Novel (A)dS metrics in 5 dimensions

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Reference: arXiv:0910.3358 by L Zhao and B Zhu

Introduction

In this talk I will be talking about some novel (A)dS spacetime in 5 dimensions which is generalization of the well-known C-metric in 4 dimensions.

- C-metric itself has a long history (dates back to 1960's);
- C-metric possesses interesting features (containing 2 black holes accelerating apart);
- C-metric is the building block for 5 dimensional black ring solutions;
- No higher dimensional C-metric spacetimes are known and actually they do not exist due to Podolsky;
- Needs some higher dimensional spacetime which keeps at least some of the properties of C-metric in order to construct explicitly black rings in dimensions $d \geq 6$.

Getting to know about C-metric

- C-metric is usually written in an unconventional coordinate system. In the absence of rotational parameters, the metric can be written as

$$ds^2 = [A(x+y)]^{-2}(-\mathcal{F}dt^2 + \mathcal{F}^{-1}dy^2 + \mathcal{G}^{-1}dx^2 + \mathcal{G}dz^2),$$
$$\mathcal{F} = -\left(\pm\frac{1}{\ell^2 A^2} + 1\right) + y^2 - 2mAy^3 + q^2A^2y^4,$$
$$\mathcal{G} = 1 - x^2 - 2mAx^3 - q^2A^2x^4.$$

  The parameters ℓ, m, q correspond to (A)dS radius, mass and charge respectively. Black hole horizons appear at roots of $\mathcal{F}$. It was known that there are 2 black holes in this metric and they accelerate apart with acceleration related to the value of A.

Getting to know about C-metric

- J. B. Griffiths, P. Krtous, and J. Podolsky, gr-qc/0609056 gave a thorough analysis on the metric in the case $\ell \to \infty$ and $q = 0$, i.e. in the absence of charge and cosmological constant;
- In the presence of charge, K. Hong and E. Teo wrote two papers gr-qc/0305089 and gr-qc/0410002 to analyze the C-metric, but not as in-depth as Griffiths *et al* did;
- The cosmological constant in C-metric was analyzed in detail by O. J. C. Dias and J. P. S. Lemos in hep-th/0210065 (AdS) and hep-th/0301046 (dS);
- Podolsky (in a talk given at Bremen in Aug. 2008) announced that there can be no higher dimensional generalization of C-metric with black holes inside.

Getting to know about C-metric

- C-metric to black ring is just as bricks to a house (R. Emparan, H. Real, hep-th/0110258, hep-th/0110260, ...
- Even the most trivial limit of C-metric (with $\ell \to \infty$, $m, q \to 0$, i.e. empty C-metric) has found important role in black ring construction, see e.g. Emparan *et al*: hep-th/0407065;
- Attempts in finding exact black rings in $d > 5$ have lead to no result, speculatively due to the lack of generalizations of C-metric in $d > 4$;
- Alternative approximate methods (matched asymptotic expansion etc) have lead to the new concept of black folds (arXiv: 0708.2081, 0902.0427, 0910.1601), but this way of research can only touch the asymptotic and approximate properties of the objects;

Getting to know about C-metric

- In all dimensions $d \geq 5$, the construction of a black ring with non-vanishing cosmological constant seems to be difficult, and I suspect that this is due to the un-matching symmetries of the usual (A)dS spacetimes and the C-metric (the former have spherical symmetries while the latter are rotationally symmetric). Therefore, a thorough understanding of C-metric like spacetimes with non-vanishing cosmological constant seems to be inevitable before a black ring with non-vanishing cosmological constant can be found.

- The solution (to Einstein equation $R_{MN} - \frac{1}{2}g_{MN}R + \Lambda g_{MN} = 0$)

$$ds^2 = \frac{1}{\alpha^2(x+y)^2}\left[-G(y)H(z)dt^2 + G(y)\frac{dz^2}{H(z)} + \frac{dy^2}{G(y)} + \frac{dx^2}{F(x)} + F(x)d\phi^2\right],$$

$$F(x) = 1 - x^2, \quad G(y) = -1 - \frac{\Lambda}{6\alpha^2} + y^2, \quad H(z) = 1 - \left(1 + \frac{\Lambda}{6\alpha^2}\right)z^2.$$

The C-metric-ish of the metric is apparent from its coordinate description, but this one has no black holes in it (no m, q parameters present).

A novel 5D (A)dS vacuum: understanding the coordinates

- two Killing coordinates t, ϕ, each can be taken as time (to make the spacetime possessing a static patch). We take t;
- signature change in $H(z)$ would result in changing the roles of t and z — horizons occur at zeros $\pm z_0$ of $H(z)$ — z plays the role of a "radial" coordinate;
- $G(y)$ appears 3 times in the metric, changing its signature corresponds to triple Wick rotations in t, z, y, resulting in a metric with two timelike variables — give up this choice and let the signature of $G(y)$ remain fixed (unless $G(y) = 0$);
- for the same reason the signature of $F(x)$ must be kept fixed unless it takes the value 0;
- the hypersurface $x + y = 0$ lies at conformal infinity, thus the spacetime must sit in one side of this hypersurface. We take $x + y \geq 0$.

A novel 5D (A)dS vacuum: understanding the coordinates

- Physical ranges of the coordinates:

$$\begin{aligned} t &\in (-\infty, \infty), \\ z &\in (-\infty, \infty), \\ y &\in [y_0, \infty), \\ x &\in [-1, 1], \\ \phi &\in [0, 2\pi), \end{aligned}$$

where

$$y_0 = \sqrt{1 + \frac{\Lambda}{6\alpha^2}}, \quad z_0 = \frac{1}{\sqrt{1 + \frac{\Lambda}{6\alpha^2}}} = \frac{1}{y_0}.$$

Horizons appear at $z = \pm z_0$ — they cease to exist for $\Lambda < -6\alpha^2$!
(Will only consider $\Lambda \geq 0$ later)

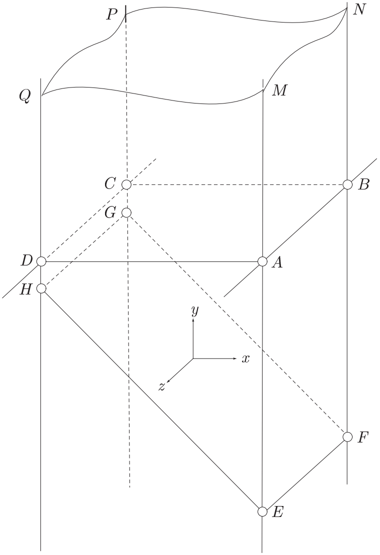


Figure: The (x, y, z) slice of the physical region of the spacetime: the static patch of the spacetime is the region bounded by the planes (A, B, C, D), (A, B, N, M), (C, D, Q, P), (A, D, Q, M) and (B, C, P, N).

A novel 5D (A)dS vacuum: understanding the coordinates

- Lack of black hole: Black holes corresponds to essential singularities which are causally disconnected from the outside world. Essential singularities signify themselves in the expressions of curvature invariants.

$$R_{MNPQ}R^{MNPQ} = \frac{10\alpha^8\Lambda^2}{9},$$

so there is no essential singularities and thus no black holes. $\Rightarrow$ horizons can only be acceleration horizons!

Horizon geometry

- metric on the horizons:

$$ds_H^2 = \frac{1}{\alpha^2 \left(y_0 - \sin\sigma \cos\theta\right)^2} \left(d\sigma^2 + \sin^2\sigma \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right),$$

  where y is replaced by σ, with $y = \frac{y_0}{\sin\sigma}$, $\sigma \in \left[0, \frac{\pi}{2}\right]$. — Clearly this is conformal to a 3-sphere;

- area of the horizon:

$$A = \pi^2 \left(\frac{6}{\Lambda}\right)^{3/2}.$$

  This area is equal to that of a 3-sphere of radius $r = (3/2\Lambda)^{1/2}$;

Horizon geometry

proper acceleration on the horizon:

- Rindler line element transverse to the horizon:

$$dl^2 = -\zeta^2 dt^2 + d\zeta^2,$$
$$\zeta = \frac{(G(y)H(z))^{1/2}}{\alpha(x+y)} \qquad (\zeta > 0).$$

- alternative coordinate (Kruskal like)

$$dl^2 = -dX^- dX^+,$$
$$X^- = -\zeta \exp(-t), \quad X^+ = \zeta \exp(t);$$

Horizon geometry

- Killing trajectory $X^M(\tau)$ along the Killing vector $\xi = \partial_t = X^+\partial_{X^+} - X^-\partial_{X^-}$: proper velocity

$$u^M = \frac{\xi^M}{\left(-\xi^2\right)^{1/2}},$$

proper acceleration

$$a^M = D_\tau u^M = u^N \nabla_N u^M, \quad a^M \partial_{X^M} = \left(X^+\right)^{-1} \partial_{X^+} + \left(X^-\right)^{-1} \partial_{X^-}.$$

- justification for the name acceleration horizon:

$$|a| = \left(g_{MN} a^M a^N\right)^{1/2} = \left(-\frac{1}{X^- X^+}\right)^{1/2} = \frac{1}{\zeta}.$$

Global structure

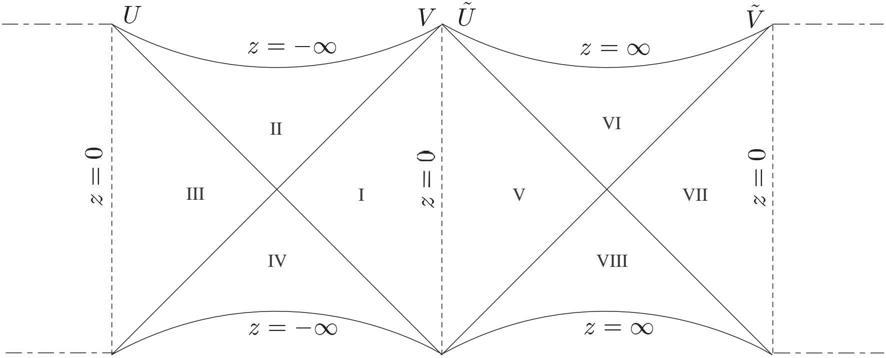
- global structure of spacetime is encoded in Kruskal coordinates. Near each horizon in the spacetime, Kruskal coordinates must be introduced separately;
- original → tortoise → Kruskal coordinates:

$$z^* = \frac{z_0}{2} \log \left| \frac{z_0 + z}{z_0 - z} \right|,$$

$$u = t - z^*, \quad v = t + z^*;$$

$$U = -\exp\left(-\frac{u}{z_0}\right), \quad V = \exp\left(\frac{v}{z_0}\right);$$

- extends cross horizons → conformal transformations → Penrose diagram:



Penrose diagram of the spacetime: Horizontal slashed lines represent repeated occurrences of the eight zones depicted in the middle.

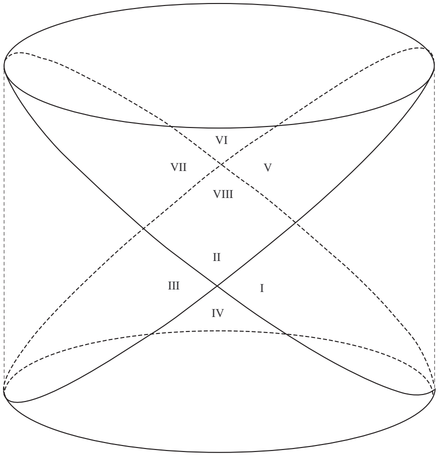


Figure: Penrose diagram drawn on a cylindrical surface

Back to 4D: KK reduction

- direct reduction:

$$ds_5^2 = e^{\varphi/\sqrt{3}} ds_4^2 + e^{-2\varphi/\sqrt{3}} d\phi^2,$$

where φ is a 4D scalar field,

$$ds_4^2 = \frac{F(x)^{1/2}}{\alpha^3(x+y)^3}\left[-G(y)H(z)dt^2 + G(y)\frac{dz^2}{H(z)} + \frac{dy^2}{G(y)} + \frac{dx^2}{F(x)}\right],$$

$$e^{-2\varphi/\sqrt{3}} = \frac{F(x)}{\alpha^2(x+y)^2};$$

Actions:

$$S_5 = \int d^5x\sqrt{-g_{(5)}}\left(R_{(5)} - \Lambda\right)$$

$$\Rightarrow$$

$$S_4 = \int d^4x\sqrt{-g_{(4)}}\left(R_{(4)} - \frac{1}{2}(\partial\varphi)^2 - \Lambda e^{\varphi/\sqrt{3}}\right).$$

Back to 4D: KK reduction

Boost + KK reduction:

- Boost:

$$t \to T = t\cosh\beta - \phi\sinh\beta,$$
$$\phi \to \Phi = -t\sinh\beta + \phi\cosh\beta,$$

velocity k and rapidity β:

$$k = \tanh\beta;$$

Back to 4D: KK reduction

- KK:

$$d\tilde{s}_5^2 = e^{\varphi/\sqrt{3}} d\tilde{s}_4^2 + e^{-2\varphi/\sqrt{3}} \left(d\Phi + \mathcal{A}\right)^2,$$

$$d\tilde{s}_4^2 = \frac{1}{\alpha^3 (x+y)^3} \left(\frac{F(x) - k^2 G(y) H(z)}{1-k^2}\right)^{1/2} \times \left[-\frac{G(y)H(z) - k^2 F(x)}{1-k^2} dT^2 + G(y) \frac{dz^2}{H(z)} + \frac{dy^2}{G(y)} + \frac{dx^2}{F(x)} \right],$$

$$\mathcal{A} = \frac{k[F(x) - G(y)H(z)]}{F(x) - k^2 G(y) H(z)} dT,$$

$$e^{-2\varphi/\sqrt{3}} = \frac{1}{\alpha^2 (x+y)^2} \frac{F(x) - k^2 G(y) H(z)}{1-k^2}.$$

- 4D action:

$$\tilde{S}_4 = \int d^4x \sqrt{-g_{(4)}} \left(R_{(4)} - \frac{1}{2} (\partial\varphi)^2 - \Lambda e^{\varphi/\sqrt{3}} - \frac{1}{4} e^{\varphi/\sqrt{3}} F_{\mu\nu} F^{\mu\nu} \right),$$

where

$$F = F_{\mu\nu} dx^\mu dx^\nu \equiv d\mathcal{A}.$$

— Einstein-Maxwell-Liouville solution!

$\Lambda = 0$

The $\Lambda = 0$ case (non-KK reduced) has a nice exterior geometric interpretation.

- metric:

$$ds^2 = \frac{1}{\alpha^2(x+y)^2}\left[-(y^2-1)(1-z^2)dt^2 + \frac{y^2-1}{1-z^2}dz^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right];$$

- Wick rotation: $t \to i\psi$:

$$ds^2 = \frac{1}{\alpha^2(x+y)^2}\left\{\frac{dy^2}{y^2-1} + (y^2-1)\left[\frac{dz^2}{1-z^2} + (1-z^2)d\psi^2\right] + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right\}.$$

$\Lambda = 0$

- The Wick rotated version of the metric is just the 5D Euclidean metric in disguise:

$$ds^2 = \sum_{i=1}^{5} dX_i^2$$

where

$$X_1 = \frac{\alpha}{B} \sin\theta \cos\phi, \quad X_2 = \frac{\alpha}{B} \sin\theta \sin\phi,$$
$$X_3 = \frac{\alpha}{B} \sinh\eta \sin\chi \cos\psi, \quad X_4 = \frac{\alpha}{B} \sinh\eta \sin\chi \sin\psi,$$
$$X_5 = \frac{\alpha}{B} \sinh\eta \cos\chi,$$

with

$$B \equiv \cosh\eta - \cos\theta,$$
$$\alpha \equiv \sqrt{a^2 - b^2}.$$

$\Lambda = 0$

- the coordinates are related via

$$x = -\cos\theta,$$
$$y = \cosh\eta,$$
$$z = \cos\chi.$$

$\Lambda = 0$

- particular value of y or η:

$$y = y_0, \quad \eta = \eta_0, \quad \cosh \eta_0 = \frac{a}{b}.$$

At this particular value of the coordinate y, the $\Lambda = 0$ metric becomes that of the embedding surface

$$X_1^2 + X_2^2 + \left(\sqrt{X_3^2 + X_4^2 + X_5^2} - a \right)^2 = b^2.$$

— This is just $S^2 \times S^2$, a 4 dimensional toric surface.

- generic values of y: for generic fixed values of y, the metric is always topologically equivalent to $S^2 \times S^2$, but local geometries can differ.

$\Lambda = 0$

- Inverse Wick rotation: Wick rotation from ψ to it corresponds to Wick rotation of X_4, or, from the point of view of embedding surfaces, corresponds to

$$X_1^2 + X_2^2 + \left(\sqrt{X_3^2 - X_4^2 + X_5^2} - a\right)^2 = b^2.$$

Therefore, the constant y hyper surfaces are all topologically equivalent to $dS_2 \times S^2$, where dS_2 is the 2-dimensional de Sitter with embedding equation $X^2 + Y^2 - Z^2 = a^2$ in 3 dimensions;

- For generic non-fixed values of y, the static patch in the $\Lambda = 0$ metric is just the usual 5D Minkowski spacetime.
- KK reduced theory is Einstein-Maxwell-dilaton theory, rather than Einstein-Maxwell-Liouville theory.

Discussions

A lot todos:

- $\Lambda < 0$;
- double Wick rotation;
- other C-metric like solutions with different foliation, e.g.

$$ds^2 = \frac{1}{\alpha^2(x+y)^2}\left[-G(y)dt^2 + \frac{dy^2}{G(y)} + \frac{dx^2}{F(x)} + F(x)\left(\frac{dz^2}{H(z)} + H(z)d\phi^2\right)\right],$$

  with

$$F(x) = 1 - x^2, \quad G(y) = -\frac{\Lambda}{6\alpha^2} - 1 + y^2, \quad H(z) = 1 - z^2.$$

- bricks are ready, where are the houses?