Multi-loop unitarity via computational algebraic geometry

ICTS, USTC, Jan. 03, 2014

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Based on

(2-loop 4-point) arXiv:1202.2019, Simon Badger, Hjalte Frellesvig and YZ

(algebraic geometry methods) arXiv:1205.5707, YZ

(3-loop 4-point) arXiv:1207.2976, Simon Badger, Hjalte Frellesvig and YZ

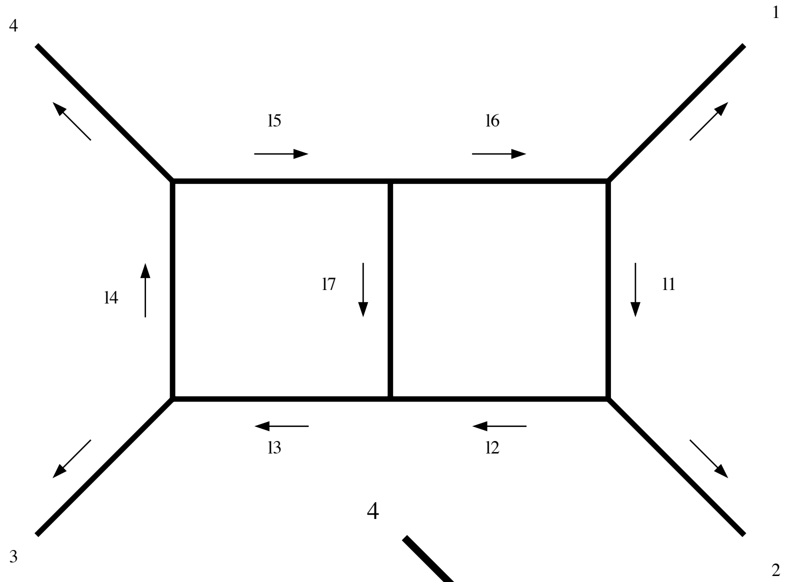
(global structure) arXiv:1302.1023, Rijun Huang and YZ

(2-loop 5-point QCD) arXiv:1310.1051, Simon Badger, Hjalte Frellesvig and YZ

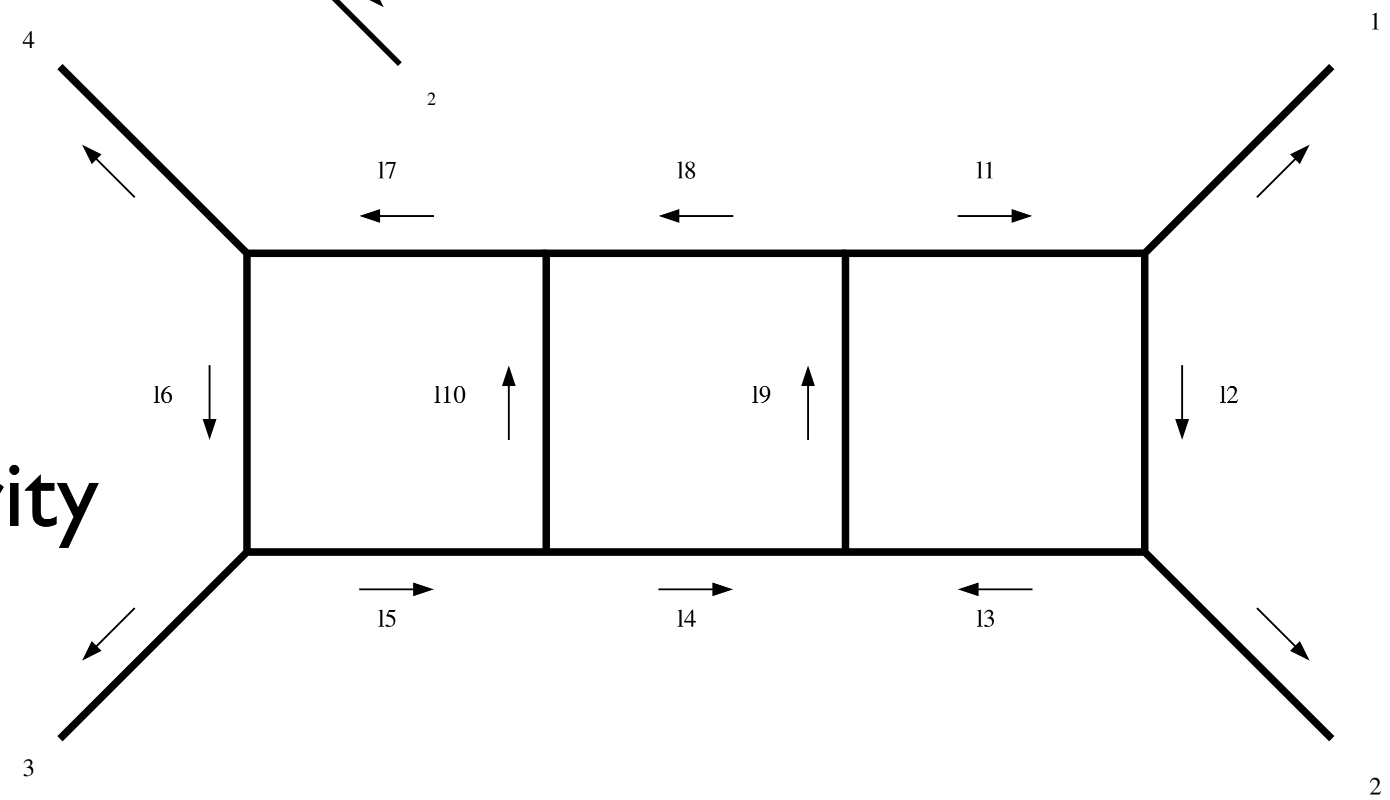
(maximal cut) arXiv:1310.6006, Mads Sogaard and YZ

and works under progress...

Outline

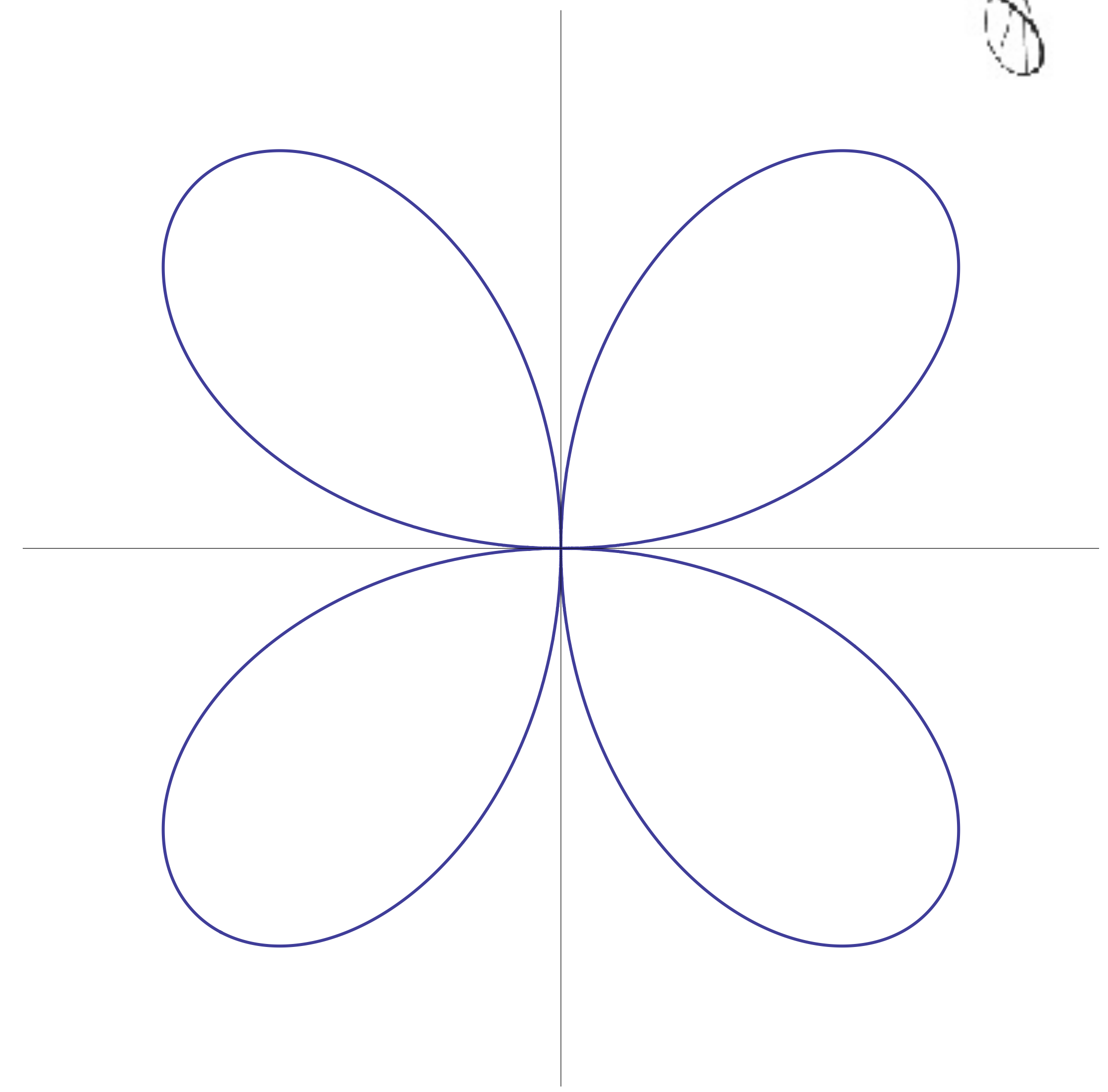


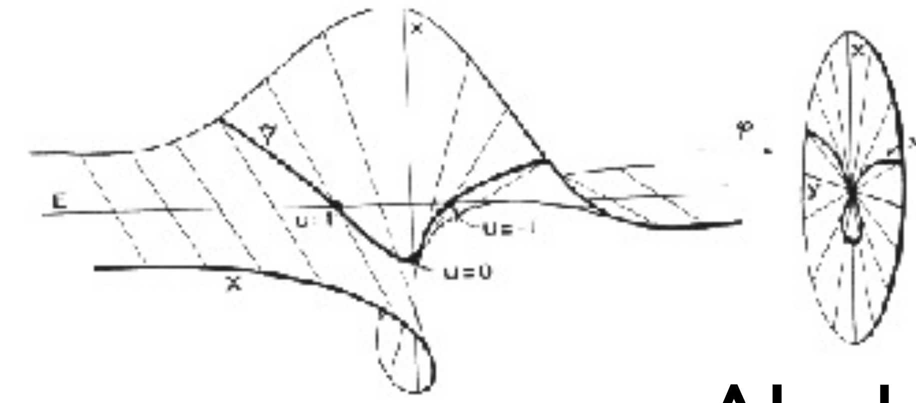
High-loop Amplitudes



Unitarity

⟹





Algebraic geometry

Gröbner Basis
Primary Decomposition
Affine Variety Structure
Multivariate residue

- Integrand reduction at one loop, review
- Integrand reduction at n loop by algebraic geometry
- Examples: 2-loop 5-gluon planar QCD, 3-loop 4-point triple-box ...

Why high loops?

- Phenomenology: NNLO correction for theoretical prediction
- Theory： deep structure in gauge theories and gravity

Computation of two-loop Feynman diagrams is complicated.

Feynman rules,
Integration-by-parts identities

- two-loop massless QCD, 2→2 process
  Anastasiou, Glover, Tejeda-Yeomans and Oleari (2000)
  Bern, Dixon, Kosower (2002) Bern, De Freitas, Dixon (2002)
- two-loop, pp→H + 1 jet
  Gehrmann, Jaquier, Glover and Koukoutsakis (2011)
- NNLO, $e^+ e^- \to 3$ jets
  Gehrmann and Glover (2008)
- NNLO, q qbar→t tbar
  Bernreuther, Czakon, Mitov (2012)
- NNLO, g g→H g
  Boughezal, Caola, Melnikov, Petriello, Schulze (2013)

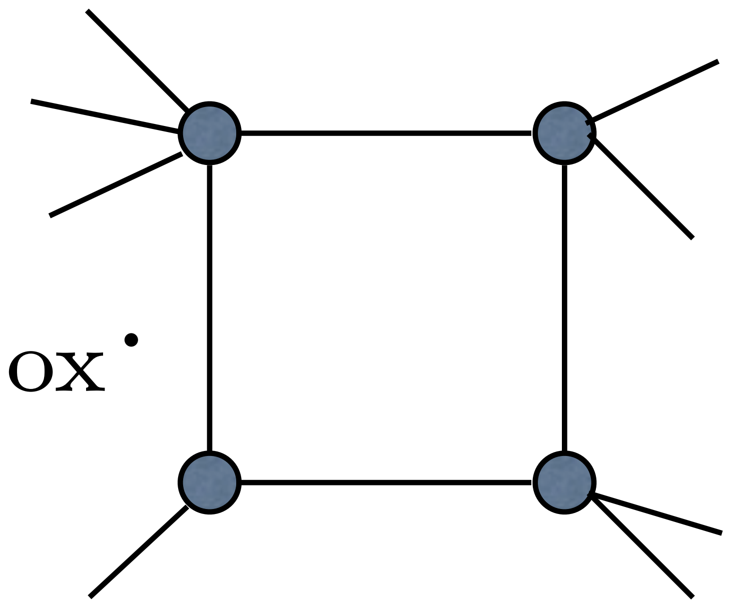
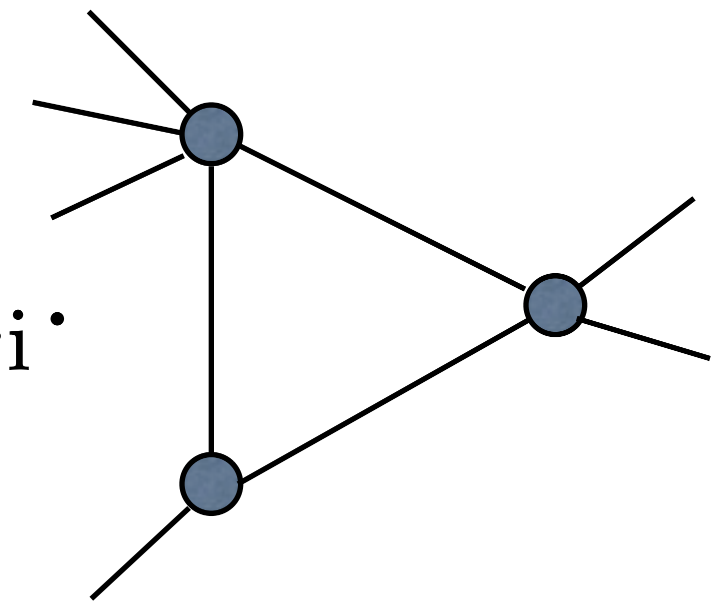
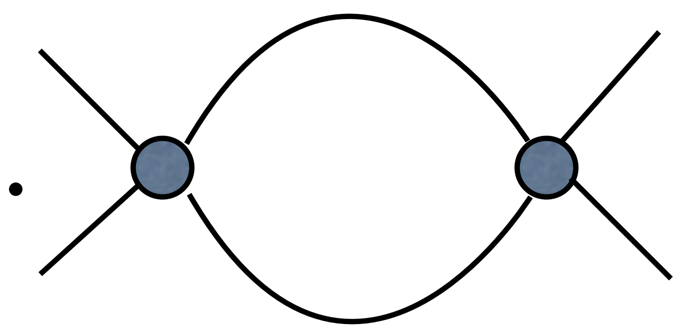
and etc.

Unitarity

multi-loop integrand reduction...

Unitarity at one-loop

$D = 4$

$A^{(1)} = c_{\text{box}} \cdot$  $+ c_{\text{tri}} \cdot$  $+ c_{\text{bub}} \cdot$  $+ \ldots$

- no pentagon, hexagon ...
- scalar integral (numerator is one.)

Unitarity:

Determine 'c' coefficients
from on-shell cut solutions
and tree amplitudes

quadruple cut $\rightarrow c_{\text{box}}$

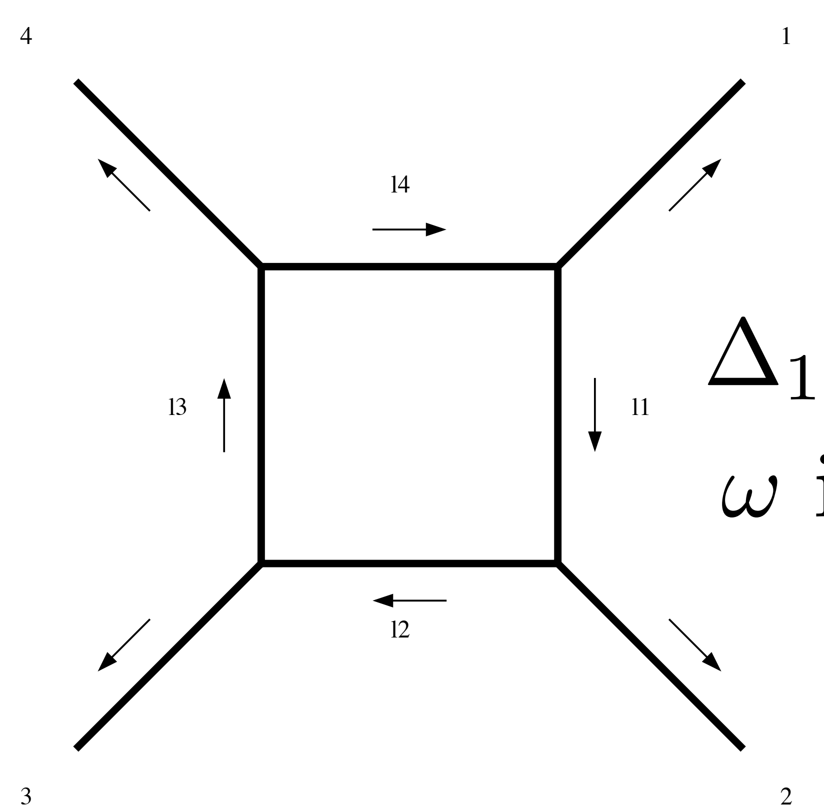
triple cut $\rightarrow c_{\text{tri}}$

double cut $\rightarrow c_{\text{bub}}$

Integrand reduction: box

Integrand-level reduction, Ossola, Papadopoulos and Pittau (OPP), 2006
Giele, Kunszt, Melnikov, 2008

$$A^{(1)} = \int \frac{d^4k}{(2\pi)^4} \frac{N(k)}{D_1 D_2 D_3 D_4}$$



$$N(k) = \Delta_{1234}(k) + \sum_{i_1<i_2<i_3} \Delta_{i_1 i_2 i_3}(k) \prod_{i \neq i_1,i_2,i_3} D_i + \sum_{i_1<i_2} \Delta_{i_1 i_2}(k) \prod_{i\neq i_1,i_2} D_i$$
$$= \Delta_{1234}(k) + O(D_1, D_2, D_3, D_4)$$

$\Delta_{1234}(k)$ is a polynomial in scalar products (SP). $\mathbb{SP} = \{k\cdot P_1, k\cdot P_2, k\cdot P_3, k\cdot\omega\}$

ω is auxiliary, $(\omega\cdot P_i) = 0$, $i = 1,2,3,4$

reducible scalar product (RSP) — irreducible scalar product (ISP)

$$\begin{aligned} 2(k\cdot P_1) &= D_4 - D_1 - P_1^2 \\ 2(k\cdot P_2) &= D_1 - D_2 + P_2^2 \\ 2(k\cdot P_3) &= D_2 - D_3 + 2P_2\cdot P_3 + P_3^2 \end{aligned}$$

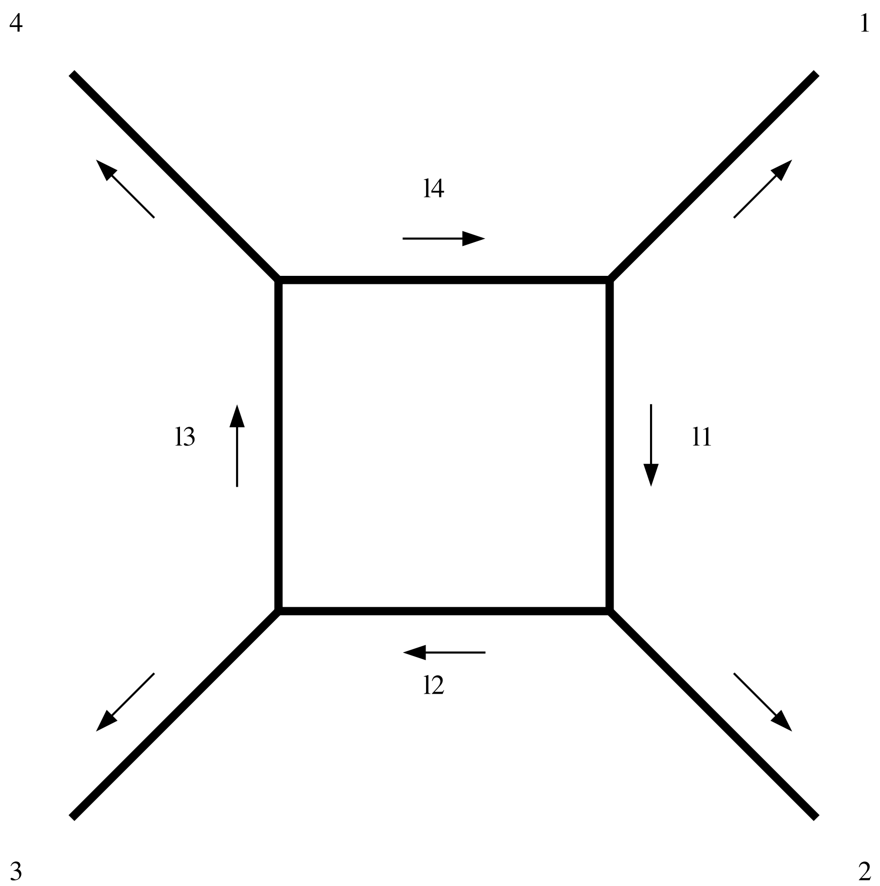
$$\Delta_{1234}(k) = \sum_i c_i (k\cdot\omega)^i$$

$\Delta_{1234}(k)$ is a polynomial in ISP only.

Integrand basis for box

$$\Delta_{1234}(k) = \sum_i c_i (k\cdot\omega)^i$$

How many terms are there?



Renormalizability $\quad i = 0, 1, 2, 3, 4$

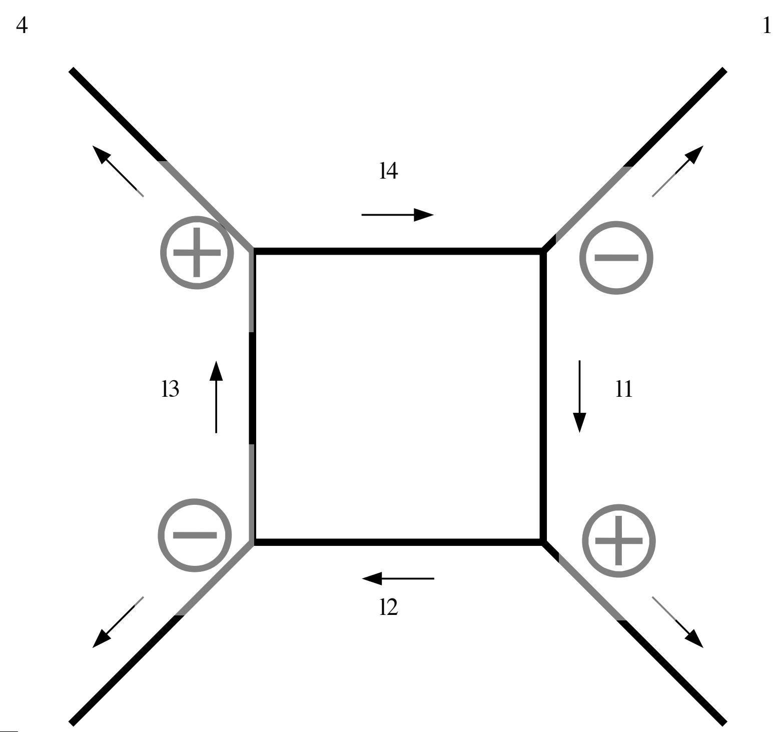
Cut-equations for ISP $\quad k^2 = D_1$

$$(k\cdot\omega)^2 = t^2/4 + O(D_1, D_2, D_3, D_4) \qquad i = 0, 1$$

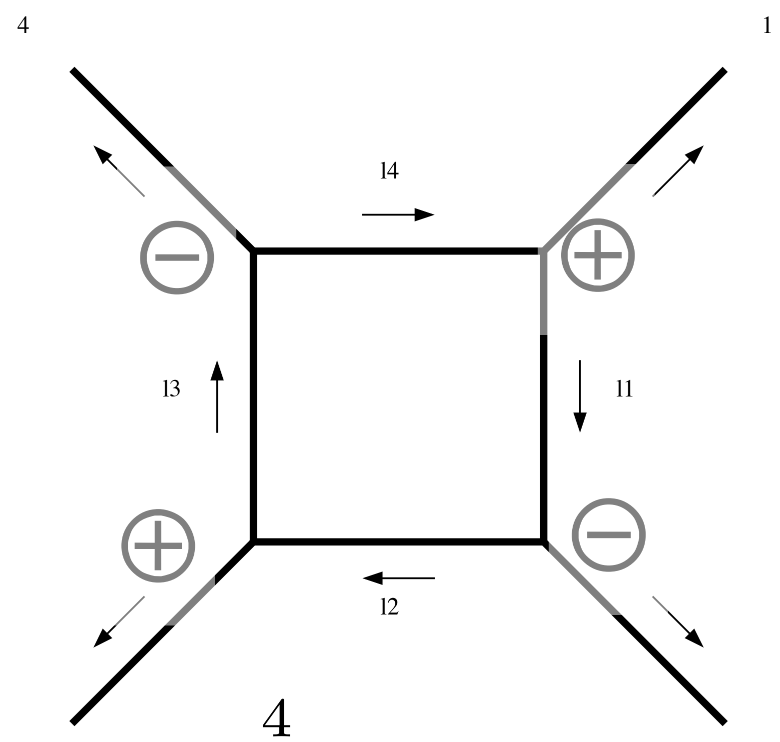
Reducible

integrand basis $\quad \boxed{\Delta_{1234}(k) = c_0 + c_1(k\cdot\omega)} \qquad \boxed{c_{\text{box}} = c_0}$

(Generalized-) Unitarity Cuts $\quad D_1 = D_2 = D_3 = D_4 = 0$



$$\prod_{i=1}^{4} A^i_{\text{tree}}(k^{(1)}) = N^{(1)}$$



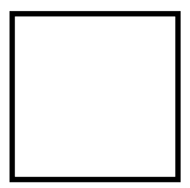
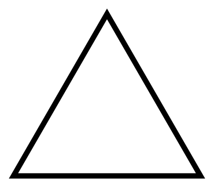
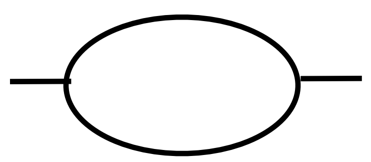
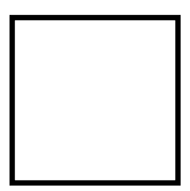
$$\prod_{i=1}^{4} A^i_{\text{tree}}(k^{(2)}) = N^{(2)}$$

Spurious term: $\displaystyle \int \frac{d^4k}{(2\pi)^4}\frac{k\cdot\omega}{D_1D_2D_3D_4} = 0$

But c_1 is crucial for fewer-propagator integrands.

$$N(k) - c_0 - c_1(k\cdot\omega) = \sum_{i_1<i_2<i_3} \Delta_{i_1i_2i_3}(k) \prod_{i\neq i_1,i_2,i_3} D_i + \dots$$

One loop, other diagrams

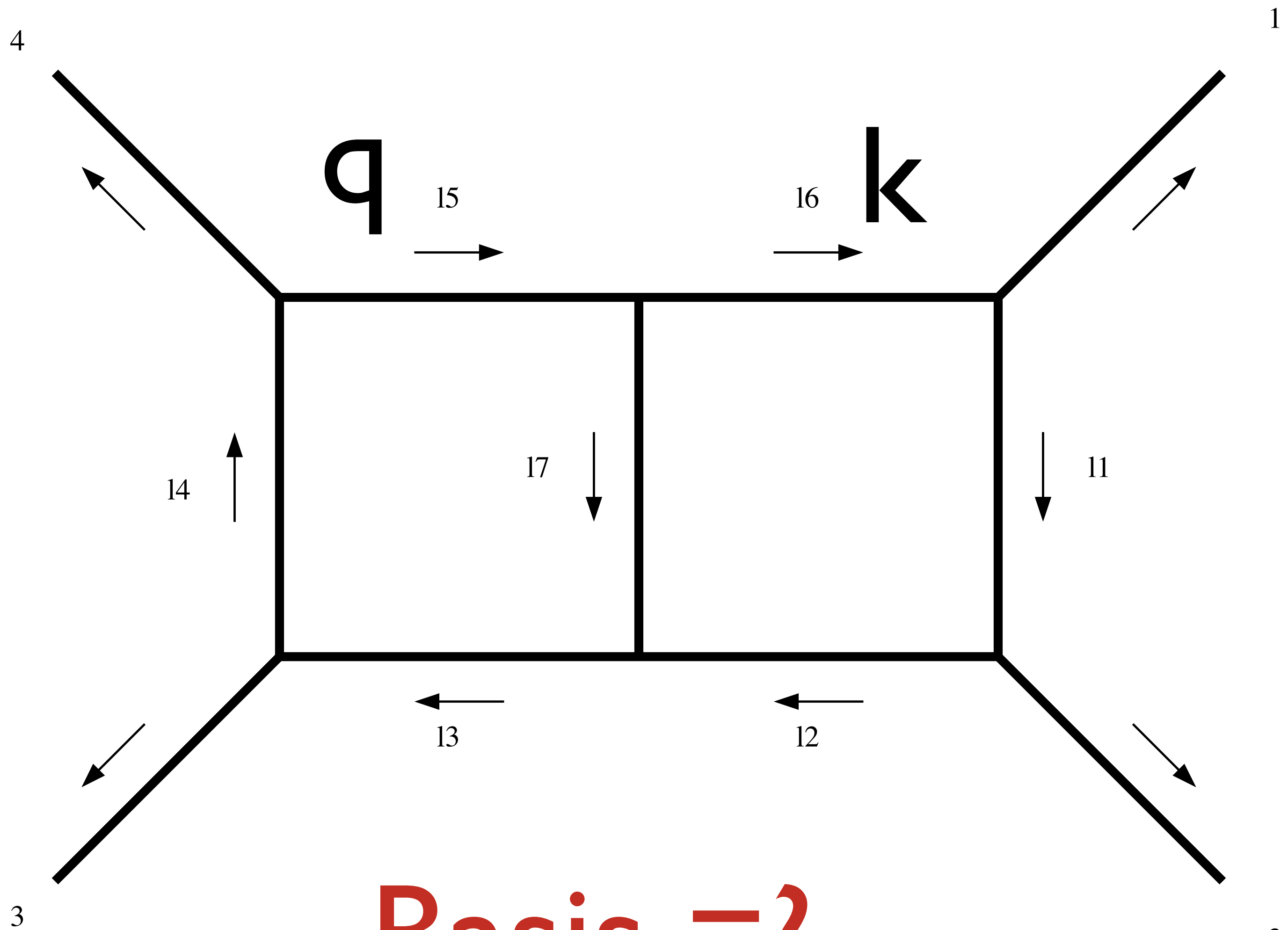
Dimension	Diagram	# SP (ISP+RSP)	#terms in integrand basis (non-spurious + spurious)	# Solutions (dimension)
4		4 (1+3)	2 (1+1)	2 (0)
4		4 (2+2)	7 (1+6)	1 (1)
4		4 (3+1)	9 (1+8)	1(2)
4-2ε		5 (2+3)	5 (3+2)	1(1)

- straightforward to obtain integrand basis, unitarity cut solutions
- all one-loop master integrals are known
- c coefficients can be automatically computed by public codes
  - 'NGluon', Badger, Biedermann, and Uwer
  - 'CutTools', Ossola, Papadopoulos, and Pittau
  - 'GoSam', Cullen, Greiner, Heinrich, Luisoni, and Mastrolia
  - ...

Generalization to higher loops?

Example: 4D massless two-loop hepta cut

P. Mastrolia, G. Ossola, 2011
S. Badger, H. Frellesvig, YZ, 2012



Basis =?

7 cut-equations in 8 SP's

$$\mathbb{SP} = \{k \cdot P_1, k \cdot P_2, k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot P_2, q \cdot P_4, q \cdot \omega\}$$

4 cut-equations to identify 4 RSP's

4 ISP's $\mathbb{ISP} = \{k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot \omega\}$

3 cut-equations for ISP's

$$(k \cdot \omega)^2 = (k \cdot P_4 - t/2)^2 \quad (1)$$

$$(q \cdot \omega)^2 = (q \cdot P_1 - t/2)^2 \quad (2)$$

$$(k \cdot \omega)(q \cdot \omega) = -\frac{t^2}{4} + \frac{t(k \cdot P_4)}{2} + \frac{t(q \cdot P_1)}{2} + \left(1 + \frac{2t}{s}\right)(k \cdot P_4)(q \cdot P_1) \quad (3)$$

Naive guessing: all renormalizable monomials which do NOT contain $(k \cdot \omega)^2$, $(q \cdot \omega)^2$ or $(k \cdot \omega)(q \cdot \omega)$.

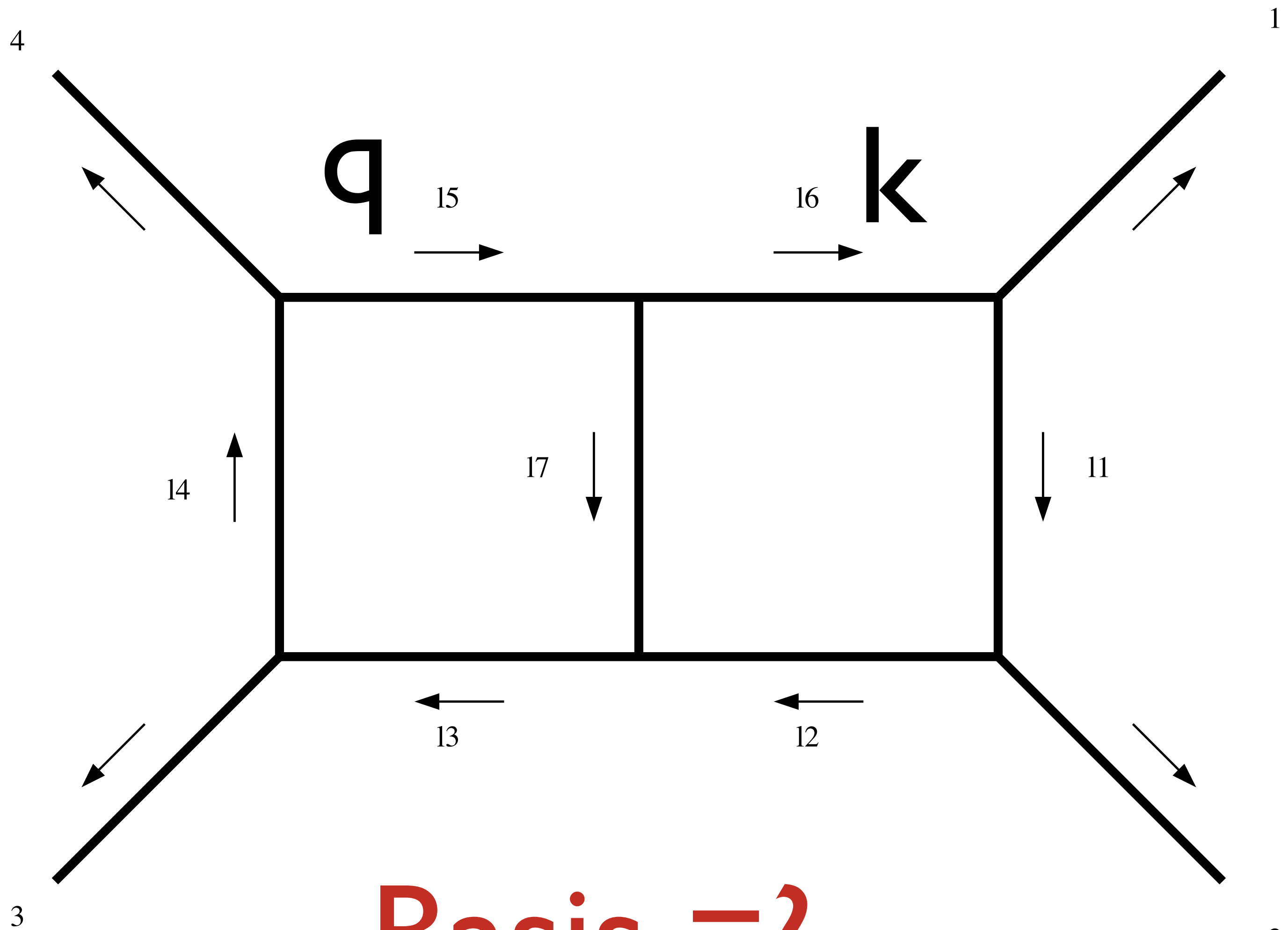
$$\Delta_{\text{dbox}} = (k \cdot P_4)^m (q \cdot P_1)^n (k \cdot \omega)^\alpha (q \cdot \omega)^\beta$$

$$m + \alpha \le 4, n + \beta \le 4, m + n + \alpha + \beta \le 6$$

$$(\alpha, \beta) = (0,0), (1,0), (0,1)$$

Example: 4D massless two-loop hepta cut

P. Mastrolia, G. Ossola, 2011
S. Badger, H. Frellesvig, YZ, 2012



7 cut-equations in 8 SP's

$$\mathbb{SP} = \{k \cdot P_1, k \cdot P_2, k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot P_2, q \cdot P_4, q \cdot \omega\}$$

4 cut-equations to identify 4 RSP's

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$$\Delta_{\mathrm{dbox}} = (k \cdot P_4)^m (q \cdot P_1)^n (k \cdot \omega)^\alpha (q \cdot \omega)^\beta$$
$$m + \alpha \leq 4, n + \beta \leq 4, m + n + \alpha + \beta \leq 6$$
$$(\alpha, \beta) = (0,0), (1,0), (0,1)$$

56 terms? wrong...

Example: 4D massless two-loop hepta cut

S. Badger, H. Frellesvig, YZ, 2012

3 cut-equations for ISP's, and their combinations

$$\begin{aligned}(k\cdot\omega)^2 &= (k\cdot P_4 - t/2)^2 \quad (1)\\ (q\cdot\omega)^2 &= (q\cdot P_1 - t/2)^2 \quad (2)\\ (k\cdot\omega)(q\cdot\omega) &= -\frac{t^2}{4}+\frac{t(k\cdot P_4)}{2}+\frac{t(q\cdot P_1)}{2}+\left(1+\frac{2t}{s}\right)(k\cdot P_4)(q\cdot P_1) \quad (3)\end{aligned}$$

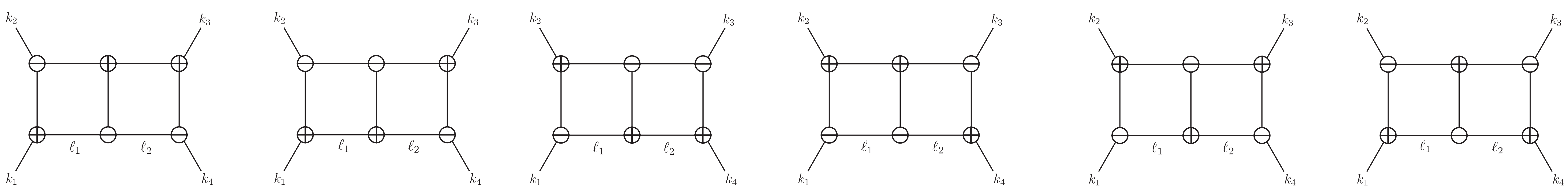
$$\boxed{(1)\times(2)-(3)^2}$$

reduced

$$4(k\cdot P_4)^2(q\cdot P_1)^2 = -2s(k\cdot P_4)^2(q\cdot P_1) - 2s(k\cdot P_4)(q\cdot P_1)^2 - st(k\cdot P_4)(q\cdot P_1)$$

We have to "exhaust" all combinations...

Finally, we determine that the basis contains 32 terms



6 families of hepta-cut solutions, Laurant series contains 38 terms

Solving 38 linear equations for 32 coefficients, done!

Messy, not automatic!

Gröbner basis and integrand basis

arXiv:1205.5707, YZ
arXiv:1205.7087, Mastrolia, Mirabella, Ossola and Peraro

$$I = \langle D_1, \ldots D_k \rangle = \{\sum_{i=1}^{k} g_i D_i | \quad \forall g_i \in R\}$$

$$\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{N}{D_1 D_2 \ldots D_7}, \quad N = Q + \Delta_{\text{dbox}}, \quad Q \in I$$

Synthetic polynomial division

N divided by $\{D_1, \ldots D_k\}$:
Define a monomial order, and recursively preform $N/D_1, \ldots, N/D_k$. Finally, the division process will stop and we have

→ Euclidean division

$$N = f_1 D_1 + \ldots f_k D_k + r'$$

where r' is the remainder. $\Delta_{\text{dbox}} = r'$???

In most cases, it does not work since it stops too early,
unless we are using Gröbner basis.

$$I = \langle D_1, \ldots, D_k \rangle = \langle g_1, \ldots, g_m \rangle$$

→ Gröbner basis
'good' generators

$$N = q_1 g_1 + \ldots q_k g_k + r$$

- r is uniquely determined.
- If $N \in I$, $r = 0$.

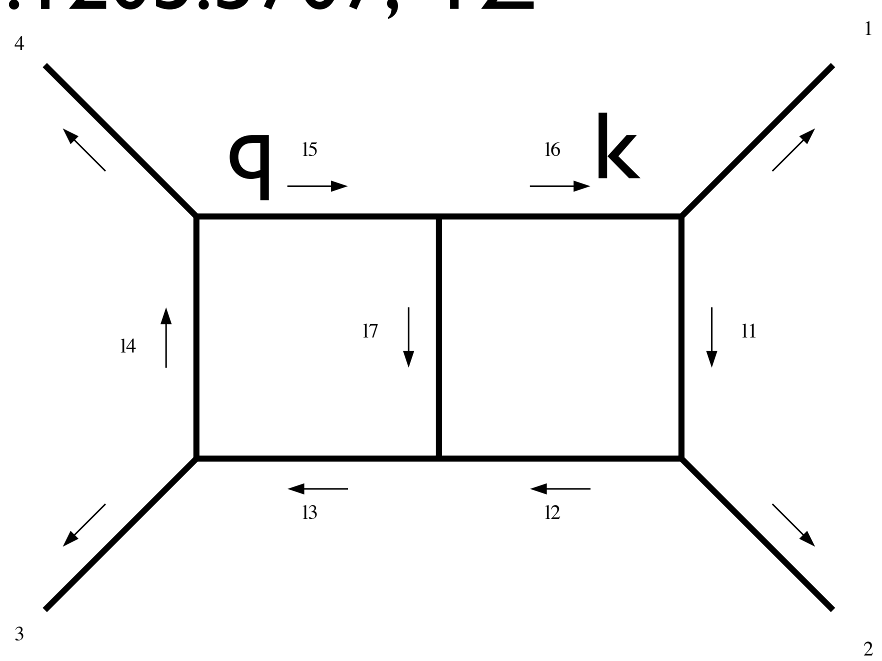
$$\boxed{\Delta_{\text{dbox}} = r}$$

$$\left(y^3 \quad x - 2y^2\right) = \left(x^3 - 2xy \quad x^2y - 2y^2 + x\right) \begin{pmatrix} -\frac{1}{4} - \frac{1}{4}xy - \frac{1}{2}y^3 & y^2 \\ \frac{1}{4}x^2 - \frac{1}{2}y + \frac{1}{2}xy^2 & 1 - xy \end{pmatrix}$$

Toy Model: $N = xy^3$, $I = \langle x^3 - 2xy, x^2y - 2y^2 + x \rangle$. Direct synthetic division of N towards $\{x^3 - 2xy, x^2y - 2y^2 + x\}$ gives $r' = xy^3$.
But the Gröbner basis is $I = \langle y^3, x - 2y^2 \rangle$, and the synthetic division of N on Gröbner basis gives $r = 0$. So $N \in I$.

Grobner basis: dbox example

arXiv:1205.5707, YZ



4 ISP's $\mathbb{ISP} = \{k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot \omega\}$

$$N = q_1 g_1 + \ldots q_k g_k + \Delta_{\text{dbox}}$$

N contains 160 terms where Δ_{dbox} contains 32 terms.

In principle, it works for arbitrary number of loops, any dimension.
Automated by the package: 'BasisDet'

http://www.nbi.dk/~zhang/BasisDet.html, YZ 2012

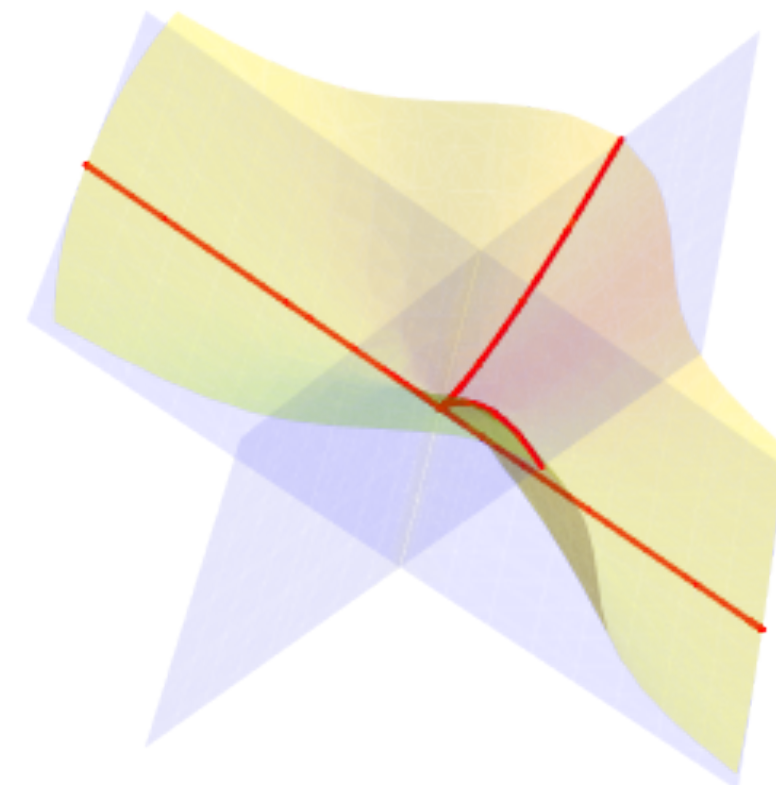
Dimension propagators, kinematics → Integrand basis

Can also find ISP automatically!

Primary decomposition

arXiv:1205.5707, YZ

Find the number of branches of unitarity solutions



$I = \langle x^2 - y^2, x^3 + y^3 - z^2 \rangle$. How many (irreducible) curves are there in $\mathcal{Z}(I)$.
Primary decomposition:

$$I = I_1 \bigcap I_2 \quad I_1 = \langle x + y, z^2 \rangle, \quad I_2 = \langle x - y, 2y^3 - z^2 \rangle$$

- AG software 'Macaulay 2'
- Numeric algebraic geometry methods

$$I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6$$

4D massless dbox hepta-cut: 6 branches of solutions

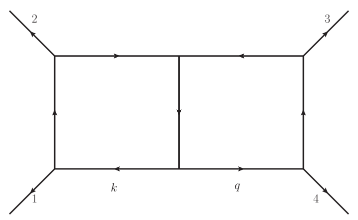
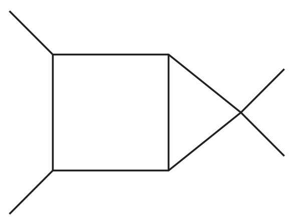
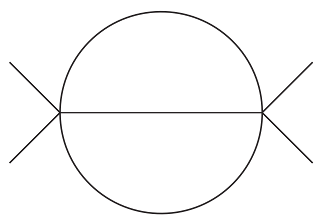
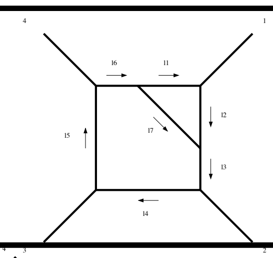
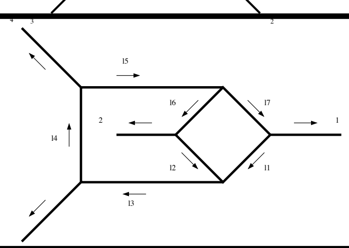
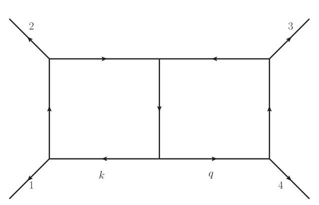
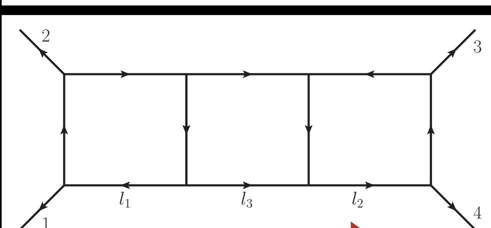
dictionary

Algebra	Geometry
height I	$\dim \mathcal{Z}(I) = n - \text{height } I$ (# free parameters)
arithmetic genus	(geometric) genus (topology)

High genus examples: arXiv:1302.1203, Rijun and YZ

works for arbitrary number of loops, any dimension

More examples

Dimension	Diagram	# SP (ISP+RSP)	#terms in integrand basis (non-spurious + spurious)	# Solutions (dimension)
4		8 (4+4)	32 (16+16)	6 (1)
4		8 (5+3)	69 (18+51)	4 (2)
4		4 (3+1)	42 (12+30)	1(5)
4		8 (3+5)	20 (10+10)	2 (2)
4		8 (4+4)	38 (19+19)	8 (1)
4-2ε		11 (7+4)	160 (84+76)	1(4)
4		12 (7+5)	398 (199+199)	14 (2)

Nontrivial dimension

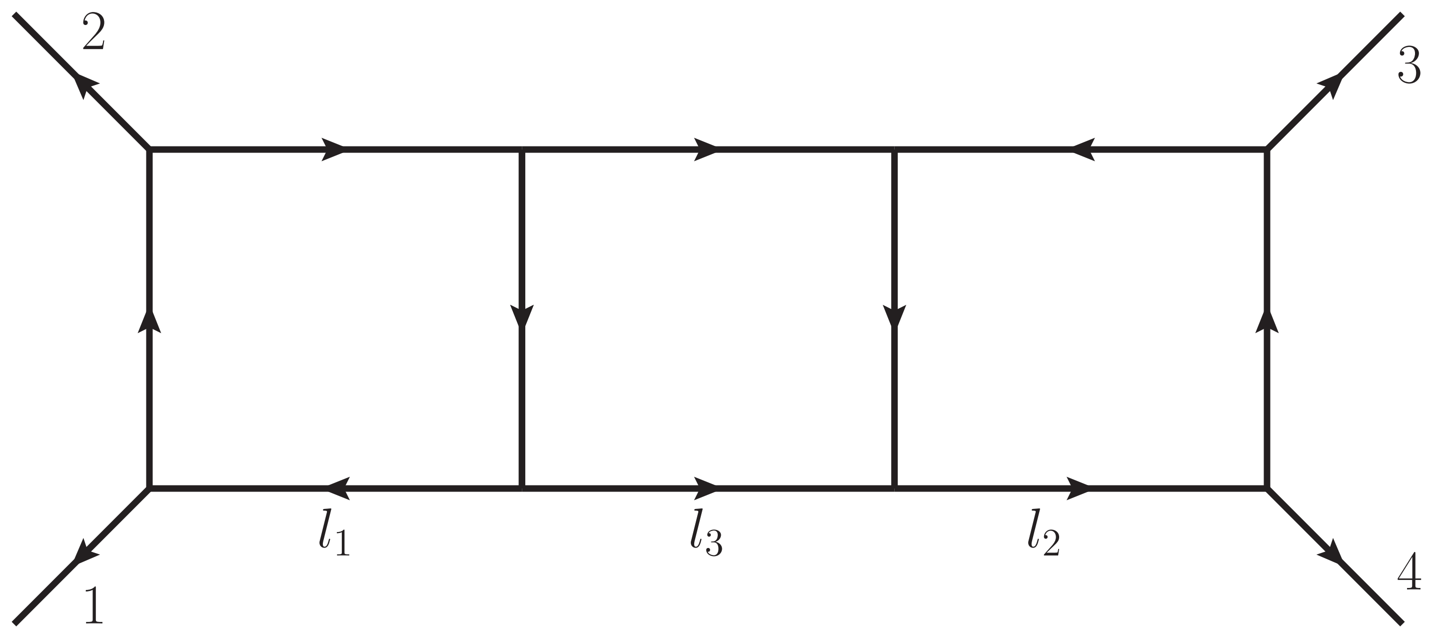
Non-planar

Three-loop!

Even more examples:
arXiv:1209.3747 Bo Feng and Rijun Huang

Triple box results

arXiv:1207.2976, Simon Badger, Hjalte Frellesvig and YZ



fit 398 \`c\` coefficients from products of 8 trees,

from 14 branches of cut-solutions

$\mathcal{N}$	n_f	n_s
0	0	0
1	1	0
2	2	1
4	4	3

Yang-Mills with n_f adjoint fermions and n_s adjoint scalars

Integration-by-parts (IBP) identities

398 terms → 3 master integrals

$$C_1 I_{\text{tribox}}[1] + C_2 I_{\text{tribox}}[l_1 \cdot p_4] + C_3 I_{\text{tribox}}[l_3 \cdot p_4]$$

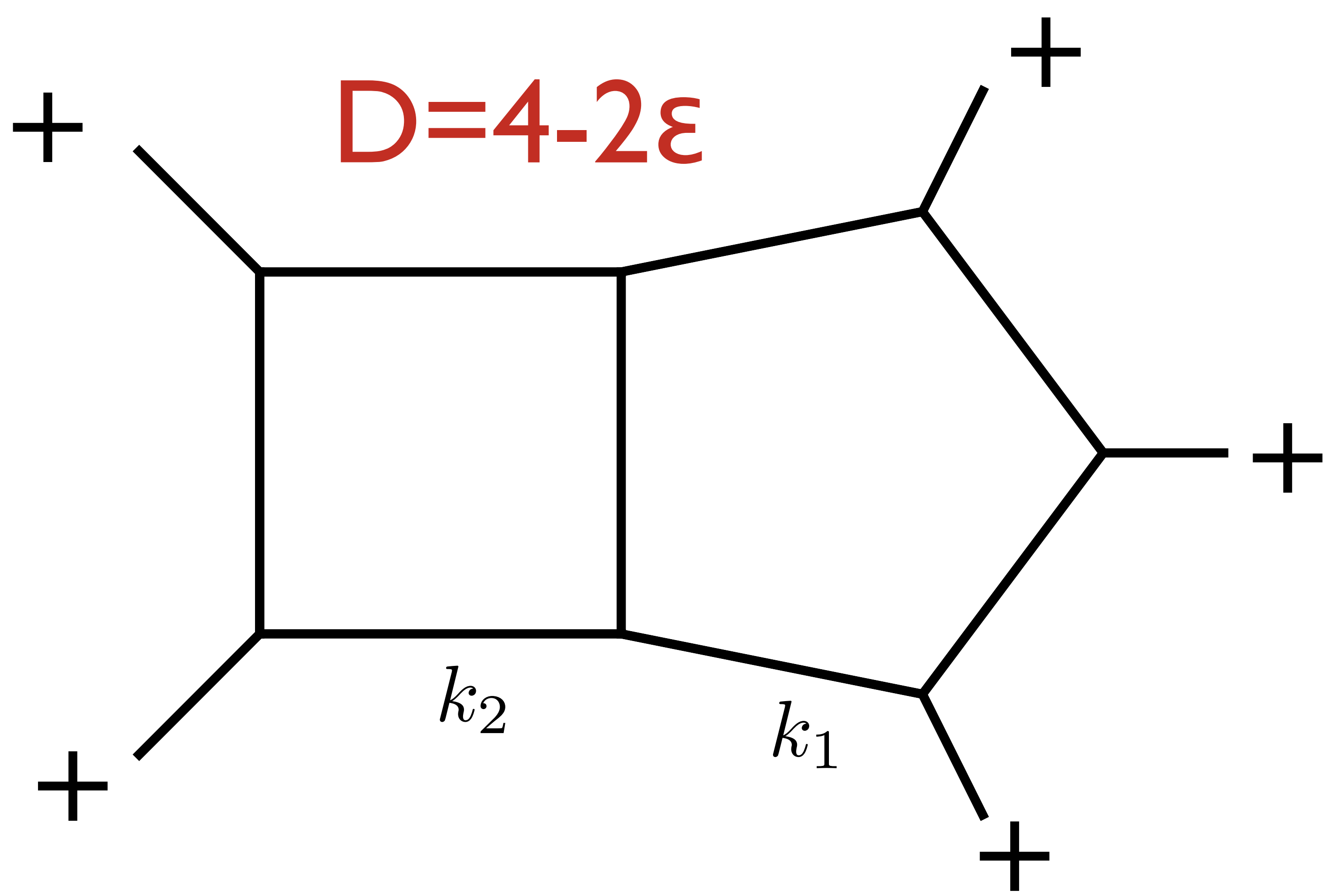
$$\begin{aligned}
C_1^{-+-+}(s,t) =& \\
& -1 + (4-n_f)\frac{st}{u^2} - 2(1+n_s-n_f)\frac{s^2t^2}{u^4} \\
& + \big(2(1-2n_s)+n_f\big)(4-n_f)\frac{s^2t(2t-s)}{4u^4} \\
& - \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{st(t^2-4st+s^2)}{8u^4} \\
C_2^{-+-+}(s,t) =& \\
& -(4-n_f)\frac{s}{u^2} + 2(1+n_s-n_f)\frac{s^2t}{u^4} \\
& - \big(2(1-2n_s)+n_f\big)(4-n_f)\frac{s^2(2t-s)}{u^4} \\
& + \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{s(t^2-4st+s^2)}{2u^4} \\
C_3^{-+-+}(s,t) =& \\
& + \big(2(1-2n_s)+n_f\big)(4-n_f)\frac{3s^2(2t-s)}{2u^4} \\
& - \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{3s(t^2-4st+s^2)}{4u^4}
\end{aligned}$$

New analytic results for non-supersymmetric gauge theory

D-dim integrand reduction

2-loop 5-point QCD

arXiv: 1310.1051: Simon Badger, Hjalte Frellesvig and YZ



$$\mu_{11} = k^2_{[-2\epsilon],1},\ \mu_{22} = k^2_{[-2\epsilon],2} \text{ and } \mu_{12} = 2(k_{[-2\epsilon],1} \cdot k_{[-2\epsilon],2})$$

$$\mu_{33} = \mu_{11} + \mu_{22} + \mu_{12}$$

$$\Delta_{431}(1^+, 2^+, 3^+, 4^+, 5^+) = \frac{i\, s_{12} s_{23} s_{45} F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}(tr_+(1345)(k_1+p_5)^2 + s_{15}s_{34}s_{45})$$

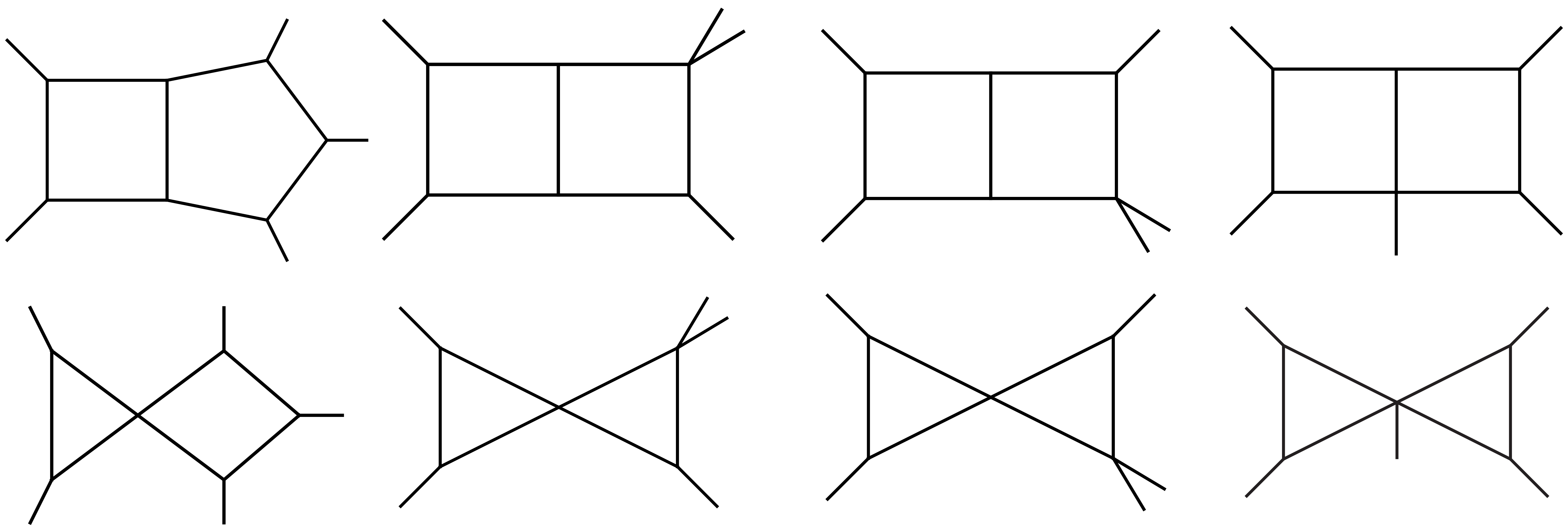
$$F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) = (D_s - 2)(\mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33}) + 4(\mu_{12}^2 - 4\mu_{11}\mu_{22})$$

- Feynman rules + cut solution
- 6D spinor helicity formalism

Momentum-twistor parametrization

$(\lambda, \tilde{\lambda}) \longrightarrow (\lambda, \mu)$ (Andrew Hodges)

2-loop 5-gluon amplitude



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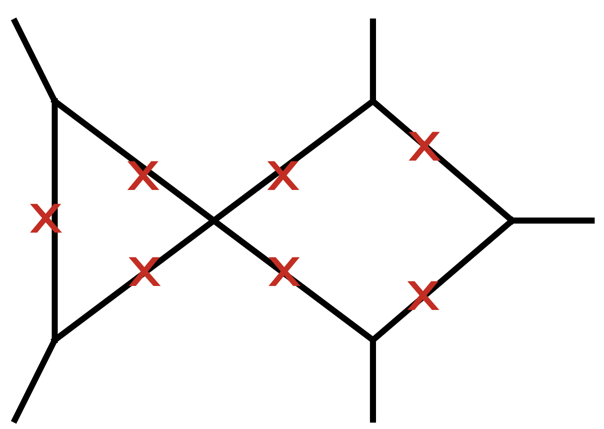
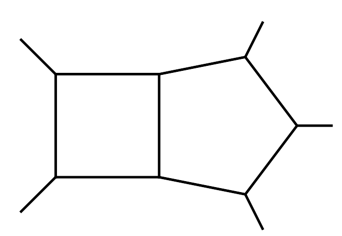
first result on 2-loop 5-gluon helicity amplitude in QCD

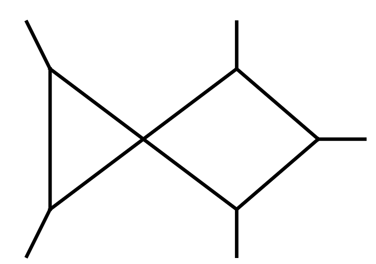
subtraction



$$- \frac{1}{(k_1 + k_2)^2} \Delta_{431}$$



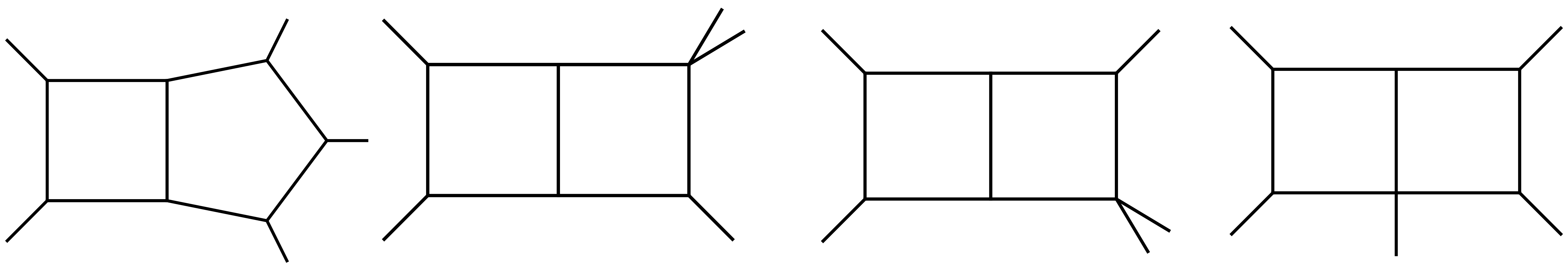
Integrand reduction →



Δ_{430}

all coefficients are analytically found
IR structure: consistent with Catani's factorization

2-loop 5-gluon amplitude



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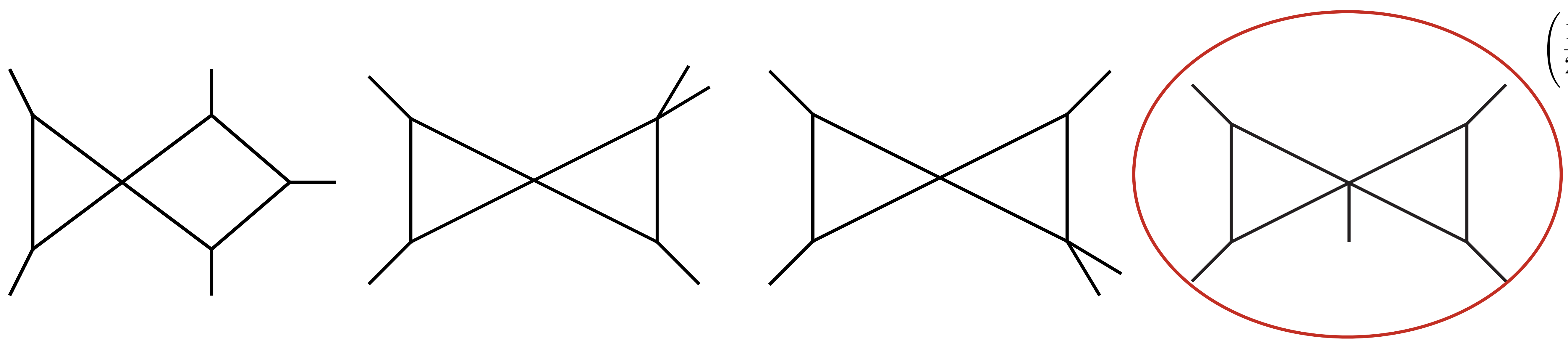
$$= F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) \times (\text{helicity factor}) \times (\mathcal{N} = 4 \text{ Integrand})$$



No corresponding $\mathcal{N} = 4$ diagrams

$$\begin{aligned}
\Delta_{330;5L}(1^+,2^+,3^+,4^+,5^+) \;=\; -\frac{i}{\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle}\times \\
\Bigg(\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)\Bigg(2(D_s-2)(\mu_{11}+\mu_{22})\mu_{12} \\
+ (D_s-2)^2\mu_{11}\mu_{22}\frac{4(k_1\cdot p_3)(k_2\cdot p_3) + (k_1+k_2)^2(s_{12}+s_{45}) + s_{12}s_{45}}{s_{12}s_{45}}\Bigg) \\
+ (D_s-2)^2\mu_{11}\mu_{22}\Bigg[(k_1+k_2)^2 s_{15} \\
+ \mathrm{tr}_+(1235)\left(\frac{(k_1+k_2)^2}{2s_{35}} - \frac{k_1\cdot p_3}{s_{12}}\left(1 + \frac{2(k_2\cdot\omega_{453})}{s_{35}} + \frac{s_{12}-s_{45}}{s_{35}s_{45}}(k_2-p_5)^2\right)\right) \qquad (1) \\
+ \mathrm{tr}_+(1345)\left(\frac{(k_1+k_2)^2}{2s_{13}} - \frac{k_2\cdot p_3}{s_{45}}\left(1 + \frac{2(k_1\cdot\omega_{123})}{s_{13}} + \frac{s_{45}-s_{12}}{s_{12}s_{13}}(k_1-p_1)^2\right)\right)\Bigg]\Bigg).
\end{aligned}$$

similar for non-planar diagrams, under progress

Simon Badger, Donal O'Connell, Hjalte Frellesvig and YZ

Momentum-twistor parametrization

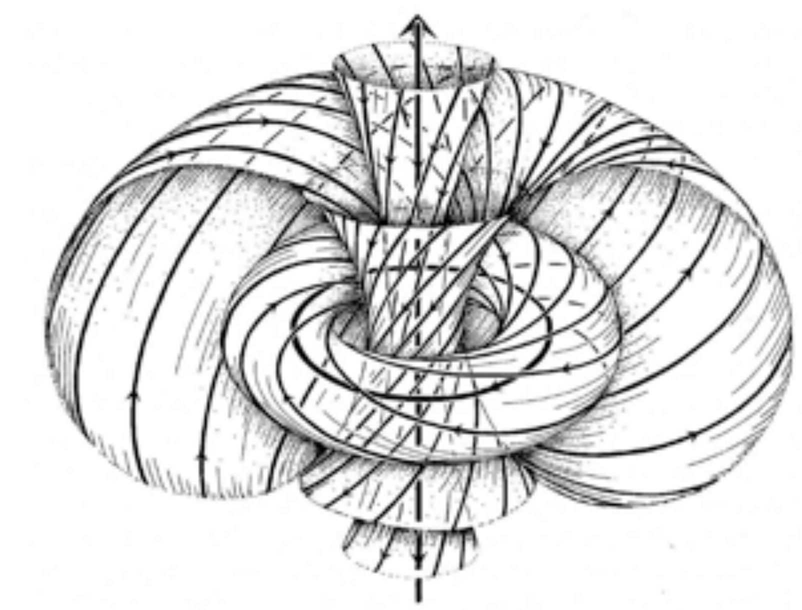
Analytic computation

Andrew Hodges

Spinor helicity formalism $(\lambda, \tilde{\lambda})$ $\longrightarrow$ Momentum-twistor parametrization (λ, μ)

- momentum conservation
- Schouten identity
- Fierz identity
- ...

all constraints resolved



$$\tilde{\lambda}_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle \langle i-1,\, i\rangle}$$

5-point

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_2} & \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_4 & 1 \\ 0 & 0 & 1 & 1 & \frac{x_5}{x_4} \end{pmatrix}$$

In the final result, it is easy to convert $\{x_1, x_2, x_3, x_4, x_5\}$ to s_{ij}, tr_5...

n-point, under progress

Conclusion

- Algebraic geometry approach to high-loop amplitudes
  - Gröbner Basis → Integrand basis
  - Primary decomposition → Global unitarity cut structure
  - Multivariate residues → maximal unitarity
- First steps towards automating high-loop amplitudes
- Promising for NNLO 2 →3, 4 processes