Scattering from Geometries

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Based on works with F. Cachazo & E. Y. Yuan (2013-15)
with N. Arkani-Hamed, Y. Bai, G.Yan (2017) …

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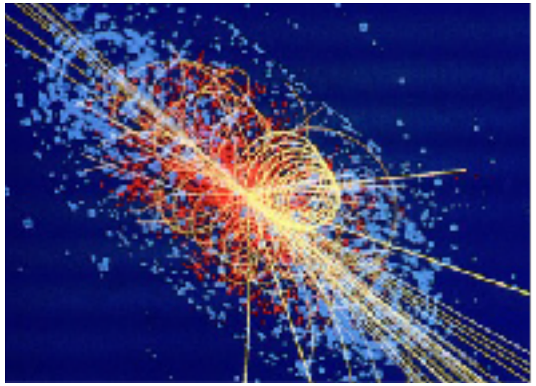
S-matrix in QFT

- Colliders at high energies need amplitudes of e.g. many gluons/quarks



gg → gg … g

- Fundamental level: understanding of QFT & gravity incomplete
  new structures & simplicity seen in (perturbative) scattering amplitudes

- Goal: new ideas & pictures of QFT & gravity from studying the S-matrix

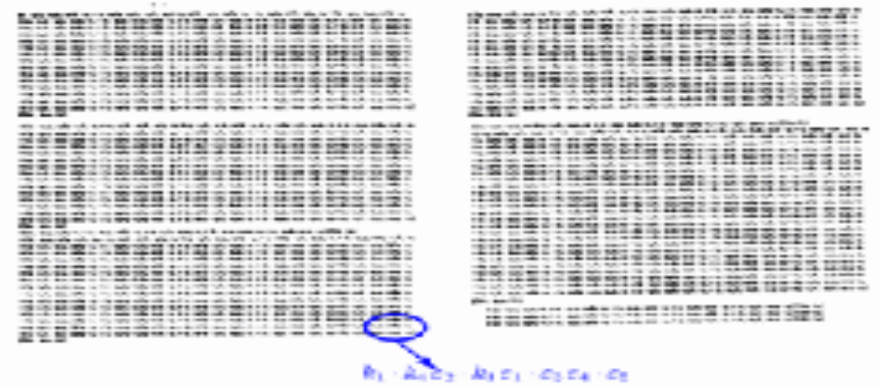
Feynman diagrams

Nice physical picture: manifest locality & unitarity, with a price to pay…

Challenging for more legs/loops: many diagrams, many many terms, no manifest gauge inv.

n-gluon scattering (tree)

n	4	5	6	7	8	9	10
# diagrams	4	25	220	2485	34300	559405	10525900



Gluons: 2 states h $= \pm$, but manifest locality requires 4 states (huge redundancies)
Much worse for graviton scattering: redundancies from diff invariance

A prior no reason to expect any simplicity or structures in the S-matrix

Parke-Taylor & Witten

There is something going on: "Maximally-Helicity-Violating" amplitudes [Parke, Taylor, 86]

$$M_n(i^-, j^-) = \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle},$$

$$k_a^\mu = (\sigma^\mu)_{\alpha,\dot\alpha}\lambda_a^\alpha\tilde\lambda_a^{\dot\alpha}, \quad \epsilon_a^\pm = \dots$$

$$\langle a\,b\rangle := \varepsilon_{\alpha,\beta}\,\lambda_a^\alpha\,\lambda_b^\beta, \quad [a\,b] := \varepsilon_{\dot\alpha,\dot\beta}\,\tilde\lambda_a^{\dot\alpha}\,\tilde\lambda_b^{\dot\beta}$$

Key observation: [Nair, 88] Parke-Taylor MHV amps = correlator on $\mathbb{CP}^1$

$$\lambda_i^\alpha \sim (z_i, 1), \qquad PT_n := \frac{1}{(z_1 - z_2)\,(z_2 - z_3)\cdots(z_n - z_1)}. \qquad j_A(z) j_B(z') = \frac{f_{AB}^C\, j_C}{z - z'} + \text{double poles} + \dots$$

Witten's twistor string theory → worldsheet model for gluon tree amplitudes

amps = string correlators with a map from $\mathbb{CP}^1$ to $\mathbb{CP}^{3|4}$(twistor space) [Witten, 2003]

Cachazo-He-Yuan formulation

Witten's twistor string very special: d=4 N=4 super Yang-Mills theory

- no supersymmetry? any spacetime dimension?
- general theories: gravity, Yang-Mills, standard model, effective field theories?
- generalizations to loop level?

CHY formulation: scattering of massless particles in any dimension [CHY 2013]

- *compact formulas* for amplitudes of gluons, gravitons, fermions, scalars, etc.
- *manifest* gauge (diff) invariance, double-copy relations, soft theorems, etc.
- *string-theory origin*: ambitwsitor string [Mason, Skinner 13] → loops from higher genus [Adamo et al 14]

Holography for S-matrix

Asymptotically flat spacetime: S-matrix is the only observable of quantum gravity!

Natural holographic question: is there a "theory at infinity" (=on-shell kinematic space) that computes S-matrix without local evolution in the "bulk"? Much harder than AdS case

Boundary of AdS=ordinary flat space (standard time & locality), only needs local QFT
No such luxuries for asymptotics of flat spacetime : no time or locality!
Mystery: what principles a holographic theory for S-matrix should be based on?

New strategy: look for fundamentally new laws (usually new math structures) -> S-matrix as the answer to entirely different kinds of questions -> "discover" unitarity & causality

Geometries in kinematic space

Encouraging success: S-matrix = Answer to (geometric) Q's in auxiliary spaces

- **Moduli space: perturbative string amps=correlators of worldsheet CFT**
- **Same picture for CHY: QFT amps=worldsheet correlators with scattering eqs**
- **Positive Grassmannian: the amplituhedron for N=4 SYM**

These auxiliary geometries have "factorizing" boundary structures, from which locality and unitarity emerge (without mentioning spacetime)!

What questions to ask, directly in kinematic space, to generate amplitudes (which encode spacetime and quantum mechanics)? Avartar of geometries?

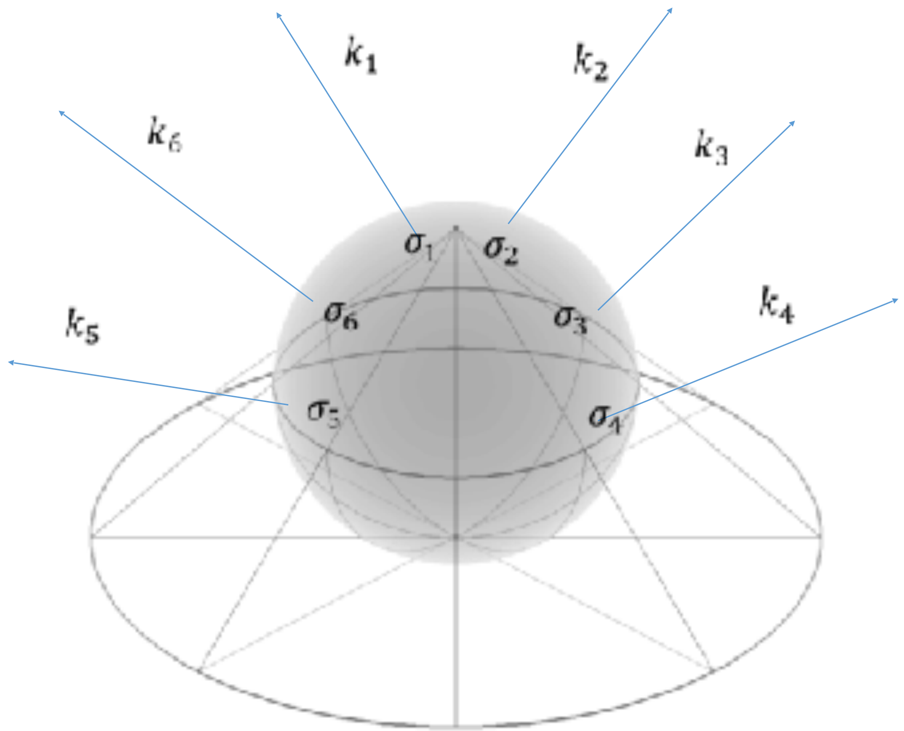
Scattering equations

$$\sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \qquad a = 1, 2, \ldots, n$$

[CHY 2013]

$SL(2,\mathbb{C})$ symmetry:

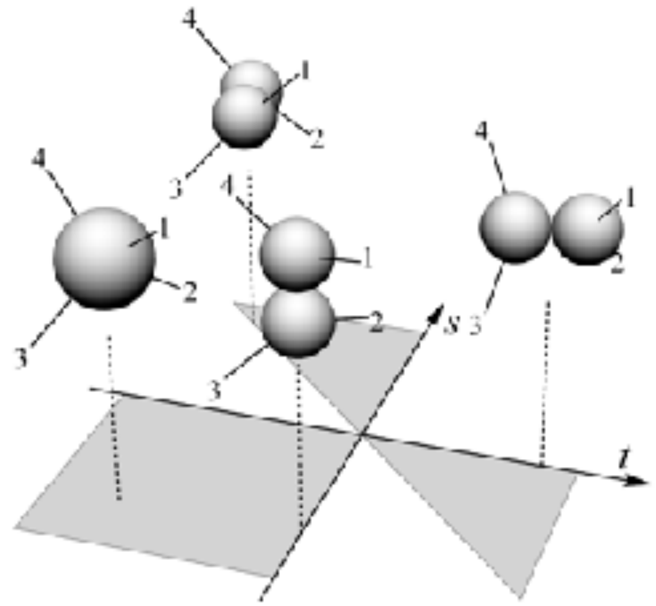
n-3 variables, n-3 equations



- universal, independent of theories (“kinetic part”): determine n punctures in terms of n null momenta
- non-trivial polynomial eqs: (n-3)! solutions [CHY; Dolan, Goddard]
- saddle-point eqs of string Koba-Nielson factor[Gross, Mende]

Geometries of moduli space

$$E_a := \sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \qquad a = 1, 2, \ldots, n$$



$$\{\sigma_2, \sigma_3, \sigma_4\} = \{0, 1, \infty\} \qquad \sigma_1 = -\frac{s_{12}}{s_{14}}$$

- Connect kinematic space of n massless particles to moduli space of n-punctured Riemann spheres
- map physical singularities to those of the moduli space
- captures universal factorization of any massless amps

CHY formulas

$$M_n = \underbrace{\int \frac{d^n\sigma}{\text{vol}\,\text{SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k,\epsilon,\sigma\}) = \sum_{\{\sigma\}\in\text{solns.}} \frac{\mathcal{I}(\{k,\epsilon,\sigma\})}{J(\{\sigma\})}$$

- n-3 integrals with n-3 delta functions -> sum over solutions, of some “CHY integrand”
- New picture: scattering of massless particles via worldsheet correlators
- Feynman diagrams, Lagrangians, even spacetime itself become emergent

$SL(2,\mathbb{C})$ symmetry: $\sigma_a \to \frac{\alpha\sigma_a+\beta}{\gamma\sigma_a+\delta}$, $E_a \to (\gamma\sigma_a+\delta)^2 E_a$

fix $\sigma_i, \sigma_j, \sigma_k$, $n-3$ variables

remove E_r, E_s, E_t, $n-3$ equations

Simplest CHY formulas

Task: find "dynamical part", i.e. CHY integrands for various theories

- Parke-Taylor factor: "half integrand" (half of the SL(2) weight of CHY integrand)

$$\mathrm{PT}[\pi] := \frac{1}{(\sigma_{\pi(1)} - \sigma_{\pi(2)})(\sigma_{\pi(2)} - \sigma_{\pi(3)}) \cdots (\sigma_{\pi(n)} - \sigma_{\pi(1)})}$$

- Simplest integrand: two copies of Parke-Taylor factors with two orderings

$$m[\pi|\rho] := \int \frac{d^n\sigma}{\mathrm{vol}\ \mathrm{SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)\ PT[\pi]\ PT[\rho]\,.$$

- Remarkably it computes simplest amps: trivalent scalar Feynman diagrams (tree)!

Scalar diagrams and ϕ^3 theory

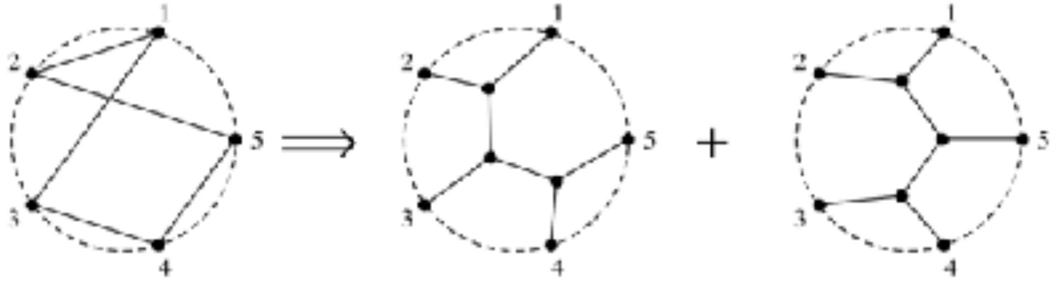
- These are "double-partial amplitudes" of a bi-adjoint scalar theory:

$$\mathcal{L}_{\phi^3} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{3!} f^{I\,J\,K} f^{I'\,J'\,K'}\, \phi^{I\,I'} \phi^{J\,J'} \phi^{K\,K'} \qquad M_n^{\phi^3} = \sum_{\pi,\rho} \mathrm{Tr}(T^{I_{\pi(1)}} \cdots T^{I_{\pi(n)}})\, \mathrm{Tr}(T^{I_{\rho(1)}} \cdots T^{I_{\rho(n)}})\, m[\pi|\rho]$$

- Sum of cubic diagrams that can be drawn on a disk with both orderings

$$m[1234|1234] = \frac{1}{s_{12}} + \frac{1}{s_{14}} \qquad m[12345|12543] = \frac{1}{s_{12}\, s_{34}} + \frac{1}{s_{12}\, s_{45}}$$

$$m[1234|1243] = \frac{1}{s_{12}}$$



Gluon scattering from CHY

- Yang-Mills? still need color part, but also a new ingredient encoding polarizations

- Inspired by the correlator of n open-string vertex operators: on the support of scattering eqs, the correlator simplifies to Pfaffian of a simple matrix

$$\mathrm{Pf}'\Psi \sim \langle V^{(0)}(\sigma_1)\ldots V^{(-1)}(\sigma_i)\ldots V^{(-1)}(\sigma_j)\ldots V^{(0)}(\sigma_n)\rangle$$

- An elegant formula for the tree-level S-matrix of n gluons in any dimensions:

$$M_n^{\mathrm{YM}}[\pi] = \int d\mu_n\, \mathrm{PT}[\pi]\, \mathrm{Pf}'\,\Psi$$

gluon amps from “heterotic strings”

The Pfaffian

- The (reduced) Pfaffian of a 2n x 2n skew matrix, with four blocks

$$\mathrm{Pf}'\Psi := \frac{\mathrm{Pf}|\Psi|^{i,j}_{i,j}}{\sigma_{i,j}} \qquad A_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases},$$

$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \qquad C_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ -\sum_{c \neq a} C_{a,c} & a = b \end{cases}$$

- The Pfaffian is gauge invariant on the support of scattering eqs: the variation under $\epsilon_a^\mu \sim \epsilon_a^\mu + \alpha k_a^\mu$ vanishes since the matrix becomes degenerate (for each solution!)

Graviton scattering from CHY

- Einstein gravity? no color but two copies of polarizations! graviton: $h^{\mu\nu} = \epsilon^{\mu}\epsilon^{\nu}$

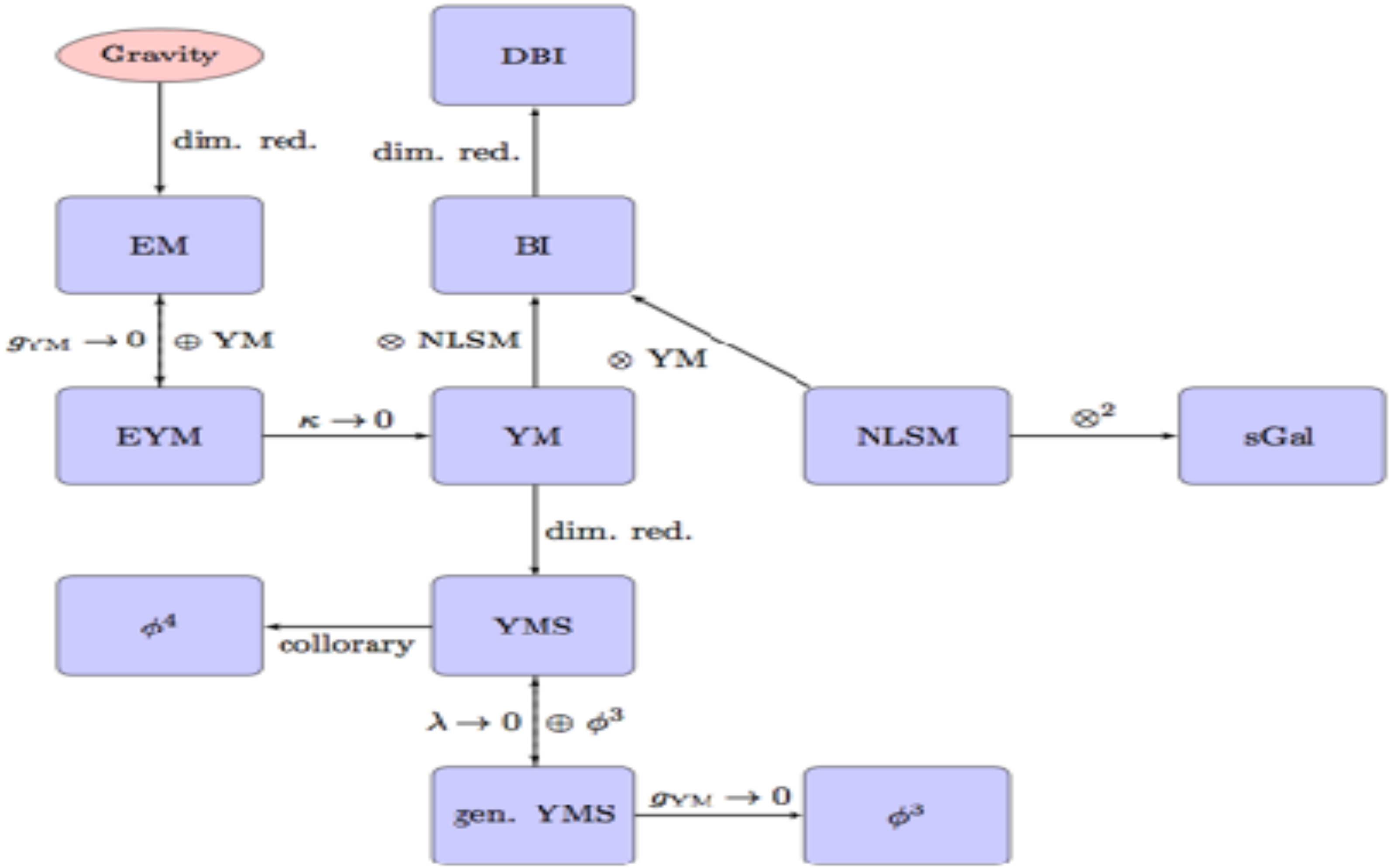
- Natural to have two copies of Pfaffians, or Pfaffian squared=determinant

$$M_n^{h+B+\phi} = \int d\mu_n \, \mathrm{Pf}'\Psi(\epsilon)\, \mathrm{Pf}'\Psi(\epsilon') \longrightarrow M_n^{\mathrm{GR}} = \int d\mu_n \, \det{}'\, \Psi(\epsilon)$$

"closed string"

- A formula for n gravitons in any dim (hidden simplicity of linearized GR)
- Diff invariance is manifest for exactly the same argument
- Double copy "$GR \sim YM \otimes YM$" or more precisely $GR = YM^2/\phi^3$

A landscape of theories



Amplitudes as differential forms

Key: scattering amplitudes as differential forms in kinematic space

For 4d gauge theories (e.g. N=4 SYM), differential form of spinors/twistors packages all helicity amps as a single object (e.g. "bosonize" superamplitude). Answer to what Q?

"volume" (canonical form) of "amplituhedron" (positive geometry of spinors/twistors)

geometry in kinematic space (avatar of G+(k,n)) -> "volume" form-> all-loop amps in N=4 SYM

This talk: identical structures for a wide range of theories in general dim

Bi-adjoint scalar amps="volume" of associahedron in kinematic space (Mandelstams)

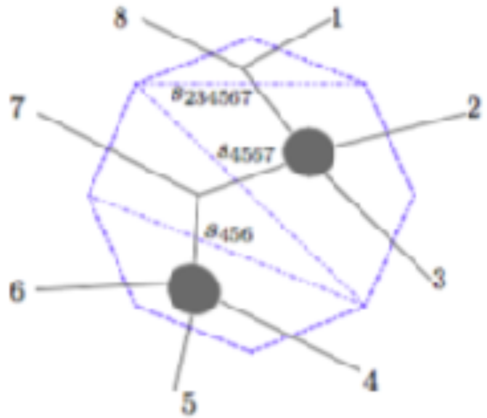
Avatar of worldsheet geometry: scattering eqs as diffeomorphism

"Geometrize" color & its duality to kinematics, forms for YM/NLSM amps etc.

Kinematic space

The kinematic space, $\mathcal{K}_n$, for n massless momenta p_i $(D \geq n-1)$ is spanned by Mandelstam variables $s_{i\,j}$'s subject to $\sum_{j \neq i} s_{i\,j} = 0$, thus $\dim \mathcal{K}_n = \binom{n}{2} - n = \frac{n(n-3)}{2}$; for any subset I, $s_I = \sum_{i<j \in I} s_{i\,j}$

Planar variables $s_{i,i+1,\cdots,j}$ for an ordering $(12\cdots n)$ are dual to $n(n-3)/2$ diagonals of a n-gon with edges $p_1, p_2, \cdots, p_n$



A planar cubic tree graph consists of $n-3$ *compatible* planar variables as poles, and it is dual to a full triangulation of the n-gon

Planar scattering form

The *planar scattering form* for ordering $(12\cdots n)$

$$\Omega_n^{(n-3)} := \sum_{\text{planar } g} \text{sign}(g) \bigwedge_{a=1}^{n-3} d\log s_{i_a, i_a+1, \cdots, j_a} \qquad e.g.\ \Omega_4^{(1)} = \frac{ds}{s} - \frac{dt}{t} = d\log\frac{s}{t}$$

Projectivity: invariant under *local* GL(1) transf. $s_{i,\cdots,j} \to \Lambda(s) s_{i,\cdots,j}$

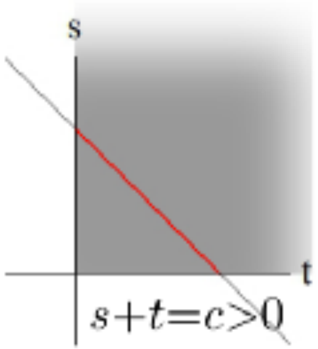
- By pullback to certain (n-3)-dim subspace, the form becomes scalar amps!
- It will be the "volume", or "canonical form" of an associahedron polytope
- Encoding universal factorization structures of any massless tree amps

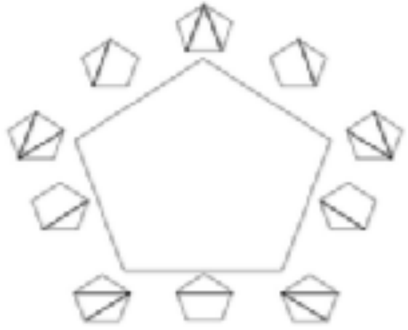
Kinematic associahedron

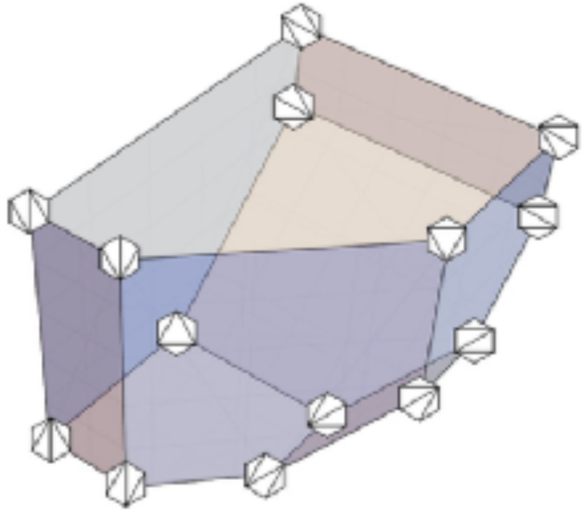
Positive region Δ_n: all planar variables $s_{i,i+1,\cdots,j} \geq 0$ (top-dimension)

Subspace H_n: $-s_{i\,j} = c_{i,j}$ as *positive constants*, for all non-adjacent pairs $1 \leq i, j < n$; we have $\frac{(n-2)(n-3)}{2}$ conditions $\implies$ dim $H_n = n-3$

Kinematic Associahedron is their intersection! $\mathcal{A}_n := \Delta_n \cap H_n$

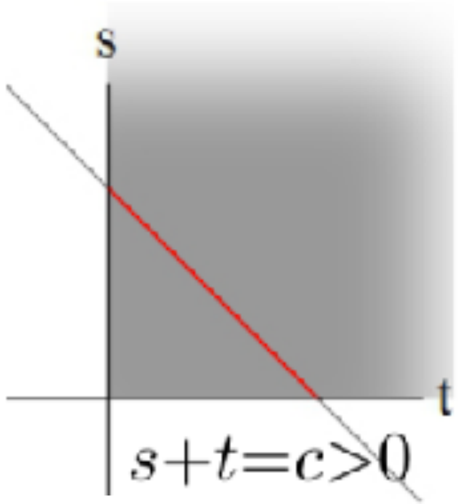


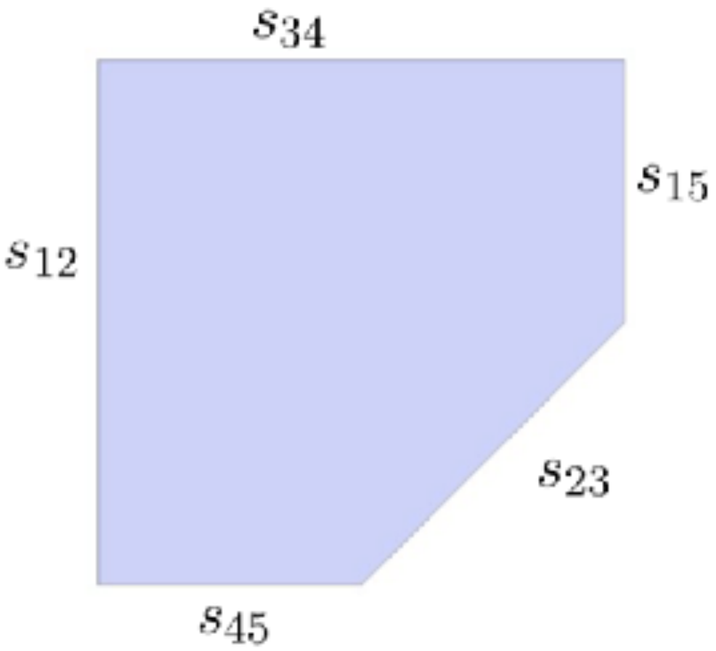


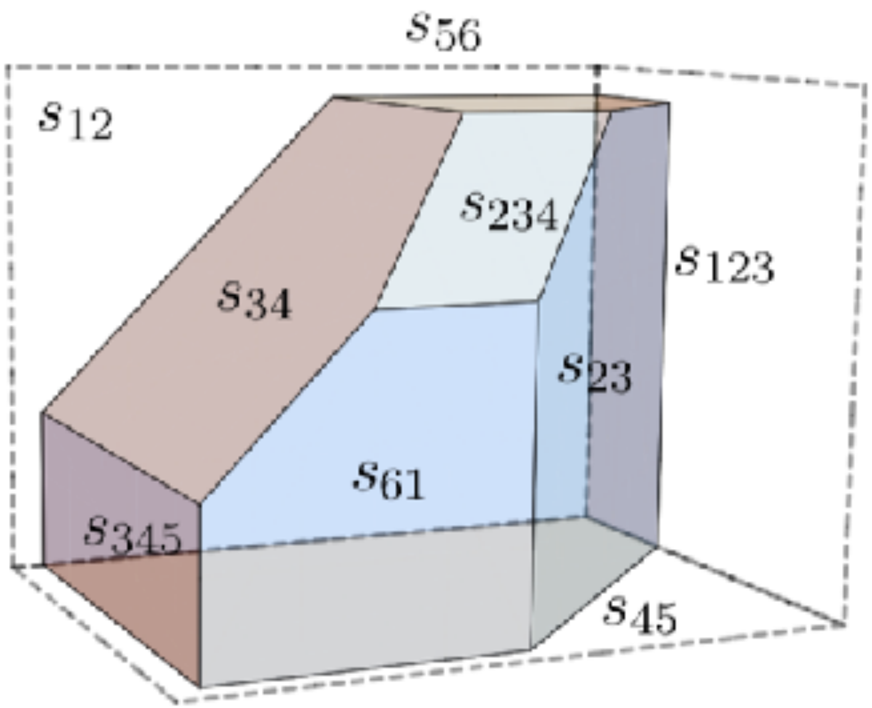


$$e.g.\ \mathcal{A}_4 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$$

$$\mathcal{A}_5 = \{s_{12}, \cdots, s_{51} > 0\} \cap \{s_{13}, s_{14}, s_{24} = \text{const} < 0\}$$







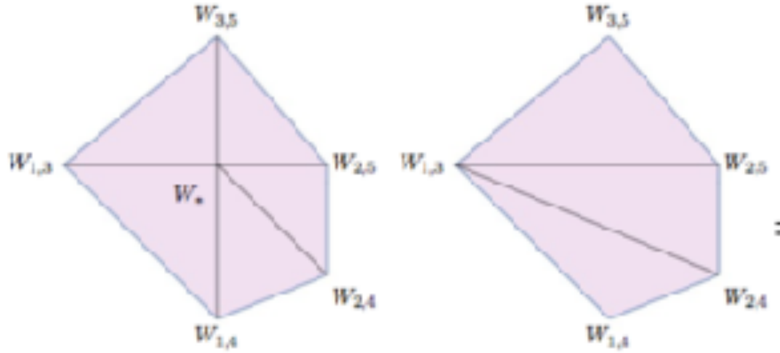
Canonical forms & amplitudes

Canonical form of $\mathcal{A}_n$ = Pullback of Ω_n to $H_n \propto$ planar ϕ^3 amplitude!

$$e.g. \qquad \Omega(\mathcal{A}_4) = \Omega_4^{(1)}|_{H_4} = \left(\frac{ds}{s} - \frac{dt}{t}\right)\Big|_{-u=c>0} = \left(\frac{1}{s} + \frac{1}{t}\right) ds$$

$$\Omega(\mathcal{A}_5) = \Omega_5^{(2)}|_{H_5} = \left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right) ds_{12} \wedge ds_{34}$$

- Associahedron is the (tree) “amplituhedron” for bi-adjoint scalar theory!
- Its canonical form, or “volume”=pullback of planar form=scalar amps
- Feynman-diagram expansion=a special triangulation, now many more new reps



$$\Omega(\mathcal{A}_5) = d^2\mathbf{s}\,\left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right)$$

$$= d^2\mathbf{s}\,\left(\frac{s_{12}+s_{51}}{s_{12}s_{34}s_{51}} + \frac{s_{12}+s_{51}}{s_{12}s_{51}s_{23}} + \frac{s_{12}-s_{45}+s_{23}}{s_{12}s_{23}s_{45}}\right)$$

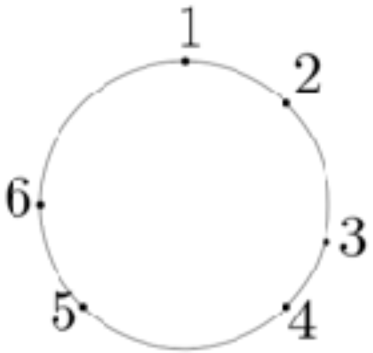
$$= \text{sum of 3 triangles of } \mathcal{A}_5 \text{ itself}$$

Worldsheet associahedron

A well-known associahedron: minimal blow-up of the open-string worldsheet $\mathcal{M}^+_{0,n} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_n\}/\mathrm{SL}(2,\mathbb{R})$ [Deligne, Mumford]

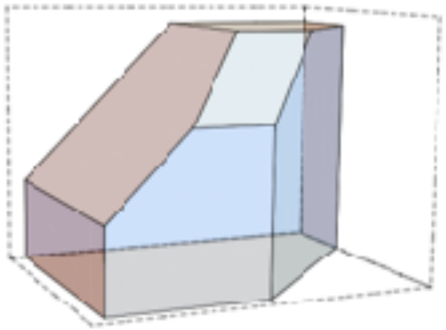
The *canonical form* of $\overline{\mathcal{M}}^+_{0,n}$ is the "Parke-Taylor" form

$$\omega_n^{\mathrm{WS}} := \frac{1}{\mathrm{vol}\,[\mathrm{SL}(2)]} \prod_{a=1}^{n} \frac{d\sigma_a}{\sigma_a - \sigma_{a+1}} := \mathrm{PT}(1,2,\cdots,n)\, d\mu_n$$



scattering equations

as a map from $\overline{\mathcal{M}}_{0,n}$ to $\mathcal{A}_n$



a geometric origin of scattering eqs & CHY

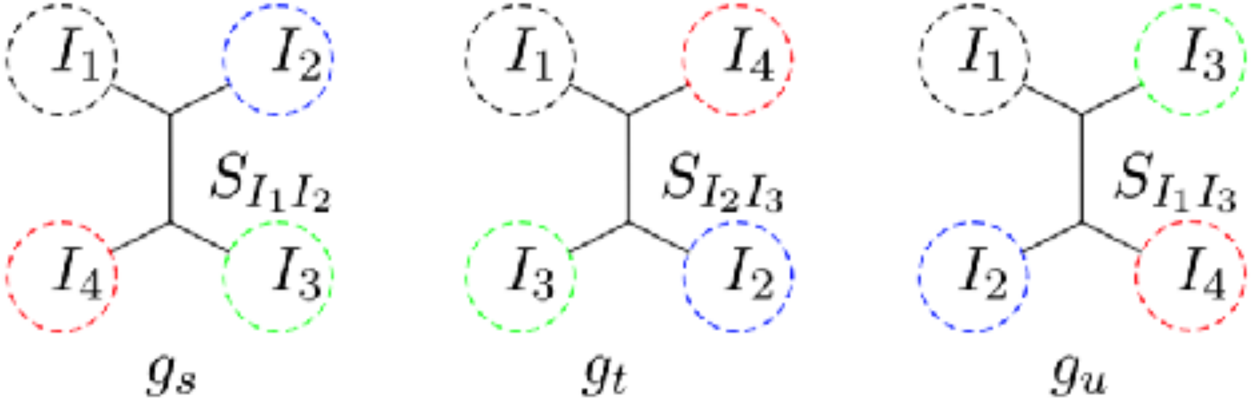
General scattering forms

Scattering forms generalize planar ones to all $(2n-5)!!$ cubic graphs:

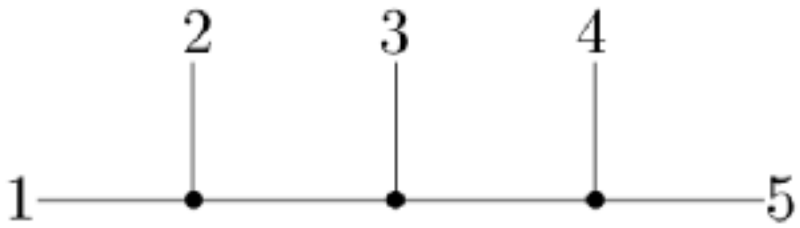
$$\Omega[N] = \sum_g N(g) \prod_I d\log s_I \,, \quad e.g.\ N_s\, d\log s + N_t\, d\log t + N_u\, d\log u$$

Projectivity ($s_I \to \Lambda(s) s_I$ inv.) $\Longleftrightarrow$ Jacobi-satisfying N's [BCJ 08]!

$$N(g_s) + N(g_t) + N(g_u) = 0 \,, \quad e.g.\ N_s + N_t + N_u = 0$$



Color is kinematics



$$C(g) = f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5}$$

$$\leftrightarrow \quad W(g) = \mathrm{d}s_{12} \wedge \mathrm{d}s_{45}$$

Duality between *color factors* & wedge product of ds for cubic graphs

$$C(g_s) + C(g_t) + C(g_u) = 0\,, \quad \leftrightarrow \quad W(g_s) + W(g_t) + W(g_u) = 0$$

Scattering forms are color-dressed amp without color factors!

$$M[N] = \sum_g N(g) C(g) \prod_I \frac{1}{s_I} \quad \leftrightarrow \quad \Omega[N] = \sum_g N(g) W(g) \prod_I \frac{1}{s_I}$$

Scattering forms for gluons & pions

Permutation invariant forms encoding full amps of gluon/pion

Gauge invariance: $\Omega_{\rm YM}$ invariant under every shift $\epsilon_i^\mu \to \epsilon_i^\mu + \alpha p_i^\mu$

Adler zero: $\Omega_{\rm NLSM}$ vanishes under every soft limit $p_i^\mu \to 0$

Key: forms are projective $\Longrightarrow$ unique $\Omega_{\rm YM}$ and $\Omega_{\rm NLSM}$!

$\Omega_{\rm YM/NLSM}$ as pushforward of **canonical, rigid worldsheet objects**:

$$\Omega_{\rm YM}^{(n-3)} = \sum_{\rm sol.} d\mu_n \, {\rm Pf}'\Psi_n(\{\epsilon, p, \sigma\}) \quad \Omega_{\rm NLSM}^{(n-3)} = \sum_{\rm sol.} d\mu_n \, {\rm det}' A_n(\{s, \sigma\})$$

Color is kinematics II

More is true for U(N)/SU(N): partial amps as pullback of forms

$$\textbf{trace decomp. } \mathbf{M}_n[N] = \sum_{\beta \in S_n/Z_n} \mathrm{Tr}(\beta(1),\ldots,\beta(n)) M_n[N;\beta]$$

$$\Longrightarrow \textbf{ partial amp. } \quad M_n[N;\beta] = \sum_{\beta-\text{planar } g} N(g|\beta) \prod_{I \in g} \frac{1}{s_I}$$

Completely parallel: Partial amplitude = pullback of scattering form to subspace $H(\beta) = \{s_{\beta(i)\,\beta(j)} = \text{const.}\}$ for non-adjacent $1 \leq i < j < n$

$$\Omega^{(n-3)}[N]|_{H[\beta]} = \left(\sum_{\beta\text{-pl. } g} N(g|\beta) \prod_{I \in g} \frac{1}{s_I} \right) dV[\beta] = M_n[N;\beta] dV[\beta]$$

Gravity amplitude & double copy

How about theories without color, such as gravity amplitude? A 0-form or equivalently top form, $\Omega^{\rm top} = M_n \times d^{n(n-3)/2}s$

Define dual forms for every scattering form: *e.g.* the dual for ϕ^3

$$\tilde{\Omega}_{\phi^3}(1,2,\cdots,n) := \bigwedge_{1\leq i<j-1<n-1} ds_{i,j}\,,$$

$\Longrightarrow$ pullback to partial amp, *e.g.* $M_n^{\rm YM}(\alpha)d^{n(n-3)/2}s = \Omega_{\rm YM}\wedge\tilde{\Omega}_{\phi^3}(\alpha)$

Natural language for BCJ double-copy : top form for *e.g.* gravity is literally the (wedge) product of a $\Omega_{\rm YM}$ and its dual $\tilde{\Omega}_{\rm YM}$:

$$\Omega^{\rm top}_{\rm GR} = \Omega^{(n-3)}_{\rm YM}\wedge\tilde{\Omega}^{(n-2)(n-3)/2}_{\rm YM} = d^{n(n-3)}s\sum_g\prod_{I\in g}\frac{N(g)\tilde{N}(g)}{s_I}$$

Summary & outlook

New picture: general massless S-matrix via punctured Riemann spheres; higher-genus for loops. A (weak-weak) QFT/String duality for S-matrix?

Applications: new relations between amps of *gluons*, *pions*, *gravitons* …
Double copy beyond amps: *classical solutions*, *gravity waves*, ……

Geometries in kinematic space: scattering amplitudes as differential forms
"volume" of associahedron = bi-adjoint scalar amp; geometric origin of CHY
general scattering forms for gluons, pions, etc. strings from geometry?

Towards a unified geometric picture for amplitudes and beyond:
cosmological polytopes, Witten diagrams, EFThedron & CFThedron…

Thank You!


FINITE
ANSATZ
AMPLITUDES
String
Integrability
BETHE
CHAIN
Spin
CORRELATION
FACTORS
PLANAR
Functions
YANG
BAXTER
MAGNONS
Quantum
LIMIT
Conformal
CLASSICS
FLUX
COUPLING
SPECTRUM
SCALAR
SPINONS
SITTER
PERTURBATION
Anti
TUBE
Loops
SPECTRAL
SCATTERING
HOLOGRAPHY
GRAVITY