^z - a^2 \sigma_1^z \sigma_2^z - iapD_1^z \cdot (\vec{\sigma}_1 \times \vec{\sigma}_2)]$$

$$H_R = \frac{4a^2 - 1}{a^2\xi^2 + a^2 - q^2} \Big[q(\xi\sigma_{2N}^x + \sigma_{2N}^z) - a^2 (\xi\sigma_{2N-1}^x + \sigma_{2N-1}^z)(\xi\sigma_{2N}^x + \sigma_{2N}^z)$$

$$-iaq(\xi D_{2N}^x + D_{2N}^z) \cdot (\vec{\sigma}_{2N} \times \vec{\sigma}_{2N-1}) \Big]$$

- New model: integrable cold atomic models

Particles with δ-function interaction: boson; fermion; mixture

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + c \sum_{j<l} \delta(x_j - x_l)$$

Bosons

E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963);

E. H. Lieb, Phys. Rev. 130, 1616 (1963);

Y. Q. Li, S. J. Gu, Z. J. Ying, U. Eckern, Europhys. Lett. 61, 368 (2003).

Fermions

M. Gaudin, Phys. Lett. A 24, 55 (1967);

C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967); **Yang-Baxter equation**

C. N. Yang, Phys. Rev. 168, 1920 (1968); **factorizableof scattering matrix**

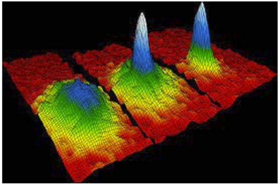
B. Sutherland, Phys. Rev. Lett. 20, 98 (1968).

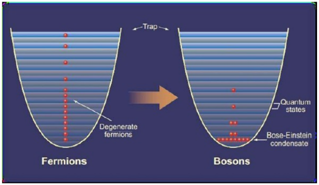
Bose-fermi mixtures

C. K. Lai, C. N. Yang, Phys. Rev. A 3, 393 (1971);

C. K. Lai, J. Math. Phys. 15, 954 (1974).

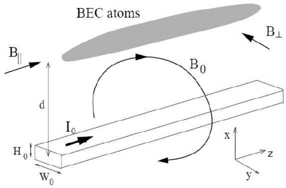
Bose – Einstein Condensation； theory： 1924-1925； experiment： 1995

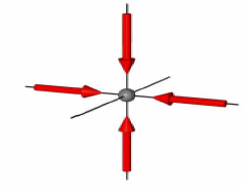


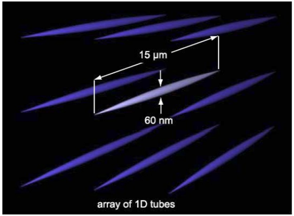


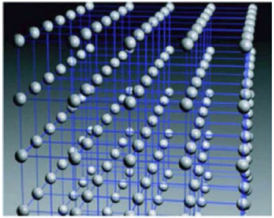
low-dimensional cold atomic systems

Optical lattices & magnetic traps



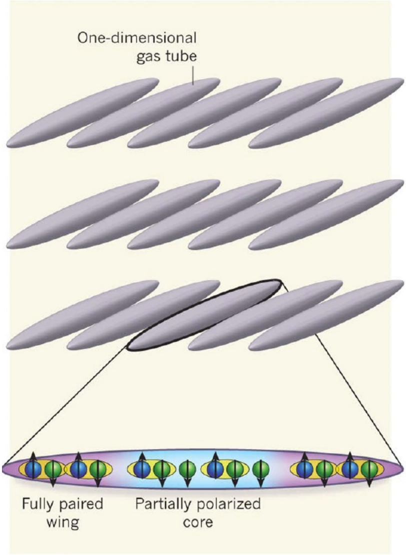


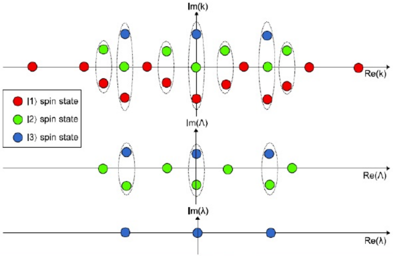


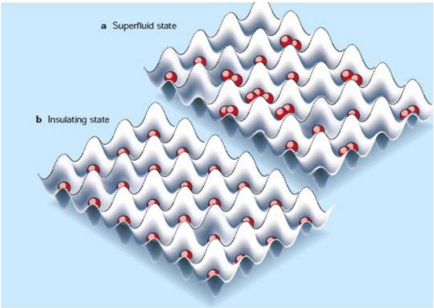


Tunable: component, interaction, dimension, lattice constant

Phase transition & pairing

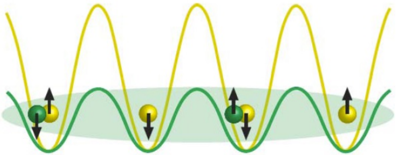






Cold atomic systems in experiments

F=1 ^{23}Na, ^{39}K, ^{87}Rb
F=2 ^{85}Rb
F=3 ^{133}Cs
F=3/2 ^{132}Cs, ^{9}Be, ^{135}Ba, ^{137}Ba
F=5/2 ^{173}Yb rare-earth



New integrable model with both contact and spin-exchanging interactions

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j}^{N} [c_0 + c_2 \vec{S}_i \cdot \vec{S}_j] \delta(x_i - x_j)$$

Atoms with arbitrary spin-s

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i<j}^{N} \left[c_0 P_{ij}^0 + c_2 \sum_{m=0,2,4,\cdots}^{2s} P_{ij}^m \right] \delta(x_i - x_j)$$

Projector operator

$$P_{ij}^m = \prod_{l=0,\neq m}^{2s} \frac{(\vec{S}_i + \vec{S}_j)^2 - l(l+1)}{m(m+1) - l(l+1)}$$

- Spin-1:

$c_2 = 0$, $SU(3)$ symmetry

$c_0 = c_2 = c$, $SU(2)$ symmetry, new states of paired Bosons

- Spin-$\frac{3}{2}$:

$c2 = 0$, $SU(4)$ symmetry

$c_0 = \frac{c}{2}$ and $c2 = -\frac{2c}{3}$, $SO(5)$ symmetry, new elemental excitations such as the heavy spinons carrying spin 0, $\frac{1}{2}$ and 1

- Hamiltonian $H=-\sum_{j=1}^{N}\partial_{x_j}^2+\sum_{\langle i,j\rangle}\sum_{lm}g_{lm}P_{ij}^{lm}\delta(x_i-x_j).$

	(pseudo-)spin	interaction	symmetry	
★	0 (boson)	$g_0=c$	$U(1)$	Lieb, *et al.*, PR130.1605(1963)
★	½ (fermion)	$g_0=c$	$SU(2)$	Yang, PRL19.1312(1967)
★	½ (boson)	$g_1=c$	$SU(2)$	Li, EPL 61. 368 (2003)
	½ (boson)	$g_{1,-1}=c_1, g_{1,1}=c_2, g_{1,0}=0.$	$U(1)$	
★	1 (boson)	$g_0=c, g_2=c$	$SU(3)$	Zhou, JPA 21. 2391; 2399 (1988)
■	1 (boson)	$g_0=-c, g_2=2c.$	$SU(2)$	Cao, *et.al*, EPL79.30005(2007)
	1 (boson)	$g_{0,0}=c, g_{2,-1}=0, g_{2,1}=0,$ $g_{2,-2}=c, g_{2,0}=c, g_{2,2}=c.$	$U(1)$	
★	1 (fermion)	$g_1=c$	$SU(3)$	Sutherland, PRL20.98(1968)
★	3/2 (fermion)	$g_0=c, g_2=c.$	$SU(4)$	Sutherland, PRL20.98(1968)
■	3/2 (fermion)	$g_0=3c, g_2=c.$	$Sp(4)$	Jiang, *et.al*, EPL87.10006(2009)
	3/2 (fermion)	$g_{0,0}=0, g_{2,2}=c_1, g_{2,-2}=c_2,$ $g_{2,-1}=0, g_{2,0}=0, g_{2,-1}=0,$	$U(1)$	
■	Integer s (boson)	$g_0=-(s-1/2)c, g_{2,4,\ldots}=c.$	$SO(2s+1)$	Jiang, *et.al*, JPA44.345001(2011)
■	Half-odd s (fermion)	$g_0=(s+3/2)c, g_{2,4,\ldots}=c.$	$Sp(2s+1)$	Jiang, *et.al*, JPA44.345001(2011)

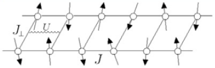
Exactly solvable models:

1. interacting particles with δ -function

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j)$$

2. spin chain and spin ladder

$$H = \frac{1}{2} J \sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} - \frac{1}{2} h \sum_{j=1}^{N} \sigma_j^z$$



3. Hubbard, supersymmetry t-J, Kondo

$$H = \sum_{i,j=1}^{N} \sum_{\sigma=\uparrow,\downarrow} t_{i,j}(C_{i,\sigma}^\dagger C_{j,\sigma} + h.c.) + U \sum_{i=1}^{N} n_{i\uparrow} n_{i\downarrow}$$

$$H = \sum_{j=1}^{N} \left\{ -t\mathcal{P} \sum_{\sigma=\pm 1} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + H.c.)\mathcal{P} + J(\mathbf{S}_j \mathbf{S}_{j+1} - \frac{1}{4} n_n n_{j+1}) \right\}$$

4. τ_2, Chiral Potts, vertex model

$$H_{P_9} = -\sum_{l=1}^{N/2} \sum_{k=1}^{8} \Omega_l^k - \sum_{l=1}^{N/2-1} \sum_{k=1}^{8} R_l^k R_{l+1}^{9-k}.$$

5. long range interaction

$$\hat{H}_n = 2g S_n^z + \sum_{m \neq n}^{N} \sum_{\alpha} \frac{2 S_n^\alpha S_n^\alpha}{\delta_n - \delta_m}$$

Gaudin model (1/r)

$$H_{CM} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j,k=1, j\neq k}^{N} \frac{\lambda^2 - \lambda P_{jk}}{(x_i - x_j)^2}$$

Calogero-Sutherland model ($1/r^2$,continue case)

Haldane-Shastry model ($1/r^2$, lattice case)

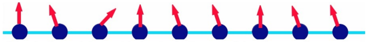
$$H_{HS} = \sum_{j<l}^{N} \frac{1}{\sin^2 \frac{\pi}{N}(j-l)} \mathbf{S}_j \cdot \mathbf{S}_l$$

- Hamiltonian

$$H = \frac{J}{2}\sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + h_1\sigma_1^z + h_N\sigma_N^z$$

Generating functional (quantum inverse scattering)

$$t(u) = tr_0\{K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\}$$



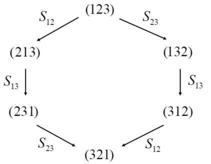
$$T_0(u) = R_{0,N}(u-\theta_N)R_{0,N-1}(u-\theta_{N-1})\cdots R_{0,1}(u-\theta_1)$$

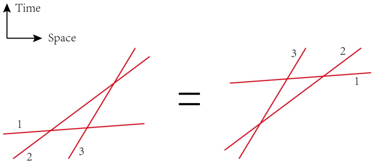
$$\hat{T}_0(u) = R_{0,1}(u+\theta_1)R_{0,2}(u+\theta_2)\cdots R_{0,N}(u+\theta_N)$$

$$R_{0,j}(u) = u + P_{0,j} = u + \frac{1}{2}(1+\vec{\sigma}_0\cdot\vec{\sigma}_j) = \begin{pmatrix} u+\frac{1}{2}(1+\sigma_j^z) & \sigma_j^- \\ \sigma_j^+ & u+\frac{1}{2}(1-\sigma_j^z) \end{pmatrix}$$

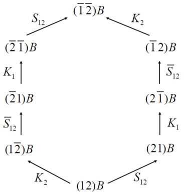
$$K_0^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix} \qquad K_0^+(u) = \begin{pmatrix} q+u+1 & \\ & q-u-1 \end{pmatrix}$$

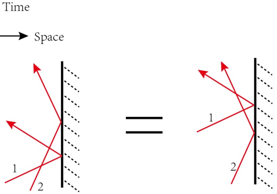
Yang-Baxter equation





Reflection equation





$$[t(u), t(v)] = 0$$

$$H = c_2^{-1} \frac{\partial \ln t(u)}{\partial u}\Big|_{u=0,\{\theta_j\}=0} - c_0$$

- Coordinate Bethe ansatz

1. Reference state

$$|0\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \cdots \otimes |\uparrow\rangle_N$$
$$H|0\rangle = E_0|0\rangle$$

2. One spin flipped state

$$|k\rangle = \sum_{x=1}^{N} \psi(x) S_x^- |0\rangle.$$

Acting H on the state $|k\rangle$, we obtain:

i) When $x \neq 1, N$,

$$J[\psi(x+1) + \psi(x-1) - 2\psi(x)] + E_0\psi(x) = E\psi(x). \quad (1)$$

Assume

$$\psi(x) = A_+ e^{ikx} + A_- e^{-ikx}. \quad (2)$$

Substituting (2) into (1), we have

$$E(k) = 2J(\cos k - 1) + E_0. \quad (3)$$

The values of quasi-momentum k are determined by the boundary conditions.

ii) When $x = 1$, the eigen-equation becomes

$$J[\psi(2) - \psi(1)] + (E_0 - 2h_1)\psi(1) = E\psi(1).$$

Substituting (2) into above equation, we obtain the relation between $A_\pm$ as

$$\frac{A_+}{A_-} = -\frac{1 - (1 - 2h_1/J)e^{-ik}}{1 - (1 - 2h_1/J)e^{ik}}. \quad (4)$$

iii) When $x = N$, the eigen-equation is

$$J[\psi(N-1) - \psi(N)] + (E_0 - 2h_N)\psi(N) = E\psi(N).$$

Similarly, we obtain

$$\frac{A_+}{A_-} = -e^{-2iNk}\frac{e^{-ik} - (1 - 2h_N/J)}{e^{ik} - (1 - 2h_N/J)}, \quad (5)$$

From Eqs.(4) and (5), we obtain the Bethe ansatz equation

$$e^{2iNk} = \frac{1 - (1 - 2h_1/J)e^{ik}}{1 - (1 - 2h_1/J)e^{-ik}}\frac{e^{-ik} - (1 - 2h_N/J)}{e^{ik} - (1 - 2h_N/J)}.$$

3. M spins flipped state

$$|k_1, \cdots, k_M\rangle = \sum_{x_1, \cdots, x_M=1}^{N} \psi(x_1, \cdots, x_M) S_{x_1}^- \cdots S_{x_M}^- |0\rangle.$$

Assume the wave-function is

$$\psi(x_1, \cdots, x_M) = \sum_{P,Q} \sum_{r_j=\pm} A_{r,P} e^{i \sum_{j=1}^{M} r_{P_j} k_{P_j} x_{Q_j}} \theta(x_{Q_1} < \cdots < x_{Q_M}).$$

Using the similar idea as $M = 1$, we obtain the energy spectrum as

$$E = -\sum_{j=1}^{M} \frac{J}{\lambda_j^2 + \frac{1}{4}} + E_0,$$

where the Bethe roots should satisfy the Bethe ansatz equation

$$\left(\frac{\lambda_j - \frac{i}{2}}{\lambda_j + \frac{i}{2}}\right)^{2N} = \frac{\lambda_j - \frac{i}{2}\mu_1}{\lambda_j + \frac{i}{2}\mu_1} \frac{\lambda_j - \frac{i}{2}\mu_N}{\lambda_j + \frac{i}{2}\mu_N} \prod_{l \neq j}^{M} \frac{\lambda_j - \lambda_l - i}{\lambda_j - \lambda_l + i} \frac{\lambda_j + \lambda_l - i}{\lambda_j + \lambda_l + i},$$

$e^{ik_j} = (\lambda_j - \frac{i}{2})/(\lambda_j + \frac{i}{2})$, $\mu_1 = 1 - J/h_1$ and $\mu_N = 1 - J/h_N$.

1. The Bethe ansatz equations can be solved

- numerically: finite system-size
- analytically: thermodynamic limit $N \to \infty$

$\Rightarrow$ physical quantities

2. Why is it called exactly solvable?

$2^N \times 2^N$ matrix (exponential wall problem)

$\Rightarrow$ N algebraic equations

- Algebraic Bethe ansatz

Transfer matrix

$$t(\lambda) = tr_0\{K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\} = tr_0\{K_0^+(\lambda)\mathcal{T}_0(\lambda)\},$$

where $\mathcal{T}_0(\lambda)$ is the double raw monodromy matrix

$$\mathcal{T}_0(\lambda) = (1-\lambda^2)^N T_0^{-1}(-\lambda)K_0^-(\lambda)T_0(\lambda) \equiv \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{pmatrix}.$$

The transfer matrix can also be written as

$$\begin{aligned} t(\lambda) &= (q+\lambda+1)\alpha(\lambda) + (q-\lambda-1)\delta(\lambda) \\ &= \frac{q-\lambda-1}{2\lambda+1}\bar{\delta}(\lambda) + (\frac{q-\lambda-1}{2\lambda+1} + q + \lambda + 1)\alpha(\lambda), \end{aligned}$$

where $\bar{\delta}(\lambda) = (2\lambda+1)\delta(\lambda) - \alpha(\lambda)$.

Reference state:

$$|0\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \cdots \otimes |\uparrow\rangle_N.$$

Direct calculation gives

$$\begin{aligned}
\gamma(\lambda)|0\rangle &= 0, \\
\alpha(\lambda)|0\rangle &= (p+\lambda)(\lambda+1)^{2N}|0\rangle, \\
\bar{\delta}(\lambda)|0\rangle &= 2(p-\lambda-1)\lambda^{2N+1}|0\rangle, \\
\alpha(\lambda)|0\rangle &\neq 0.
\end{aligned}$$

Assume the eigenstate of transfer matrix $t(\lambda)$ is

$$|\lambda_1, \cdots \lambda_M\rangle = \beta(\lambda_1)\cdots\beta(\lambda_M)|0\rangle.$$

Acting the transfer matrix $t(\lambda)$ on the assumed state

$$t(\lambda)\,|\lambda_1, \cdots \lambda_M\rangle = [\bar{c}_1\bar{\delta}(\lambda) + \bar{c}_2\alpha(\lambda)]\beta(\lambda_1)\cdots\beta(\lambda_M)|0\rangle.$$

Reflection equation

$$R_{12}(\lambda-\mu)\mathcal{T}_1(\lambda)R_{12}(\lambda+\mu)\mathcal{T}_2(\mu)=\mathcal{T}_2(\mu)R_{12}(\lambda+\mu)\mathcal{T}_1(\lambda)R_{12}(\lambda-\mu),$$

Commutation relations among the matrix elements of monodromy matrix

$$\begin{aligned}
\bar{\delta}(\lambda)\beta(\mu) &= \frac{(\lambda-\mu+1)(\lambda+\mu+2)}{(\lambda-\mu)(\lambda+\mu+1)}\beta(\mu)\bar{\delta}(\lambda)-\frac{2(\lambda+1)}{(\lambda-\mu)(2\mu+1)}\beta(\lambda)\bar{\delta}(\mu)\\
&\quad+\frac{4(\lambda+1)u}{(2\mu+1)(\lambda+\mu+1)}\beta(\lambda)\alpha(\mu),\\
\alpha(\lambda)\beta(\mu) &= \frac{(\lambda+\mu)(\lambda-\mu-1)}{(\lambda-\mu)(\lambda+\mu+1)}\beta(\mu)\alpha(\lambda)-\frac{1}{(\lambda+\mu+1)(2\mu+1)}\beta(\lambda)\bar{\delta}(\mu)\\
&\quad+\frac{2\mu}{(\lambda-\mu)(2\mu+1)}\beta(\lambda)\alpha(\mu),\\
[\beta(\lambda),\beta(\mu)] &= 0.
\end{aligned}$$

Finial resluts

$$t(\lambda) \mid \lambda_1, \cdots \lambda_M\rangle = \Lambda(\lambda) \mid \lambda_1, \cdots \lambda_M\rangle + \textit{unwanted terms}.$$

Eigenvalues

$$\Lambda(\lambda) = \left(\frac{q-\lambda-1}{2\lambda+1} + q + \lambda + 1\right)(p+\lambda)(\lambda+1)^{2N} \prod_{j=1}^{M} \frac{(\lambda+\lambda_j)(-\lambda-\lambda_j-1)}{(\lambda-\lambda_j)(\lambda-\lambda_j+1)}$$
$$+2\frac{q-\lambda-1}{2\lambda+1}(p-\lambda-1)\lambda^{2N+1} \prod_{j=1}^{M} \frac{(\lambda-\lambda_j+1)(\lambda+\lambda_j+2)}{(\lambda-\lambda_j)(\lambda+\lambda_j+1)}.$$

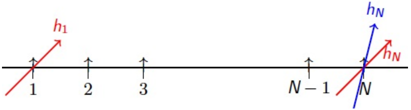
Bethe ansatz equation

$$\frac{(q+\lambda_j)(p+\lambda_j)}{(\lambda_j+1-q)(\lambda_j+1-p)} \left(1+\frac{1}{\lambda_j}\right)^{2N} \prod_{l\neq j} \frac{(\lambda_j+\lambda_l)(\lambda_j-\lambda_l-1)}{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)} = 1.$$

Eigen-energy is

$$E = \frac{1}{4pq} \frac{d\Lambda(\lambda;\lambda_1\cdots\lambda_M)}{d\lambda}\bigg|_{\lambda=0} - \frac{1}{2}N.$$

When the boundary magnetic fields are unparallel,



the general Hamiltonian is

$$H = \sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + h_1 \sigma_1^z + h_N^x \sigma_N^x + h_N^z \sigma_N^z$$

In this case

$$R_{0,j}(u) = u + \frac{1}{2}\left(1 + \vec{\sigma}_0 \cdot \vec{\sigma}_j\right),$$

$$K_0^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \qquad K_0^+(u) = \begin{pmatrix} q+u+1 & \xi(u+1) \\ \xi(u+1) & q-u-1 \end{pmatrix}.$$

The spin of quasi-particle could be changed after the boundary reflections. Thus the particle number of fixed spin is not conserved.

The system is still integrable.

Quantum integrable models without $U(1)$ symmetry

- Anti-periodic boundaries
- XYZ spin chain, eight vertex model

- Lacking the reference state

Due to the $U(1)$ symmetry-broken, there is no obvious reference state. Traditional Bethe ansatz does not work. Although the model has been proved to be integrable, the exact solutions are difficult to be obtained.

- Polynomial analysis

$t(u)$ is a operator polynomial of u with the degree $2N+2$

$$t(u) = \hat{O}_{2N+2}u^{2N+2} + \hat{O}_{2N+1}u^{2N+1} + \cdots + \hat{O}_1 u + \hat{O}_0$$

$$\Lambda(u) = O_{2N+2}u^{2N+2} + O_{2N+1}u^{2N+1} + \cdots + O_1 u + O_0$$

which can be determined by its values at $2N+3$ points.

We need $2N+3$ constraints to determine the values of the coefficients.

Fusion and operator product identities

$$t(\theta_j)t(\theta_j-1)=a(\theta_j)d(\theta_j-1),\quad j=1,\cdots,N$$

In the homogeneous limit $\{\theta_j=0\}$

$$[t(u)t(u-1)]^{(n)}|_{u=0}=[a(u)d(u-1)]^{(n)}|_{u=0},\quad n=1,\cdots,N$$

$$crossing\ symmetry: t(u)=t(-u-1)$$

$$degree: 2N+2\rightarrow N+1$$

$$constraints: N+2$$

$$asymptotic\ behavior: t(u)|_{u\rightarrow\pm\infty}=2u^{2N+2}+\cdots$$

$$t(0)=2\,p\,q\prod_{j=1}^{N}(1-\theta_j)(1+\theta_j)$$

- Inhomogeneous $T - Q$ relation with $\{\theta_j = 0\}$

$$\Lambda(u) = a(u)\frac{Q(u-1)}{Q(u)} + d(u)\frac{Q(u+1)}{Q(u)} + \frac{2[1-(1+\xi^2)^{\frac{1}{2}}]u^{2N+1}(u+1)^{2N+1}}{Q(u)}$$

Bethe roots

$$Q(u) = \prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+1)$$

Bethe ansatz equations

$$a(\lambda_j)Q(\lambda_j - 1) + d(\lambda_j)Q(\lambda_j + 1) = -2[1-(1+\xi^2)^{\frac{1}{2}}]\lambda_j^{2N+1}(\lambda_j+1)^{2N+1},$$
$$j = 1, \cdots, N$$

Eigenstates and separation of variables

Construct the complete bases of the Hilbert space by using the N inhomogeneous parameters $\{\theta_j\}$ as

$$|\theta_{p_1}, \cdots, \theta_{p_n}\rangle = \bar{\mathcal{A}}(\theta_{p_1}) \cdots \bar{\mathcal{A}}(\theta_{p_n})|\Omega\rangle_\xi, \qquad 1 \leq p_1 < p_2 < \cdots < p_n \leq N, \tag{6}$$

$$\langle -\theta_{q_1}, \cdots, -\theta_{q_n}| =_\xi \langle\bar{\Omega}|\bar{\mathcal{D}}(-\theta_{q_1}) \cdots \bar{\mathcal{D}}(-\theta_{q_n}), \qquad 1 \leq q_1 < \cdots < q_n \leq N. \tag{7}$$

Here the reference is

$$|\Omega\rangle_\xi = \otimes_{j=1}^N |1\rangle_j, \quad {}_\xi\langle\bar{\Omega}| = \otimes_{j=1}^N \langle 2|_j.$$

where

$$\begin{aligned} |1\rangle_n &= \frac{\sqrt{1+\xi^2}+1}{2\xi\sqrt{1+\xi^2}}|\uparrow\rangle_n + \frac{1}{2\sqrt{1+\xi^2}}|\downarrow\rangle_n, \quad n = 1, \cdots, N, \\ \langle 2|_n &= \xi\langle\uparrow|_n - (\sqrt{1+\xi^2}+1)\langle\downarrow|_n, \quad n = 1, \cdots, N. \end{aligned}$$

These states are also orthogonal

$$\langle a|_j b\rangle_k = \delta_{a,b}\,\delta_{j,k}, \quad a, b = 1, 2, \quad j, k = 1, \cdots, N.$$

The orthogonal and complete bases are obtain based on

$$\begin{aligned} t(\lambda) &= K^{+}_{11}(\lambda)\,\mathcal{A}(\lambda) + K^{+}_{12}(\lambda)\,\mathcal{C}(\lambda) + K^{+}_{21}(\lambda)\,\mathcal{B}(\lambda) + K^{+}_{22}(\lambda)\,\mathcal{D}(\lambda) \\ &= \bar{K}^{+}_{11}(\lambda)\,\bar{\mathcal{A}}(\lambda) + \bar{K}^{+}_{22}(\lambda)\,\bar{\mathcal{D}}(\lambda). \end{aligned}$$

According to the possible choices of p_n in the right state (6) and the choices of q_n in the left state (7), we obtain

$$\sum_{n=0}^{N} \frac{N!}{(N-n)!n!} = 2^N,$$

which is exactly the number of the dimension of the Hilbert space of the system. Thus both the right state (6) and the left state (7) are complete.

For the arbitrary inhomogeneous parameters $\{\theta_j\}$, the right and left states are orthogonal

$$\langle -\theta_{q_1}, \cdots, -\theta_{q_m} | \theta_{p_1}, \cdots, \theta_{p_n} \rangle = f_n(\theta_{p_1}, \cdots, \theta_{p_n}) \delta_{m+n,N} \delta_{\{q_1,\cdots,q_m\};\{p_1,\cdots,p_n\}},$$

where $f_n(\theta_{p_1}, \cdots, \theta_{p_n})$ is the normalized coefficient,

$$\delta_{\{q_1,\cdots,q_m\};\{p_1,\cdots,p_n\}} = \begin{cases} 1 & \text{if } \{q_1, \cdots, q_m, p_1, \cdots, p_n\} = \{1, \cdots, N\}, \\ 0 & \text{otherwise,} \end{cases}$$

Assume the eigenstate of the system is $\langle\Psi|$, which can be expanded by the complete bases $\{|\theta_{p_1},\cdots,\theta_{p_n}\rangle\}$ or $\{\langle -\theta_{p_1},\cdots,-\theta_{p_n}|\}$. The expansion coefficients are

$$\bar{F}_n(\theta_{p_1},\cdots,\theta_{p_n}) = \langle\Psi|\theta_{p_1},\cdots,\theta_{p_n}\rangle,$$
$$n = 0,\cdots,N, \quad 1 \le p_1 < p_2 < \cdots < p_n \le N.$$

In order to calculate the values of coefficients, we consider the physical quantity $\langle\Psi|t(\theta_{p_{n+1}})|\theta_{p_1},\cdots,\theta_{p_n}\rangle$.

Acting the transfer matrix $t(\theta_{p_{n+1}})$ to the left and to the right, we obtain

$$\bar{F}_n(\theta_{p_1},\cdots,\theta_{p_n}) = \left\{\prod_{j=1}^{n}\frac{(2\theta_{p_j}+\eta)\Lambda(\theta_{p_j})}{(2\theta_{p_j}+\eta)\bar{K}^+_{11}(\theta_{p_j})+\eta\bar{K}^+_{22}(\theta_{p_j})}\right\}\bar{F}_0, \tag{8}$$

where $\bar{F}_0 = \langle\Psi|\Omega\rangle_\xi$ is a scalar factor. Therefore, we have retrieved the eigenstates by using the obtained eigenvalues.

Now, we construct the traditional Bethe state. Define

$$ {}_B\langle\lambda_1,\cdots,\lambda_N| = \langle 0| \left\{ \prod_{j=1}^{N} \frac{\bar{\mathcal{C}}(\lambda_j)}{(-1)^N \bar{K}^-_{21}(\lambda_j) d(\lambda_j) d(-\lambda_j-\eta)} \right\}, \tag{9} $$

where

$$ \langle 0| = \langle\uparrow|_j \otimes_{j=1}^{N} . $$

Expand the Bethe states by the complete bases, and the coefficients are

$$ \begin{aligned} & {}_B\langle\lambda_1,\cdots,\lambda_N|\theta_{p_1},\cdots,\theta_{p_n}\rangle \\ & = \left\{ \prod_{j=1}^{n} (-1)^N (\theta_{p_j}+p)\, a(\theta_{p_j}) d(-\theta_{p_j}-\eta) \frac{Q(\theta_{p_j}-\eta)}{Q(\theta_{p_j})} \right\} \langle 0|\Omega\rangle_\xi, \\ & n = 0,\cdots,N, \quad 1 \le p_1 < p_2 < \cdots < p_n \le N. \end{aligned} $$

Comparing these coefficients with (8), we find that the only difference is a scalar factor. Therefore, the Bethe state (9) is indeed the eigenstate of the transfer matrix and Hamiltonian, where the Bethe roots $\{\lambda_j\}$ should satisfy the Bethe ansatz equations.

Exact physical quantities in the thermodynamic limit

Ground state energy density, elementary excitations, surface energy, free energy at finite temperature

- Degenerate points, at which the inhomogeneous term in the $T - Q$ relations is zero

- $t - W$ scheme

Main ides of fusion

The R-matrix

$$R_{12}(u) = u + P_{12} = \begin{pmatrix} u+1 & & & \\ & u & 1 & \\ & 1 & u & \\ & & & u+1 \end{pmatrix},$$

Properties

$$\begin{aligned}
\textit{regularity} &: \quad R_{12}(0) = P_{12}, \\
\textit{unitarity} &: \quad R_{12}(u)R_{21}(-u) = \rho_1(u) \times \mathrm{id}, \quad \rho_1(u) = -(u-1)(u+1), \\
\textit{crossing} - \textit{unitarity} &: \quad R_{12}^{t_1}(u)R_{21}^{t_1}(-u-2) = \rho_2(u) \times \mathrm{id}, \quad \rho_2(u) = -u(u+2), \\
\textit{Fusion conditions} &: \quad R_{12}(-1) = -2P_{12}^{(-)}, \quad R_{12}(1) = 2P_{12}^{(+)}.
\end{aligned}$$

Here $P_{12}^{(-)}$ is a one-dimensional projector with the base

$$|f_1\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle).$$

$P_{12}^{(+)}$ is a three-dimensional projector with the bases

$$|g_1\rangle = |11\rangle, \quad |g_2\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \quad |g_3\rangle = |22\rangle. \tag{10}$$

The properties of projection operators give

$$[P_{12}^{(-)}]^2 = P_{12}^{(-)}, \quad [P_{12}^{(+)}]^2 = P_{12}^{(+)}. \tag{11}$$

The the bases (10) and (11) are complete and orthogonal

$$P_{12}^{(-)} + P_{12}^{(+)} = 1, \quad P_{12}^{(-)}P_{12}^{(+)} = P_{12}^{(+)}P_{12}^{(-)} = 0.$$

If $u=\pm1$, then $R_{12}(\pm1)=\pm2P_{12}^{(\pm)}$. From the Yang-Baxter equation, we obtain

$$P_{12}^{(\pm)}R_{23}(u)R_{13}(u\pm1)P_{12}^{(\pm)}\equiv R_{\langle12\rangle3}(u)\equiv R_{\bar{1}3}(u).$$

1) The operator $P_{12}^{(\pm)}R_{23}(u)R_{13}(u\pm1)P_{12}^{(\pm)}$ defined in the tensor space $V_1\otimes V_2\otimes V_3$ can be projected into the subspace $V_{\langle12\rangle}\otimes V_3$, where $V_{\langle12\rangle}\equiv V_{\bar{1}}$ is the projected space of $V_1\otimes V_2$.

2) The fused R-matrix $R_{\bar{1}3}(u)$ also satisfies the Yang-Baxter equation, which means that the fusion does not break the integrability.

3) Fusion is used to obtain the high-dimensional representation of certain algebras.

4) If we take the fusion both in auxiliary and in quantum (physical) spaces, from the resulted R-matrix $R_{\langle12\rangle\langle34\rangle}(u)$ we can construct some new interesting integrable models.

Fusion of the Hubbard model?

By using fusion, we obtain

$$
\begin{aligned}
& t(u)t(u-1) \\
& \quad = [\rho_2(2u-1)]^{-1} tr_{12}\{[P_{12}^{(-)} K_2^+(u-1) R_{12}(-2u-1) K_1^+(u) P_{21}^{(-)}] \\
& \qquad \times [P_{21}^{(-)} T_1(u) T_2(u-1) P_{21}^{(-)}] \\
& \qquad \times [P_{21}^{(-)} K_1^-(u) R_{21}(2u-1) K_2^-(u-1) P_{12}^{(-)}] \\
& \qquad \times [P_{12}^{(-)} \hat{T}_1(u) \hat{T}_2(u-1) P_{12}^{(-)}]\} \\
& \qquad + [\rho_2(2u-1)]^{-1} tr_{12}\{[P_{21}^{(+)} K_2^+(u-1) R_{12}(-2u-1) K_1^+(u) P_{12}^{(+)}] \\
& \qquad \times [P_{12}^{(+)} T_1(u) T_2(u-1) P_{12}^{(+)}] \\
& \qquad \times [P_{12}^{(+)} K_1^-(u) R_{21}(2u-1) K_2^-(u-1) P_{21}^{(+)}] \\
& \qquad \times [P_{21}^{(+)} \hat{T}_1(u) \hat{T}_2(u-1) P_{21}^{(+)}]\} \\
& \quad = t_1(u) + t_2(u).
\end{aligned} \tag{12}
$$

The first term is the fusion by the one-dimensional projectors, and the results is the quantum determinant

$$t_1(u) = tr_{12}\{K^+_{\langle 12\rangle(-)}(u)T_{\langle 12\rangle(-)}(u)K^-_{\langle 12\rangle(-)}(u)\hat{T}_{\langle 12\rangle(-)}(u)\}$$

$$= [\rho_2(2u-1)]^{-1}h_2(u-1)h_1(u)$$

$$\times\prod_{j=1}^{N}(u-\theta_j+1)(u-\theta_l-1)(u+\theta_j+1)(u+\theta_l-1).$$

The second term is the fusion by the three-dimensional projectors. Detailed calculation gives

$$t_2(u) = [\rho_2(2u-1)]^{-1}(-2u^2)\prod_{j=1}^{N}(u-\theta_j)(u+\theta_j)$$

$$\times tr_{12}\{K^+_{\langle 12\rangle(+)}(u-\frac{1}{2})T_{\langle 12\rangle(+)}(u-\frac{1}{2})K^-_{\langle 12\rangle(+)}(u-\frac{1}{2})\hat{T}_{\langle 12\rangle(+)}(u-\frac{1}{2})\}.$$

From Yang-Baxter relations and reflection equations one can prove that the transfer matrices $t(u)$, $t_1(u)$ and $t_2(u)$ commutate with each other,

$$[t(u), t_1(u)] = [t(u), t_2(u)] = [t_1(u), t_2(u)] = 0.$$

Therefore, they have common eigenstates and can be diagonalized simultaneously.

$t - W$ relation

From Eq.(12), we have the following $t - W$ relation

$$t(u)t(u-1) = \Delta_q(u) \times \mathbf{id} + \bar{d}(u)\mathbf{W}(u), \tag{13}$$

where $\Delta_q(u)$ is the quantum determinant.

The transfer matrix $t(u)$ is an operator-valued degree $2N+2$ polynomial of u. The operator $\mathbf{W}(u)$ is an operator-valued degree $2N+2$ polynomial of u.

As we have shown that $t(u)$ and $\mathbf{W}(u)$ commutate with each other,

$$[\mathbf{W}(u), t(u)] = 0.$$

Therefore, they have common eigenstates.

Acting (13) on a common eigenstate $|\Psi\rangle$ we have

$$\Lambda(u)\Lambda(u-1) = \Delta_q(u) + \bar{d}(u)W(u), \tag{14}$$

where $W(u)$ is the eigenvalue of $\mathbf{W}(u)$.

Because $\Lambda(u)$ is a polynomial of u with degree $2N+2$ and satisfies the crossing symmetry $\Lambda(u)=\Lambda(-u-1)$, we parameterize $\Lambda(u)$ as

$$\Lambda(u)=\bar{\Lambda}_0\prod_{j=1}^{N+1}(u-z_j-\frac{1}{2})(u+z_j-\frac{1}{2}),\quad \bar{\Lambda}_0=2.$$

Thus $\Lambda(u)$ has $2N+2$ roots, $z_j+\frac{1}{2}$ and $-z_j+\frac{1}{2}$. Here we have put all the inhomogeneous parameters in zero, $\{\theta_j\}=0$.

$W(u)$ is a polynomial of u with degree $2N+2$, and we put

$$W(u)=\bar{W}_0\prod_{l=1}^{2N+2}(u-w_l),\quad \bar{W}_0=3-\xi^2.$$

An important fact is that (14) is a degree $4N+4$ polynomial equation and thus gives $4N+5$ independent equations for the coefficients to determine the $N+1$ z_j and $2N+2$ w_l completely.

Let $u = z_j + \frac{1}{2}$ in (14), we obtain

$$\bar{a}(z_j + \frac{1}{2})\bar{d}(z_j - \frac{1}{2}) = -\bar{d}(z_j + \frac{1}{2})(3 - \xi^2) \prod_{l=1}^{2N+2} \sinh(z_j - w_l + \frac{1}{2}), \quad j = 1, \cdots, N+1. \tag{15}$$

Let $u = -z_j + \frac{1}{2}$ in (14), we obtain

$$\begin{aligned} \bar{a}(-z_j + \frac{1}{2})\bar{d}(-z_j - \frac{1}{2}) &= -\bar{d}(-z_j + \frac{1}{2}) \\ &\times (3 - \xi^2) \prod_{l=1}^{2N+2} \sinh(-z_j - w_l + \frac{1}{2}), \quad j = 1, \cdots, N+1. \end{aligned} \tag{16}$$

Let $u = w_l$ in (14) we obtain

$$\begin{aligned} 4 \prod_{j=1}^{N+1} &(w_l - z_j - \frac{1}{2})(w_l + z_j - \frac{1}{2})(w_l - z_j - \frac{3}{2})(w_l + z_j - \frac{3}{2}) \\ &= \bar{a}(w_l)\bar{d}(w_l - 1), \quad l = 1, \cdots, 2N+2. \end{aligned} \tag{17}$$

Eqs.(15)-(17) are the Bethe ansatz equations. There are $3N+3$ equations and $3N+3$ unknowns.

- Distribution of zero-roots with finite system-size

- Thermodynamic limit

- Density of zero-roots

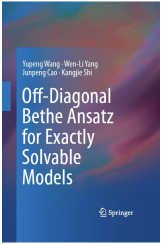
- Ground state energy density

- Surface energy

- Free energy at finite temperature

Summary

- Universal method to exactly solve the quantum integrable systems
- Anisotropic interactions and strongly correlated electronic systems
- Arbitrary integrable boundary conditions
- High spin and high rank (A_n, B_n, C_n, D_n)
- Non-Hermite physics and integrability (non-Hermite integrable models with interaction)



Thank you for your attention!