# Static Equilibrium and Chaotic Motion Bound

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# Introduction

Newtonian Gravity $F = -\frac{m_1 m_2}{r^2}$

Coulomb's electrostatic force $F = \frac{e_1 e_2}{r^2}$.

Thus charged massive particles can achieve static equilibrium, which is independent of the separation.

In particular, for the ensemble of static $e/m = 1$, there is no force between any of these particles.

The situation becomes much more complicated in Einstein gravity, since the theory becomes highly nonlinear.

## Charged black holes in Einstein-Maxwell theory

Charged particles become black holes in Einstein-Maxwell theory provided that

$$M \geq Q \,,$$

for appropriate charge convention. The black hole has temperature until the inequality is saturated for which it becomes extremal.

No force condition restores for the extremal black holes and two extremal black holes can be positioned arbitrarily.

For non-extremal black holes, the situation becomes almost unsolvable.

The problem can be simplified if one of the black hole is so small that it can be treated as a test particle, in which case, we only need to consider geodesic motion.

## Test particles and chaos

It turns out that test particles can exhibit chaotic motion.

Furthermore, an unstable equilibrium typically signals chaos.

Charged test particles can have stable equilibrium for some appropriate charge/mass ratio.

In the vicinity of the horizon, the unstable equilibrium has the universal radial perturbation

$$\epsilon \sim e^{\pm\kappa t}\,.$$

When the motion in other coordinates is open , the full nonlinear system implies that the system become chaotic with an upper bound Lyapunov exponent

$$\lambda \leq \kappa\,.$$

It was conjectured as a universal bound.
[Hashimoto, Tanahashi,1610.06070]

This bound was first proposed by Maldacena et al for many-body quantum thermal system. [Maldacena, Shenker, Stanford, 1503.01409]

## Loopholes

- A bound derived from a many-body system may not be applicable in a single (test) particle situation.
- There is no decoupling limit of near-horizon geometry for non-extremal black holes.

A more detailed analysis beyond the near-horizon geometry is needed.

## The set up and general discussions

Consider a charged black hole

$$ds_D^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + \rho(r)^2 d\Omega_{D-2,k}^2\,, A = \psi(r)dt\,, \cdots\,,$$

The horizon is located at $r = r_+$ such that $h(r_+) = f(r_+) = 0$. The surface gravity is then simply

$$\kappa = \frac{1}{2}\sqrt{h'f'}\Big|_{r=r_+}\,,$$

A charged particle is governed by

$$S = -m\int d\tau\left(\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} + \frac{e}{m}A_\mu\frac{dx^\mu}{d\tau}\right)\,,$$

## Radial motion and equilibrium

Restricted (consistently) to Radial motion

$$S = m\int dt L\,, \qquad L = -\sqrt{h(r) - \frac{\dot r^2}{f(r)}} - \frac{e}{m}\psi(r)\,,$$

Assuming there is an equilibrium at $r = r_0$. For small perturbation $|\dot r| \ll 1$, we have

$$L = \frac{\dot r^2}{2\sqrt{h(r)}\,f(r)} - V_{\rm eff}(r)\,, \qquad V_{\rm eff} = \sqrt{h(r)} + \frac{e}{m}\psi(r)\,.$$

Thus for the equilibrium position located at $r_0 \ge r_+$, the charge/mass ratio must satisfy

$$\frac{e}{m} = -\frac{(\sqrt{h})'}{\psi'}\Big|_{r=r_0}\,.$$

(for non-extremal black hole, there is no massive particle can have equilibrium on the horizon.)

## Radial motion and linear perturbation

Let $r = r_0 + \epsilon(t)$, we have

$$L = \frac{1}{2\sqrt{h(r_0)f(r_0)}}\left(\dot{\epsilon}^2 + \lambda^2\epsilon^2\right) + \mathcal{O}(\epsilon^3)\,,$$

where

$$\lambda^2 = \sqrt{h}f\left(\frac{\psi''}{\psi'}(\sqrt{h})' - (\sqrt{h})''\right)\Big|_{r=r_0}\,.$$

The characteristics of the equilibrium is specified by the sign choice of $\lambda^2$. Specifically, we have

$$\begin{cases} \lambda^2 > 0: & \text{unstable, with} \qquad \epsilon \sim e^{\lambda t}\,; \\ \lambda^2 = 0: & \text{marginal;} \\ \lambda^2 < 0: & \text{stable, with} \qquad \epsilon \sim \cos(\sqrt{-\lambda^2}\,t)\,. \end{cases}$$

Now is there an universal property of $\lambda$?

## The near-horizon property

Near-horizon Taylor expansion

$$\begin{aligned} f(r) &= f_1(r - r_+) + \cdots , \\ h(r) &= h_1(r - r_+) + \cdots , \\ \psi(r) &= \psi_0 + \psi_1(r - r_+) + \cdots , \end{aligned}$$

Then

$$r_0 = r_+ + \frac{m^2 h_1}{4e^2\psi_1^2} .$$

Perturbing around this $r = r_0$, one finds an universal expression

$$\lambda = \kappa .$$

But is this maximum?

## The near-horizon property: one more order

$$\begin{aligned} f(r) &= f_1(r-r_+) + f_2(r-r_+)^2 + \cdots\,, \\ h(r) &= h_1(r-r_+) + h_2(r-r_+)^2 + \cdots\,, \\ \psi(r) &= \psi_0 + \psi_1(r-r_+) + \psi_2(r-r_+)^2 + \cdots\,. \end{aligned}$$

Equilibrium

$$\frac{e}{m} = -\frac{\sqrt{h_1}}{2\psi_1\sqrt{r-r_+}} + \frac{-3h_2\psi_1 + 4h_1\psi_2}{4\sqrt{h_1}\psi_1^2}\sqrt{r-r_+} + \cdots\,.$$

The Lyapunov exponent is now modified to become

$$\begin{aligned} \lambda^2 &= \kappa^2 + \gamma(r_0 - r_+) + \mathcal{O}((r-r_+)^2)\,, \\ \gamma &= \frac{1}{4}(f_2h_1 - f_1h_2) + 4\kappa^2\frac{\psi_2}{\psi_1}\,. \end{aligned}$$

The sign of $\gamma$ becomes crucial; no energy-condition can dictate the sign.

## A special case

Schwarzschild-like $h = f$ and $\rho = r$, then we have

$$\psi = \frac{q}{r^{D-3}}\,,$$

where $q$ is the charge parameter. In this case, we have

$$\gamma = -\frac{2(D-2)\kappa^2}{r_+} < 0\,.$$

There is a local bound near the horizon for $\lambda$.

It does not say much about the equilibrium away from the horizon.

There does not seem to have a general smart way approach. We are going to consider this case by case.

# RN black holes

We begin with Einstein-Maxwell gravity, coupled to a bare cosmological constant $\Lambda_0$:

$$\mathcal{L} = \sqrt{-g}(R - \frac{1}{4}F^2 - 2\Lambda_0)\,,$$

where $F = dA$ is the field strength. For simplicity of our presentation, we shall focus our discussion on $D = 4$ dimensions.

First consider asymptotically-flat solution with $\Lambda_0 = 0$.

$$h = f = 1 - \frac{2M}{r} + \frac{q^2}{r^2}\,, \qquad \psi = \frac{2q}{r}\,, \qquad \rho = r\,,$$

$$Q_e = \frac{1}{16\pi}\int {*F} = \frac{1}{2}q\,.$$

Mass and charge can be expressed in terms of $r_\pm$:

$$M = \frac{1}{2}(r_+ + r_-)\,, \qquad q = \sqrt{r_+ r_-}\,.$$

The surface gravity on the outer horizon is

$$\kappa = \frac{r_+ - r_-}{2r_+^2}\,.$$

The requirement that $\kappa \geq 0$ implies that mass and charge must satisfy the inequality

$$M \geq q = 2Q_e\,,$$

which is saturated in the extremal limit where the $r_\pm$ coalesce.

The equilibrium $r_0$ is determined by

$$\frac{e}{m} = \frac{(r_+ + r_-)r_0 - 2r_+r_-}{4\sqrt{r_+r_-(r_0 - r_+)(r_0 - r_-)}}\,.$$

As was discussed, for $r_+ > r_-$, the equilibrium $r_0$ cannot be located on the horizon for any massive particle. For $r_0$ lying in the region $(r_+, \infty)$, we must have

$$\frac{e}{m} \geq \frac{r_+ + r_-}{4\sqrt{r_+r_-}} = \frac{M}{2q} = \frac{M}{4Q_e} \geq \frac{1}{2}\,.$$

Thus equilibrium outside of the horizon is only possible for electron-like charged particles that violate the black hole condition. The Lyapunov exponent can be obtained:

$$\lambda = \frac{r_+ - r_-}{2r_0^2} \leq \frac{r_+ - r_-}{2r_+^2} = \kappa\,.$$

## Asymptotically AdS

AdS radius $\ell$:

$$\kappa = \frac{\left(r_+ - r_-\right)\left(k\ell^2 + r_-^2 + 3r_+^2 + 2r_-r_+\right)}{2\ell^2 r_+^2}.$$

The equilibrium $r_0$ is related to the mass/charge ratio by

$$\frac{e}{m} = \frac{r_0^2 f'(r_0)}{4q\sqrt{f(r_0)}}.$$

The corresponding Lyapunov exponent is

$$\lambda^2 = \frac{-2r_0^6 + 3\left(r_-^2 + r_+^2\right)r_0^4 - 6r_-^2 r_+^2 r_0^2 + r_-^2 r_+^2\left(r_-^2 + r_+^2\right)}{\ell^4 r_0^4}$$
$$+ \frac{2M\left(-3r_0^4 + 4\left(r_- + r_+\right)r_0^3 - 6r_-r_+r_0^2 + r_-r_+\left(r_-^2 - r_+r_- + r_+^2\right)\right)}{\ell^2\left(r_- + r_+\right)r_0^4}$$
$$+ \frac{M^2\left(r_- - r_+\right)^2}{\left(r_- + r_+\right)^2 r_0^4}.$$

# Extremal case $r_- = r_+$

In this case, $\kappa = 0$.

$$\frac{e}{m} = \frac{\sqrt{r_+}\left(\ell^2 M + r_+^3 + r_0 r_+^2 + r_0^2 r_+ + r_0^3\right)}{2\ell q\sqrt{\ell^2 M + r_+\left(r_+ + r_0\right)^2}},$$

$$\lambda^2 = -\frac{\left(r_0 - r_+\right)^3\left(\ell^2 M\left(r_+ + 3r_0\right) + 2r_+\left(r_+ + r_0\right)^3\right)}{\ell^4 r_+ r_0^4}.$$

Thus we see that for the ratio

$$\frac{e}{m} = \frac{1}{2}\sqrt{1 + \frac{3r_+^4}{\ell^2 q^2}},$$

the equilibrium is located on the horizon, for which $\lambda = 0$, giving rise to the marginally stable equilibrium. The concept of no-force condition breaks down for the asymptotically-AdS extremal black holes. For particles with the larger $e/m$ ratio, the equilibrium $r_0$ is located outside of the horizon and all these equilibria are stable since $\lambda^2 < 0$. It is intriguing that charged massive particles can be trapped in these hypersurfaces. This may provide a mechanism of matter condensation.

## Non-extremal case $r_- < r_+$

For the non-extremal black holes with $r_+ > r_-$, we have

$$r_0 \to r_+ : \qquad \lambda^2 = \kappa^2 - \frac{4\kappa^2}{r_+}(r_0 - r_+) + \mathcal{O}\Big((r_0 - r_+)^2\Big),$$
$$r_0 \to \infty : \qquad \lambda^2 = -\frac{2r_0^2}{\ell^4} + \mathcal{O}(1).$$

Thus we see, not surprisingly, that $\lambda^2 > 0$ near the horizon, but it is always negative at asymptotic infinity. In fact for the non-extremal RN-AdS black holes, there exists $r_0^* > r_+$ where $\lambda = 0$. In the region $(r_+, r_0^*)$, the equilibria are unstable with $\lambda^2 > 0$. For $r_0 > r_0^*$, we have $\lambda^2 < 0$ and the equilibria become all stable.

We always have $\lambda^2 \leq \kappa^2$ in this case.

## Asymptotically dS

$$f=-\frac{1}{3}\Lambda_0\,r^2+1-\frac{2M}{r}+\frac{q^2}{r^2}\,,\qquad \psi=\frac{2q}{r}\,,\qquad \rho=r\,.$$

$$\begin{aligned}
M &= \frac{\left(r_-+r_+\right)\left(r_c+r_-\right)\left(r_c+r_+\right)}{2\left(r_c^2+\left(r_-+r_+\right)r_c+r_-^2+r_+r_-+r_+^2\right)}\,,\\
q &= \sqrt{\frac{r_-r_+r_c\left(r_c+r_-+r_+\right)}{r_-\left(r_c+r_+\right)+r_c^2+r_+r_c+r_-^2+r_+^2}}\,,\\
\Lambda_0 &= \frac{3}{r_c^2+\left(r_-+r_+\right)r_c+r_-^2+r_+r_-+r_+^2}\,.
\end{aligned}$$

$r_c > r_+ \geq r_-$ are the cosmological, outer and inner horizons. Surface gravity on the event horizon

$$\kappa=\frac{\left(r_+-r_-\right)\left(r_c-r_+\right)\left(r_c+r_-+2r_+\right)}{2r_+^2\left(r_c^2+r_+^2+r_-^2+r_-r_c+r_+r_c+r_+r_-\right)}\,.$$

## $\boxed{\textbf{Extremal case: } r_c > r_+ = r_-}$

Equilibria:

$$\frac{e}{m} = -\frac{r_+^2\,(r_0 - 2r_c) + r_+\left(r_0^2 - r_c^2\right) + r_0^3}{2r_+\sqrt{r_c\left(r_c + 2r_+\right)(r_c - r_0)\left(r_c + 2r_+ + r_0\right)}}\,.$$

$$\lambda^2 \;=\; \frac{(r_0 - r_+)^3}{r_0^4(r_c^2 + 2r_c r_+ + 3r_+^2)^2}\Big[(r_0 - r_+)(r_0 + r_+)^2$$
$$+2(3r_0^2 + 4r_0 r_+ + r_+^2)(r_c - r_0) + (3r_0 + r_+)(r_c - r_0)^2\Big] \geq 0\,.$$

All equilibria are unstable.

Since $\kappa = 0$, the violation of $\lambda/\kappa \leq 1$ is maximum!

## Non-extremal case: $r_c > r_+ > r_-$, an example

$$(M, q, \Lambda_0) = (\frac{6}{5}, \frac{6}{5}, \frac{3}{25}), \quad \kappa = \frac{1}{25}, \quad f = \frac{(r-1)(r-2)(3-r)(6+r)}{25r^2}.$$

$(r_c, r_+, r_-) = (3, 2, 1)$, so the normal space region is $r \in (2, 3)$.

$$\kappa^2 - \lambda^2 = \frac{2(r_0 - 2)(3 - r_0)(18 - 5r_0 - r_0^2)}{625r_0^2}.$$

Since the last bracket in the numerator has one native root, and one positive root $r_0^*$:

$$2 < r_0^* = \frac{1}{2}(\sqrt{97} - 5) < 3,$$

It follows that the chaotic bound is violated not near the horizon $r_+$, but in the region $r_0 \in (r_0^*, 3)$. In particular, the maximal violation occurs at $r_0 = 2.74$, which is not near to any horizon. In fact, we have $\kappa = \lambda$ on both the event and cosmological horizons.

## Neutral dS black hole

$$h = f = -\frac{1}{3}\Lambda_0 r^2 + 1 - \frac{2M}{r}, \qquad \rho = r\,.$$

$$M = \frac{r_c r_+ (r_c + r_+)}{2(r_c^2 + r_+^2 + r_c r_+)}, \qquad \Lambda_0 = \frac{3}{r_c^2 + r_+^2 + r_c r_+}\,.$$

It is easy to verify that the equilibria is located at

$$r_0 = \Big(\frac{3M}{\Lambda_0}\Big)^{\frac{1}{3}} = \Big(\frac{1}{2} r_c r_+ (r_c + r_+)\Big)^{\frac{1}{3}}, \text{with} \lambda^2 = \Lambda_0 - (9M^2\Lambda_0^4)^{\frac{1}{3}}\,.$$

Requiring that $r_c > r_+ > 0$ implies that $\lambda^2 > 0$ in general and hence the equilibrium is unstable, causing chaotic motion for general perturbation. The Lyapunov index however satisfy the bound.

Applying this result to our galaxy, whose effective Schwarzschild radius is about 0.2 light year, the unstable equilibria is located at 2.7 million light years away, with Lyapunov exponent $\lambda = 0.004 s^{-1}$.

Do we have examples of violation that is asymptotically flat?

## Charged black holes in EMD theories

Einstein-Maxwell-Dilaton Theory

$$\mathcal{L} = \sqrt{-g}\Big(R - \frac{1}{4}e^{a\phi}F^2 - \frac{1}{2}(\partial\phi)^2\Big)\,. \qquad a^2 = \frac{4}{N} - \frac{2(D-3)}{D-2}\,.$$

In supergravities, $N = 1, 2, 3, 4$ integers.

$$\begin{aligned} ds^2 &= -H^{-\frac{D-3}{D-2}N}\tilde{f}dt^2 + H^{\frac{N}{D-2}}\Big(\frac{dr^2}{\tilde{f}} + r^2 d\Omega^2\Big)\,,\\ A &= \psi dt\,, \qquad \phi = \frac{1}{2}Na\log H\,,\\ \tilde{f} &= 1 - \frac{\mu}{r^{D-3}}\,, \qquad \psi = \frac{\sqrt{N\,q(\mu+q)}}{r^{D-3}H}\,, \qquad H = 1 + \frac{q}{r^{D-3}}\,. \end{aligned}$$

In other words, we have

$$h = H^{-\frac{D-3}{D-2}N}\tilde{f}\,, \qquad f = H^{-\frac{N}{D-2}}\tilde{f}\,, \qquad \rho = r^2 H^{\frac{N}{D-2}}\,.$$

$$\kappa = \frac{D-3}{2r_+}\Big(1 + \frac{q}{r_+^{D-3}}\Big)^{-\frac{1}{2}N}\,.$$

The black hole thermodynamical quantities can be easily obtained by standard procedure, and they are

$$\begin{aligned}
M &= \frac{(D-2)\Omega}{16\pi}\Big(\mu + \frac{D-3}{D-2}Nq\Big)\,, \qquad\qquad T = \frac{\kappa}{2\pi}\,,\\
S &= \frac{1}{4}\Omega\, r_+^{D-2} H(r_+)^{\frac{1}{2}N}\,,\\
\Phi &= \psi(r_+)\,, \qquad\qquad Q = \frac{(D-3)\Omega}{16\pi}\sqrt{Nq(\mu+q)}\,.
\end{aligned}$$

It is easy to verify that the first law $dM = TdS + \Phi dQ$ of black hole thermodynamics is satisfied.

## Equilibria

$$\frac{e}{m} = \frac{\Omega}{32\pi} \frac{h'(r_0) H(r_0)^2}{Q\sqrt{h(r_0)}} r_0^{D-2} .$$

It turns out that the equilibria are all unstable, and the Lyapunov exponent is given by

$$\begin{aligned}
\lambda^2 \; = \; & \frac{N^2(D-3)^2 r_0^{2(5-2D)}}{64(D-2)^2 H(r_0)^{N+2}} \Big[ 4(D-3)^2 H(r_0)^2 (r_0 r_+)^{2(D-3)} \\
& +4(D-2)(D-3)a^2 \Big( 1 + (r_0 r_+)^{2(D-3)} + 2q r_0^{2(D-3)} r_+^{2(D-3)} \\
& +q^2 \big( 2 r_0^{D-3} (r_0^{D-3} - r_+^{D-3}) + r_+^{2(D-3)} \big) \Big) \\
& +(D-2)^2 H(r_0) \, (r_0 r_+)^{D-3} \Big( (r_0 r_+)^{D-3} + q(4 r_0^{D-3} - 3 r_+^{D-3}) \Big) \Big] ,
\end{aligned}$$

which is positive definite for $r_0 > r_+$. For two cases, the expression $\lambda/\kappa$ becomes particularly simple:

$$a = 0 : \qquad \frac{\lambda}{\kappa} = \left( \frac{q + r_+^{D+3}}{q + r_0^{D-3}} \right)^{\frac{D-2}{D-3}} ,$$

$$a = 1 , \quad D = 4 : \qquad \frac{\lambda}{\kappa} = \left( \frac{q + r_+}{q + r_0} \right)^2 .$$

## General dimensions

For general $D$ and $N$, we have the leading order of the near-horizon expansion

$$\kappa^2 - \lambda^2\Big|_{x\to 0} = \frac{(D-2)(D-3)^2}{2r_+^{D-1}H(r_+)^{1+N}}\Big(r_+^{D-3} - \frac{(D-4)a^2 q}{(D-2)a^2 + 2(D-3)}\Big)x\,.$$

$$r_0 = (1+x)r_+\,.$$

Thus we see that unless $a=0$ or $D=4$, the quantity can be negative for sufficiently large $q$.

In $D=4$, $\lambda^2 < \kappa$ for $N=1,2,4$, but can be violated for $N=3$.

## $\boxed{\text{Further examples examined}}$

Einstein-Born-Infeld

Einstein-Gauss-Bonnet

Violations can be found

## Conclusions

Unstable equilibria exist for charged particles outside the charged black holes.

The chaotic motion bound $\lambda \le \kappa$ holds for RN and RN-AdS black holes, but violated in RN-dS black holes, and in many other charged black holes in other theories include EMD, EMBI and EMGB theories.

However the violations seem all to occur when $\kappa$ is small. For the examples we studied, we find that there may exist a modified chaotic motion bound, namely for sufficiently large $\kappa$, then we have

$$\frac{\lambda}{\kappa} < \mathcal{C}\,,$$

where $\mathcal{C}$ is some order-one constant.