Generating classical string solutions

In collaboration with
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Prelude

• Classical string solutions play an important role in understanding various aspects of the AdS/CFT.

• One can use integrability methods to check and maybe even prove the correspondence.

• One particular class of classical string solutions are the solitonic ones (at the classical level we can map those solutions to solitons in sin(h)-Gordon)

• Some examples of solitonic solutions

-GKP string
-Kruczenski (application to scattering amplitudes at strong coupling)
-Berkovits-Maldacena
-Giant magnons in various spaces

We will focus on the last example

Outline

In this talk I will briefly describe the objects we are interested in (Giant Magnons) and then I will present a powerful method (Dressing method) that gives recursion relations that relate N magnons to $N-1$. Using determinant manipulations it is possible to find a compact and simple formula for the scattering of N magnons. We briefly comment on the classical time delay due to the scattering of those magnons. Our method can be applied to a great class of various spaces of interest as well as to Wilson loops.

- **Describe the objects we are interested in (Giant Magnons)**
- **Describe a method to find them**
  (Dressing method (1978))
- **Find the N-magnon solution**
- **Talk about some of the properties of the N-soliton solution**
- **Apply the dressing method to Wilson loops and BTZ**
- **Conclude**

Introduction to AdS/CFT

• Solving QCD is an interesting problem (LHC phenomenology, gluon scattering amplitudes, ...), but also very hard.

• In order to simplify the problem we look at a different (simpler) theory with more structure and symmetry, the $\mathcal{N} = 4$ supersymmetric Yang-Mills with $SU(N)$ gauge group.

• Then the problem of solving $\mathcal{N} = 4$ consists of finding all correlation functions of local gauge invariant operators.

• The problem can be further simplified by considering the large N-limit ['t Hooft].

• The large N-limit indicates that gauge theories can be related to string theories.

• The first non-trivial example was given more than 10 years ago [Maldacena; Gubser, Klebanov, Polyakov; Witten] and relates $\mathcal{N} = 4$ to strings in $AdS_5 \times S^5$.

AdS/CFT dictionary

$\mathcal{N}=4$ SYM	stings on $AdS_5 \times S^5$
't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$	string tension $T = \frac{\sqrt{\lambda}}{2\pi}$
number of colors N	string coupling $g_s = \frac{\lambda}{4\pi N}$
large N limit	free strings
strong coupling	classical strings
local operators	string states
scaling dimension Δ	Energy E
amplitudes	certain Wilson loops
$\langle \mathcal{O}_A(x)\,\mathcal{O}_B(y)\rangle = \frac{M\,\delta_{A,B}}{(x-y)^{2\,\Delta_A(\lambda,\frac{1}{N})}}$	$\mathcal{H}_{\mathrm{string}}\,\lvert\mathcal{O}_A\rangle = E_A(\frac{R^2}{\alpha'}, g_s)\,\lvert\mathcal{O}_A\rangle$

strong-weak correspondence

Integrability in AdS/CFT

- String theory in $AdS_5 \times S^5$ is classically integrable [Mandal, Suryanarayana, Wadia; Bena, Polchinski, Roiban].

- We can use integrability methods to study the spectrum of string theory. We expect agreement with gauge theory results.

- In the gauge theory we can view the problem of determining the anomalous dimensions of single trace operators as the problem of determining the spectrum of certain spin-chains [Minahan, Zarembo]. A well-known method is the Bethe ansatz.

- A feature of integrable theories (like $\mathcal{N} = 4$) is that can be solved if we know the 2-particle scattering. Exact S-matrix has been proposed for the planar theory [Beisert, Eden, Staudacher].

- Following Witten's discovery of twistor string theory we have seen a lot of progress in SYM calculations even relevant for collider physics.

Giant magnon limit [Hofman, Maldacena (2006)]

According to the AdS/CFT dictionary states with $E-J=0$ correspond to a long chain of Z fields

$$E-J=0 \quad \Leftrightarrow \quad \mathrm{tr(Z^J)}.$$

One can also consider states with finite $E-J$

$$E-J=\text{finite} \quad \Leftrightarrow \quad \mathcal{O}_p=\sum_l e^{ipl}(\ldots ZZZ\underset{\overset{\uparrow}{l}}{W}ZZZ\ldots),$$

Using supersymmetry, Beisert has shown that

$$E-J=\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$$

in the large coupling limit

$$E-J=\frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right|.$$

The giant magnon limit is defined

$$E,J\to\infty, \quad \lambda=g_{YM}^2 N=\text{fixed}, \quad p=\text{fixed}, \quad E-J=\text{finite}.$$

Review of giant magnon solution

Nambu-Goto in spherical coordinates

$$S = \frac{\sqrt{\lambda}}{2\pi} \int dt d\phi \sqrt{\cos^2\theta\theta'^2 + \sin^2\theta}$$

solution $\sin\theta = \frac{\sin\theta_0}{\cos\phi}$, where θ_0 is the integration constant

energy is the same as in gauge theory ($E - J = \frac{\sqrt{\lambda}}{\pi}\sin\frac{\Delta\phi}{2}$) with $\Delta\phi \equiv p$

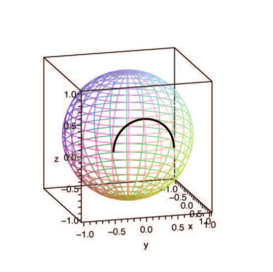

Figure: Plot of the HM elementary giant magnon for $p = 1$ (p is the angle between the endpoints, not a Noether charge).

Pohlmeyer reduction (1976)

We would like to solve eom+Virasoro for the string sigma model. Many different techniques have been developed (depending on the problem we want to solve) including

- ansatz
- dressing method (for example in the case of giant magnons)
- Pohlmeyer reduction
- ...

We use the Pohlmeyer reduction method. One can view the Pohlmeyer reduction as a sophisticated gauge choice where we are left with a model that only involves physical degrees of freedom. The reduced model inherits integrable structures of the original sigma model.

Let us review the Polhmeyer reduction for S^2.

S^2 example

$$\text{eom}: \quad \partial\bar{\partial}X = (\partial X \cdot \bar{\partial}X)X$$

$$\text{Vir}: \quad (\partial X)^2 = (\bar{\partial}X)^2 = 1$$

$$\text{length}: \quad X^2 = 1$$

$$\rightleftarrows$$

sin-Gordon

$$\partial\bar{\partial}\alpha = \sin\alpha$$

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- Demand that $\partial(\bar{\partial}\mathrm{Basis}) = \bar{\partial}(\partial\mathrm{Basis})$ (compatibility condition)

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$\rightleftarrows$

sin-Gordon

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$\sin(h)$*-Gordon should contain all information we need in boundary behavior and location of the poles.*

Some more examples

$$
\begin{aligned}
S^2 \text{ strings} &\longleftrightarrow \sin\text{Gordon} \\
S^3 \text{ strings} &\longleftrightarrow \text{complex } \sin\text{Gordon} \\
AdS_5 \text{ strings} &\longleftrightarrow \text{generalized } \sinh\text{Gordon} \\
CP^3 \text{ strings} &\longleftrightarrow \text{known} \\
AdS_5 \times S^5 \text{ strings} &\longleftrightarrow \text{system of generalized } \sin(\text{h}) \text{ Gordon}
\end{aligned}
$$

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strings in $AdS_5 \times S^5$

ansatz

dressing method

Bäcklund transformation

...

$\rightleftarrows$

generalized $\sin(\text{h})$-Gordon

Hitchin equations

ansatz

Bäcklund transformation

...

Pohlmeyer map of giant magnons [1976]

At the classical level there is a correspondence between

sine-Gordon solitons $\leftrightarrow$ giant magnons

$$\cos\alpha = \partial X \cdot \bar{\partial} X$$

We can compare energy, time delay, and phase shift for the two theories

sine-Gordon	giant magnon
$E_{\mathrm{sG}} = \gamma$	$E_{\mathrm{magnon}} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\gamma}$
$\Delta T_{CM} = \frac{2}{\gamma v}\log v$	$\Delta T_{12} = \frac{2}{\gamma_1 v_1}\log v_{cm}$

$$v = \cos\frac{p}{2}, \quad \gamma^{-2} = 1 - v^2, \quad \text{phase shift} = \int dE_1 \Delta T_{12}$$

Dyonic giant magnons [Chen, Dorey, Okamura (2006)]

Magnons with polarizations in an $SU(2)$ subsector carry a second conserved $U(1)$ R-charge, J_2, and they can form boundstates with exact dispersion relation

$$E - J = \sqrt{J_1^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$$

that correspond to operators of the form

$$\mathcal{O}_p = \sum_l e^{ipl} (\ldots ZZZ \underset{\substack{\uparrow \\ l}}{W}{}^{J_1} ZZZ \ldots)$$

The dressing method [Zakharov, Mikhailov, Shabat (1978)]

• In 1967 the inverse scattering method (ISM) was discovered by Gardner, Green, Kruskal, Miura. The task of enumerating nonlinear differential equations integrable by this method became fundamental.

• In 1978 an algorithm for constructing exact solutions of new classes of equations integrable by ISM was given. This is the so called dressing method.

• We would like to study scattering and bound states of giant magnons. A candidate method to study these states is the dressing method. In one line we can say that the dressing method generates **new soliton solutions** from old ones.

• The last years the dressing method has been successfully used in the context of giant magnons by many authors [David, Hollowood, Jevicki, C.K., Miramontes, Papathanasiou, Sahoo, Spradlin, Suzuki, Volovich, ...].

We now proceed to explain how the dressing method works.

- We start with the lagrangian of the principal chiral model

$$\mathcal{L} = \mathrm{Tr}[(\partial_\mu g g^{-1})^2]$$

where g is an element of a Lie group. For example for strings in S^3 we can take

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ X_3 - iX_4 & X_1 - iX_2 \end{pmatrix} \in SU(2),$$

where X_i are embedding coordinates of S^3.

- More generally, we can impose constraints on g, such that g becomes an element of a coset (for example for $CP^3 = SU(4)/U(3)$ we have that $gg^\dagger = g\theta g\theta = 1$, where $\theta = \mathrm{diag}(-1,1,1,1)$).
- The eom of the principal chiral model for the matrix field $g(z,\bar{z})$ are

$$\bar{\partial}(\partial g g^{-1}) + \partial(\bar{\partial} g g^{-1}) = 0.$$

Then we are looking for a new solution of the form

$$\underbrace{g'(z,\bar{z})}_{\text{new solution}} = \underbrace{\chi(z,\bar{z})}_{\text{dressing factor}} \times \underbrace{g(z,\bar{z})}_{\text{known solution}}$$

Finding the dressing factor χ

The eom for g

$$\bar{\partial}(\partial g g^{-1}) + \partial(\bar{\partial} g g^{-1}) = 0$$

is the compatibility condition $(\partial(\bar{\partial}\Psi) = \bar{\partial}(\partial\Psi))$ of the first order linear system

$$\partial\Psi(\lambda) = \frac{\partial g g^{-1}\Psi(\lambda)}{1-\lambda}\,, \qquad \bar{\partial}\Psi(\lambda) = \frac{\bar{\partial} g g^{-1}\Psi(\lambda)}{1+\lambda}$$

for $\lambda = 0$. The complex parameter λ is called the spectral parameter. Suppose we know a solution to the above system, $\Psi(0) = g$. We will now find a new one. We make the ansatz

$$\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda), \quad \chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1}P,$$

where λ_1 is an arbitrary complex constant. It turns out that P is a projector onto the subspace spanned by $\Psi(\bar{\lambda}_1)e_1$ for arbitrary constant vector e_1. Then the new solution that is labeled by (λ_1, e_1) is

$$g' = \Psi'(0).$$

Comments on the dressing method

- transforms second order differential equation to a system of first order equations
- we can start with any known solution (a choice that usually simplifies the problem is the vacuum of the theory)
- construct one solution and all other can be obtained with algebraic methods
- has the flexibility to approach the same problems with several different ways (for example we can see the 5-sphere either as the coset $SO(6)/SO(5)$ or as $SU(4)/Sp(2)$)
- can be applied to a great variety of different spaces (including all symmetric spaces)

The N-magnon solution in $R \times S^3$

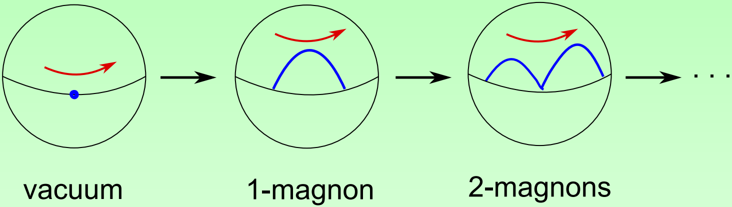

Figure: In order to find the N-magnon solution we start by applying the dressing method to the the vacuum, a point particle that moves with the speed of light around an equator of the S^3. The endpoints of the dressed solutions also move around the equator with the speed of light. Applying the dressing methods N-times we can get the N-magnon solution that corresponds to an N-magnon excitation in the gauge theory side. The angle seperation of the endpoints corresponds to the momentum of the magnon in the gauge theory side. The solution is dual to the N-soliton solution in the complex sine-Gordon model.

N-magnon solution

• The dressing method gives us (complicated) recursion relations that relate N magnons to $N-1$ magnons that we write explicitly as

$$\Psi_N(\lambda) = \left(1 + \frac{\lambda_N - \bar{\lambda}_N}{\lambda - \lambda_N} P\right) \Psi_{N-1}(\lambda), \quad P = \frac{\Psi_{N-1}(\bar{\lambda}_N) e e^\dagger \Psi_{N-1}^{-1}(\lambda_N)}{e^\dagger \Psi_{N-1}^{-1}(\lambda_N) \Psi_{N-1}(\bar{\lambda}_N) e}$$

• The proof is based on properties of determinants (they take care of redundancy and how to go from N to $N-1$ magnons)

$$\begin{vmatrix} a_1 + \lambda b_1 & b_1 & \cdots & c_1 \\ a_2 + \lambda b_2 & b_2 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n + \lambda b_n & b_n & \cdots & c_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & \cdots & c_1 \\ a_2 & b_2 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & b_n & \cdots & c_n \end{vmatrix}$$

$$\begin{vmatrix} 1 & x_1 & \cdots & x_n \\ 0 & a_1 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_n & \cdots & c_n \end{vmatrix} = \begin{vmatrix} a_1 & \cdots & c_1 \\ \vdots & \ddots & \vdots \\ a_n & \cdots & c_n \end{vmatrix}.$$

The N magnon solution for CP^n

After choosing the appropriate variables (generally a difficult step) and after using elementary determinant manipulations the N-magnon solution can be written in a compact form as

$$Z_N = \left(\det(\alpha_{ij}) + \sum_{i,j=1}^{N} (-1)^{i+j} M_{ij} h_j h_i^\dagger\right) Z_0,$$

where

- Z_i the embedding coordinates for the i'th magnon,
- M_{ij} is the minor formed by removing the i'th row and j'th column of the matrix with elements α_{ij}, $i, j = 1, \ldots, N$,
- $h_i = \theta \Psi_0(\bar{\lambda}_i) e_i, \; \theta = \mathrm{diag}(-1, 1, \ldots, 1)$
- $\alpha_{ij} = -\frac{\lambda_i \beta_{ij}}{\lambda_i - \bar{\lambda}_j} - \frac{\gamma_{ij}}{\lambda_i \bar{\lambda}_j - 1}, \; \beta_{ij} = h_i^\dagger h_j, \; \gamma_{ij} = h_i^\dagger \theta \Psi(0) h_j$

where λ_j is the spectral parameter and e_i the polarization vector.

General comments on our N-magnon solution

$$Z_N = \left(\det(\alpha_{ij}) + \sum_{i,j=1}^{N} (-1)^{i+j} M_{ij} h_j h_i^\dagger\right) Z_0,$$

- We have not specified what Z_0 is. In fact it can be any solution of the string model. It can be taken for example to be the 1- or 2-magnon solution or even a non-solitonic one.
- It is a general observation that N-soliton solutions of various integrable systems can be written as $N \times N$ determinants. Therefore our formula is not of a surprise. The difficult part is to find the appropriate variables in which we can express the solution in a nice form.
- We have not found all possible solutions of the system. It may also possess not solitonic ones.

Asymptotic behavior

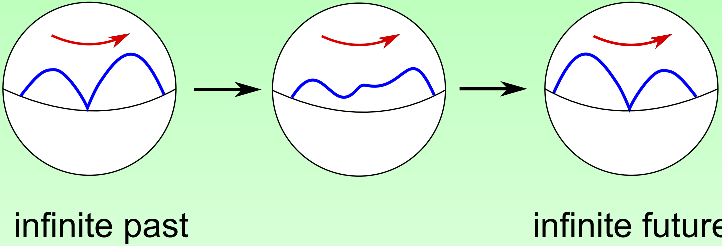

Figure: In the $x \to \pm\infty$ the magnon touches the equator with $p = \sum p_i$. In the $t \to \pm\infty$ limit we can prove that asymptotically the N-magnons split into N-single magnons, with the effect of the interaction being encoded only in a relative time delay (the shape of the magnons remain the same after scattering). The N-magnons exhibit the property of factorized scattering as expected by the integrability of the σ-model. The time delay agrees with the $\sin$-Gordon time delay. The phase shift agrees with the large λ limit of the phase in gauge theory [Arutyunov, Frolov, Staudacher].

Wilson Loops on S^2

The dressing method can be used to find new Wilson loops. The Wilson loops we are interested in are 1/8-BPS and couple to three of the six real scalar fields $\vec{\Phi}$ of $\mathcal{N} = 4$ SYM

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \oint dt \left(i\dot{x}^\mu A_\mu + (\vec{x} \times \dot{\vec{x}}) \cdot \vec{\Phi} \right), \tag{1}$$

where $x^\mu = (\vec{x}, 0), \quad \vec{x}^2 = 1.$

The string duals of these Wilson loops are contained in an $AdS_4 \times S^2$ subspace of $AdS_5 \times S^5$. We write the metric of the subspace as

$$ds^2 = \frac{1}{z^2} dx^i dx^i + z^2 dy^i dy^i, \quad z^2 = \frac{1}{y^i y^i}. \tag{2}$$

The supersymmetric constraints are

$$\vec{x}^2 + z^2 = 1, \quad \vec{x} \cdot \vec{y} = C = \mathrm{const}, \quad z^2 \partial_\alpha (\vec{x} \times \vec{y}) = \epsilon_{\alpha\beta} \partial_\beta \vec{x}. \tag{3}$$

If we set $\xi^A = (\vec{\xi}, \xi^4), \quad \vec{\xi} = z\vec{y} \times \vec{x}, \quad \xi^4 = \sqrt{1 + C^2} z$, we get

$$\partial^2 \xi^A + \xi^A \partial_A \xi^B \partial_A \xi^B = 0. \tag{4}$$

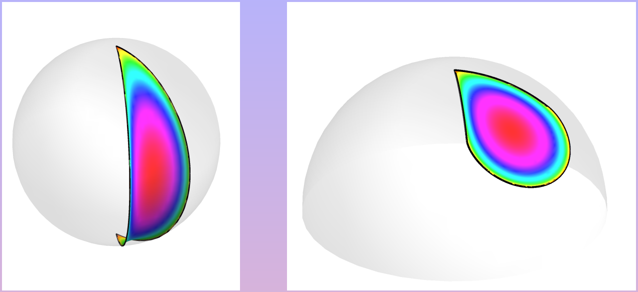

Figure: By dressing the known longitude solution once we get a new solution that we call the petal solution.

Generating String Solutions in BTZ

One can find applications of the dressing method to the problem of string solutions in BTZ. The problem can be regarding as finding new string solutions that move in $SU(1,1)$. Then the dressing factor takes the form

$$\Psi_1(\lambda) = \left(1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P\right) \Psi_0(\lambda), \quad P = \frac{\Psi_0(\bar{\lambda}_1) e e^\dagger \Psi_0^{-1}(\lambda_1) M}{e^\dagger \Psi_0^{-1}(\lambda_1) M \Psi_0(\bar{\lambda}_1) e},$$

where

$$M = \mathrm{diag}(-1,\ 1).$$

The N-solition solution similarly follows.

Concluding words

- We have applied the dressing method to the spaces $SU(n),\ SO(n)/SO(n-1),\ CP^n,\ SU(1,1)$. The N-soliton solutions were found.
- Other spaces where the method can be applied are $Sp(n)/U(n),\ SO(n),\ SO(2n)/U(n)$ and many more.
- The dressing method can be applied to a variety of problems like for example 1/8-BPS Wilson loops and string solutions in BTZ.
- It would be interested to use the dressing method to AdS space as well as to construct even more Wilson loops by dressing more than once or by dressing other known solutions.
- It would be interested to find giant-magnon type solutions in other spaces of interest like $S^3 \times S^3$.