Calabi-Yau Manifolds and Modularity

Minxin Huang

University of Science and Technology of China

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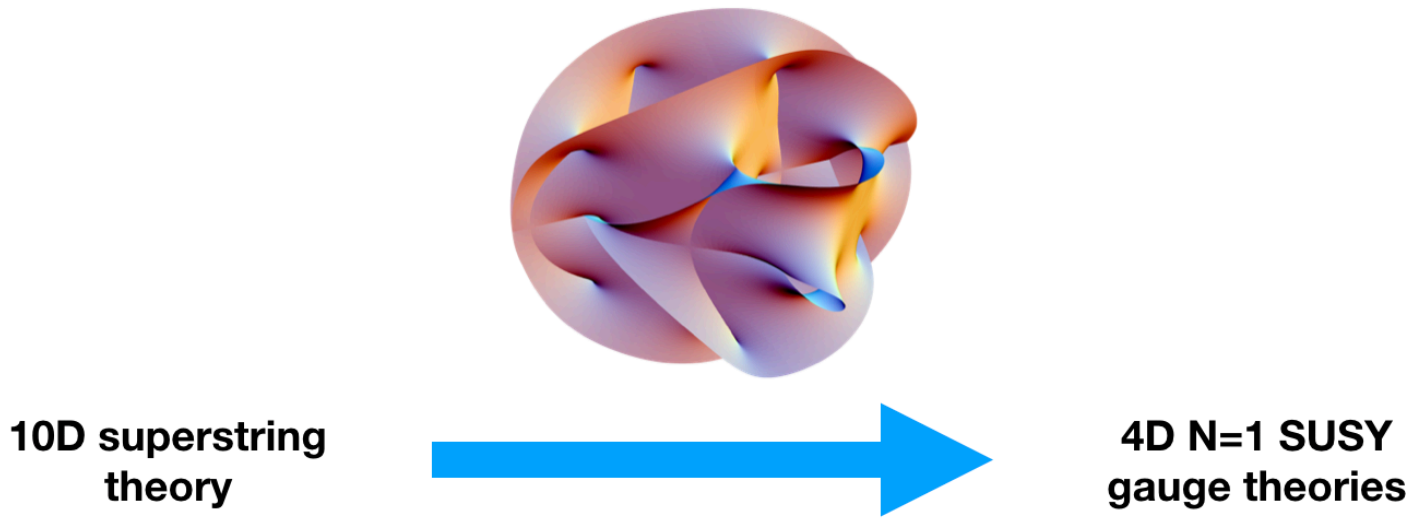
This is a review talk based on my discussions with collaborators A. Klemm, S Katz, and some preliminary ideas.

Calabi-Yau manifolds



- Calabi-Yau manifolds appear as phenomenologically interesting compactification of superstring theories.

- Mirror Symmetry: a pair of Calabi-Yau manifolds, with exchanged Hodge numbers. This provides a fertile ground at the interface of mathematics and physics.

Modularity

- This is a different kind of modularity from what I studied before. More about arithmetic, related to number theory.

- The simplest Calabi-Yau manifolds are elliptic curves. A famous result: elliptic curves over $\mathbb{Q}$ are modular (Conjectured by Taniyama, Shimura, Weil, proven by Wiles, Taylor et al). We will explain more details in a moment.

- This implies Fermats last theorem: the following equation has no positive integer solutions for $n > 3$,

$$a^n + b^n = c^n$$

- It is very interesting to generalize to higher dimensional Calabi-Yau manifolds, find connections with mirror symmetry.

Local Zeta Function

- Let X be a smooth projective variety. Consider counting points over finite fields. For a prime number p, we have $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and more generally $\mathbb{F}_{p^n} = \{\sum_{k=0}^{n-1} c_k e^{\frac{2\pi ik}{n}} | c_k \in F_p\}$.

- Define the local Zeta function

$$Z(X_p, T) = \exp[\sum_{n=1}^{\infty} \#X_p(\mathbb{F}_{p^n}) \ \frac{T^n}{n}] \tag{1}$$

Some simple examples:

1. X is a point. We have $\#X_p(\mathbb{F}_{p^n}) = 1$, $Z(X_p, T) = \frac{1}{1-T}$.

2. X is a projective line $(z_1, z_2) \sim \lambda(z_1, z_2)$. Counting points $(1, 0), (z, 1)$ with $z \in \mathbb{F}_{p^n}$. We have $\#X_p(\mathbb{F}_{p^n}) = p^n + 1$, $Z(X_p, T) = \frac{1}{(1-T)(1-pT)}$.

3. X is an elliptic curve. Counting points on $y^2 = x^3 + Ax + B$, $A, B \in \mathbb{Z}$. We have $Z(X_p, T) = \frac{1-a_pT+pT^2}{(1-T)(1-pT)}$, with $a_p = p + 1 - \#X(\mathbb{F}_p)$.

Weil Conjecture

- The Weil Conjectures (1949): The local Zeta function for a (complex) d-dimensional projective variety X is a rational function

$$Z(X_p, T) = \prod_{i=0}^{2d} P_i(X_p, T)^{(-1)^{i-1}}, \tag{2}$$

  where $P_i(X_p, T)$ is a polynomial of degree $b_i(X)$ (Betti numbers) with integral coefficients, and all roots of absolute value $p^{-\frac{i}{2}}$. Certain functional equations analogous to the Riemann Zeta function.

- We can check the simple examples in the previous slide.

- The conjectures, relating topology and number theory, are highly influential and motivated mathematical developments for several decades. It is still called "conjecture" though it is now proven.

Global Zeta Function

- X is now a projective variety over $\mathbb{Q}$. One defines the Hasse-Weil zeta function, which is the product of the global L-functions

$$L_i(X,s) = \prod_p P_i(X_p, p^{-s})^{-1}, \qquad \Re(s) \gg 0,$$
$$\zeta(X,s) = \prod_{i=0}^{2d} L_i(X,s)^{1-i} = \prod_p Z(X_p, p^{-s}), \tag{3}$$

where the product factors over some exceptional "bad prime" number are defined differently from previous slides.

- A simple example: X is a point. This is just the Riemann Zeta function

$$\zeta(X,s) = L_0(X,s) = \prod_p \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{4}$$

- A general Riemann conjecture: The global L-function $L_i(X,s)$ can be analytically continued to a meromorphic function of the complex plane, satisfies a functional equation $s \to i+1-s$, and all non-trivial zeros on the critical line $\Re(s) = \frac{i+1}{2}$.

- For $d=1$ dimension, $L_0(X,s), L_2(X,s)$ are simply related to Riemann zeta function, and the remaining one is $L_1(X,s)$. For elliptic curve

$$L(X,s) \equiv L_1(X,s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{1-2s}} \tag{5}$$

- The rational points on an elliptic curve E form a finitely generated Abelian group, known as Mordell-Weil group. A finitely generated Abelian group is isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_n}$.

- Birch-Swinnerton-Dyer conjecture (a Millennium Prize Problem): For an elliptic curve E over $\mathbb{Q}$, the L-function $L(E,s) = 0$ if and only if E has infinitely many rational points. Furthermore, the order of zero at $s=1$ is the rank fo the Mordell-Weil group.

Modular Forms

- Modular forms are holomorphic functions over upper half plane, transform according to

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z), \tag{6}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $SL(2,\mathbb{Z})$ matrix, and k is called the modular weight. Sometimes we also consider congruence subgroups of $SL(2,\mathbb{Z})$, e.g. $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | c \equiv 0 \text{ mod } N\}$.

- Modular forms of weight k forms a finite dimensional space. An example

$$\mathbb{G}_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k} = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $q = e^{e\pi iz}$, $k > 2$ an even integer, B_k is the Bernoulli number, and $\sigma_k(n)$ denotes the sum of k's powers of positive divisors of n.

Hecke Theory

- Hecke operators T_m: a linear operator on modular forms of weight k

$$T_m f(z) = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} \frac{1}{d^k} \sum_{b=0}^{d-1} f(\frac{az+b}{d}). \tag{7}$$

  All $T_m (m \in \mathbb{N})$ commute with each others.

- Hecke eigenforms: simultaneous eigenstates of all Hecke operators T_m. Suppose $f(q) = \sum_{n=0}^{\infty} a_n q^n$ is a Hecke eigenform, usually normalized $a_1 = 1$. Then we have

$$T_m f = a_m f, \quad a_m a_n = \sum_{r|(m,n)} r^{k-1} a_{mn/r^2}, \quad (m, n > 0). \tag{8}$$

- An example of Hecke eigenform, $T_m \mathbb{G}_k(z) = \sigma_{k-1}(m) \mathbb{G}_k(z)$.

- In particular, for a Hecke eigenform $f(q) = \sum_{n=0}^{\infty} a_n q^n$, we have

$$a_{mn} = a_m a_n \qquad (m,n) = 1$$
$$a_{p^{n+1}} = a_p a_{p^n} - p^{k-1} a_{p^{n-1}}, \qquad p \text{ prime}, n \geq 1. \tag{9}$$

- Define a Hecke L-series

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{10}$$

- Due to equation (9), we have the product form

$$L(f,s) = \prod_p (1 + \sum_{n=1}^{\infty} a_{p^n} p^{-ns}) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}} \tag{11}$$

  For weight $k = 2$, this is very similar to the L-function of an elliptic curve!

- For eigenforms on $\Gamma_0(N)$, the product factors for $p|N$ need suitable modifications.

- Modularity conjecture/theorem: The L-function of an elliptic curve is the L-function of a weight two modular form of $\Gamma_0(N)$ for some N. (Taniyama, Shimura, Weil, Wiles, Taylor, et al)

- An example: consider elliptic curve $E : Y^2 - Y = X^3 - X^2$. The L-function is $L(E,s) = \prod_p \frac{1}{1-a_p p^{-s}+p^{1-2s}}$. Taking into account the "point at infinity", we have

$$a_p = p - |\{(x,y) \in (\mathbb{Z}_p)^2 | y^2 - y = x^3 - x^2\}| \tag{12}$$

For example $a_2 = -2, a_3 = -1, a_5 = 1, \cdots$. So the L-function is

$$\begin{aligned} L(E,s) &= (1+\frac{2}{2^s}+\frac{2}{2^{2s}})^{-1}(1+\frac{1}{3^s}+\frac{3}{3^{2s}})^{-1}(1-\frac{1}{5^s}+\frac{5}{5^{2s}})^{-1}\cdots \\ &= 1-\frac{2}{2^s}-\frac{1}{3^s}+\frac{2}{4^s}+\frac{1}{5^s}+\cdots \end{aligned} \tag{13}$$

- This is the L-function of the following modular form in $\Gamma_0(11)$

$$\begin{aligned} f(z) &= \eta(z)^2\eta(11z)^2 = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \\ &= q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots \end{aligned} \tag{14}$$

Higher dimensions

- Suppose X is a Calabi-Yau d-fold. We are interested in the L-function of the middle cohomology, namely $L(X,s) \equiv L_d(X,s)$.

- Dimension 2: K3 surfaces. The non-vanishing Betti numbers are $b_0 = b_4 = 1, b_2 = 22$. The middle cohomology is again the only non-trivial one. For the special case of singular K3, there is now a theorem relating the L-function to that of a weight 3 modular form of a congruence subgroup.

- Dimension 3. The simplest case is the rigid Calabi-Yau threefolds, i.e. $h^{1,2} = 0$. It is conjectured that the L-function is that of a weight 4 modular form of a congruence subgroup. Review by N. Yui, arXiv:1212.4308

- The general case $h^{1,2} > 0$ is difficult. However, there are some expectations that somethings nice happen at the attractor points. Moore, Candelas et al

Rigid Calabi-Yau threefolds

- For a Kahler manifold we have $h^{1,1} \geq 1$. Naively a rigid Calabi-Yau threefold ($h^{1,2} = 0$) can not have a mirror. This is remedied by a generalized notion of mirror symmetry in terms of non-linear sigma model with non-geometric target space.

- Rigid Calabi-Yau threefolds are very rare, so far about 50 examples are known. (Meyer, 2005)

- Let $(x_0, x_1, x_2, x_3, x_4)$ be homogeneous coordinates of $\mathbb{P}^4$. Consider the quintic equation

$$P(x_0, x_1, x_2, x_3, x_4) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0 \quad (15)$$

For generic ψ ($\psi^5 \neq 1, 0, \infty$), the hypersurface is a smooth Calabi-Yau manifold with $h^{1,1} = 1, h^{1,2} = 101$.

- The equation is invariant under a group $(\mathbb{Z}_5)^3$ action

$$(x_0, x_1, x_2, x_3, x_4) \to (x_0, x_1\xi^{\lambda_1}, x_2\xi^{\lambda_2}, x_3\xi^{\lambda_3}, x_4\xi^{\lambda_4}), \tag{16}$$

where $\lambda_i \in \mathbb{Z}_5$, $\sum_{i=1}^{4} \lambda_i = 0 \mod 5$, and ξ is a fixed primitive 5th root of unity. We can consider the orbifold under the group action and resolve the singularity. For generic ψ, this gives the mirror of quintic with $h^{1,1} = 101, h^{1,2} = 1$.

- An example: Schoen's quintic. Consider ψ a 5th root of unity, for example $\psi = 1$. Then there are 125 singularities, namely $(1, 1, 1, 1, 1)$ and its image under the $(\mathbb{Z}_5)^3$ action. (The singularities are the points with $P(x_0, x_1, x_2, x_3, x_4) = 0$ and $\partial_{x_i} P = 0, i = 0, 1, \cdots 4.$) Resolving these singularity gives a rigid Calabi-Yau with $h^{1,1} = 25, h^{1,2} = 0$.

- One way to see is Euler number $= -200 + 2 \cdot 125 = 50$. The hodge numbers can be inferred by counting points on $\mathbb{F}_p$ for a large p, e.g. $(p = 31)$.

Application in the Swampland

- Swampland (Ooguri, Vafa et al): Quantum field theories that become inconsistent when quantum gravity effects are considered. Some examples

  1. Distance conjecture. If we go over a distance in the scalar field space $\Delta\phi \gg 1$ in Planck unit, then there are towers of light particles with masses $m \sim e^{-a\Delta\phi}$ descend from UV, so that the effective field theory is no longer valid.

  2. De Sitter conjecture. Effective potential satisfies a universal bound $|\nabla V| \geq \frac{c}{M_p} V$. This conjecture exclude de Sitter local minimum or maximum in string theory.
     This is later refined/weakened to alternatively $\min(\nabla_i \nabla_j V) \leq -\frac{c'}{M_p^2} V$, which excludes only de Sitter local minimum, but not maximum.

- This is very controversial due to conflict with KKLT construction of de Sitter vacuum in string theory.

Theta-problem and rigid Calabi-Yaus

- Another swampland conjecture: No free continuous parameter in string theory.
  Cecotti, Vafa, arXiv: 1808.03482: Consider the θ-angle in QED
  $$L = \sqrt{-g}(\frac{1}{2}R - \frac{1}{4e^2}F^2 + \frac{\theta}{32\pi^2}F\tilde{F}) \tag{17}$$
  The θ angle is only observable in gravitational physics. But it seems to be a free continuous parameter.

- Consider type IIB on a rigid Calabi-Yau threefold. The $\mathcal{N} = 2$ supergravity theory has a gravity multiplet with the graviphoton, no vector multiplet, and many hypermultiplets. The coupling constant $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ for the graviphoton is computed by
  $$\tau = -(\int_{\gamma_2} \Omega)/(\int_{\gamma_1} \Omega), \tag{18}$$
  where Ω is the holomorphic three-form, and $\gamma_{1,2}$ are the two independent integral 3-cycle.

- We can compute the coupling constant for various rigid Calabi-Yau models. It turns out for all models

$$j(\tau) \in \mathbb{R} \;\Rightarrow\; \theta = 0 \text{ or } \pi \tag{19}$$

- Future works: More calculations can determine the imaginary part, or the fine structure constant $\frac{e^2}{4\pi}$. Some examples in Cynk, Van Straten, arXiv: 1709.09751 [math.AG].

Relations to Mirror Symmetry

- Consider again the quintic $P(x,\psi) = \sum_{i=0}^{4} x_i^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0$. It is well known the periods $\int \Omega$ of mirror quintic is described by a 4th order Picard-Fuchs linear differential equation, and the solutions are

$$
\begin{aligned}
\omega_0(\lambda) &= f_0(\lambda) = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} \lambda^m, \\
\omega_1(\lambda) &= f_0(\lambda) \log(\lambda) + f_1(\lambda), \\
\omega_2(\lambda) &= f_0(\lambda) \log^2(\lambda) + 2 f_1(\lambda) \log(\lambda) + f_2(\lambda), \\
\omega_3(\lambda) &= f_0(\lambda) \log^3(\lambda) + 3 f_1(\lambda) \log^2(\lambda) + 3 f_2(\lambda) \log(\lambda) + f_3(\lambda),
\end{aligned}
$$

  where $\lambda = (5\psi)^{-5}$.

- These periods determine the mirror map and count holomorphic spheres in the quintic. However, it turns out that they can be used to count points (of finite fields) on quintic. Candelas, de la Ossa, Villegas, hep-th/0012233, hep-th/0402133.

- Denote ${}^n f_j$ as the truncation of the series f_j to n+1 terms, e.g. ${}^n f_0(\lambda) = \sum_{m=0}^{n} \frac{(5m)!}{(m!)^5} \lambda^m$. Further define a semiperiod $f_4(\lambda)$ from the extension of differential operator to 5th order.

- It is found that the number of counting points has a congruence relation

$$
\begin{aligned}
& |\{x \in \mathbb{F}_p^5 \mid P(x,\psi) = 0\}| \\
= \;& {}^{(p-1)}f_0(\lambda^{p^4}) + (\frac{p}{1-p})^{(p-1)} f_1'(\lambda^{p^4}) + \frac{1}{2!}(\frac{p}{1-p})^2 \; {}^{(p-1)} f_2''(\lambda^{p^4}) \\
& + \frac{1}{3!}(\frac{p}{1-p})^3 \; {}^{(p-1)} f_3'''(\lambda^{p^4}) + \frac{1}{4!}(\frac{p}{1-p})^4 \; {}^{(p-1)} f_4''''(\lambda^{p^4}) \qquad \bmod p^5,
\end{aligned}
$$

- Some explanations: we expand fractional numbers in terms p-adic expansion $\sum_{i=k}^{\infty} a_i p^i$. The p-adic expansion is inverse of usual decimal expansion, quite counter-intuitive. For example, to compute the 5-adic expansion of $\frac{1}{3}$ up to 4th order, we check $\frac{2 \cdot 5^4 + 1}{3} = 417$ is integer, so $\frac{1}{3} = 417 = 2 + 3 \cdot 5 + 1 \cdot 5^2 + 3 \cdot 5^3 \pmod{5^4}$.

Congruence to Modular Forms

- The hypergeometric series in the mirror period has a congruence relation to modular form. (Conjectured by Villegas, complete proof by Long, Tu, Yui, Zudilin, arXiv: 1705.01663.)

- Consider the hypergeometric series

$$ {}_4F_3\left[{\alpha_1,\alpha_2,\alpha_3,\alpha_4 \atop 1,1,1}; \lambda\right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k(\alpha_4)_k}{k!^4}\lambda^k, \tag{20} $$

  where $(\alpha)_k = \Gamma(\alpha+k)/\Gamma(\alpha)$. This appears in a class of 14 one-parameter Calabi-Yau models, e.g. $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$ for the quintic.

- There is a weight 4 modular form $f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{n=1}^{\infty} a_n q^n$, with $a_1 = 1$, such that

$$ {}_4F_3\left[{\alpha_1,\alpha_2,\alpha_3,\alpha_4 \atop 1,1,1}; \lambda\right]_{p-1} = a_p \mod p^3, \tag{21} $$

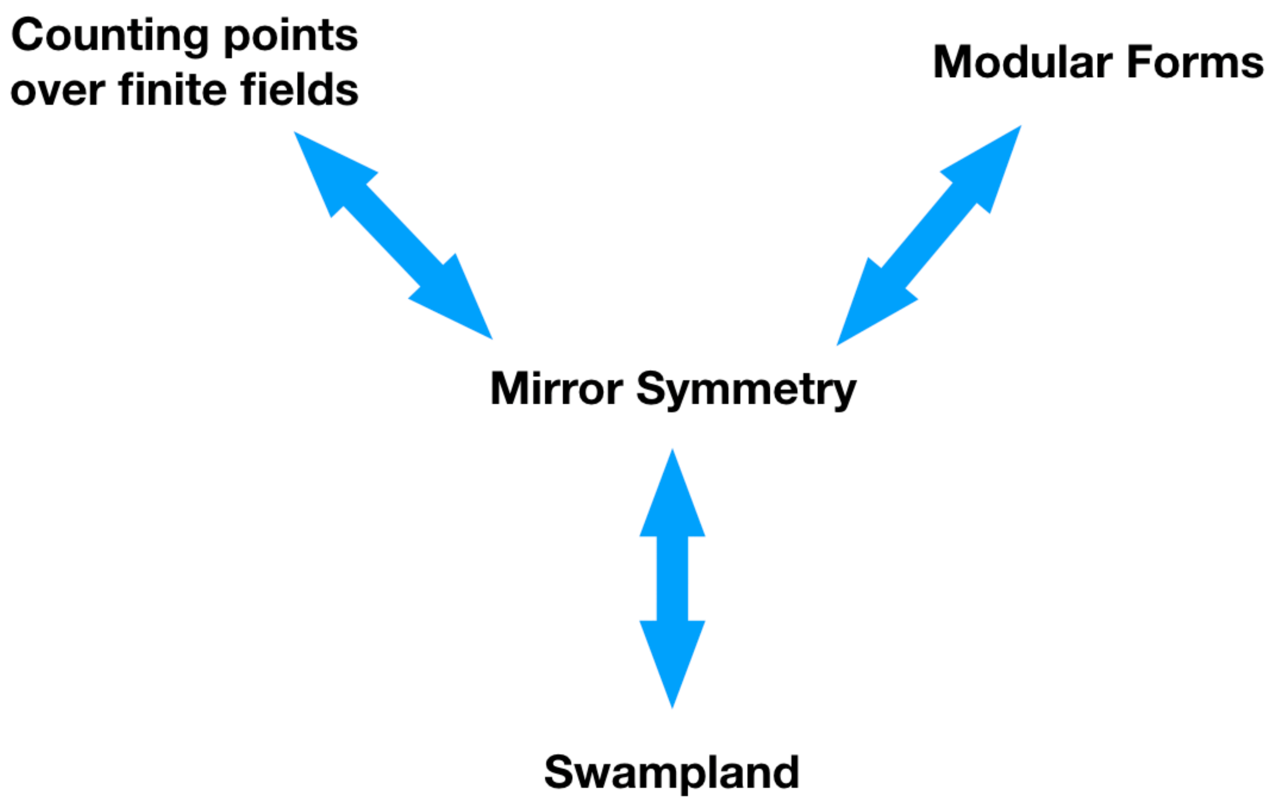
where subscript $_{p-1}$ means truncation of the series to pth term. This congruence relation can be used to determine the modular form. For example, for the quintic we have $f = \frac{\eta(5z)^{10}}{\eta(z)\eta(25z)} + 5\eta(z)^2\eta(5z)^4\eta(25z)^2$, a modular form of $\Gamma_0(25)$.

- What about the L-function? This is related to the analytic continuation of period matrix from large volume point to conifold point . It was known that after taking into account the usual vanishing conditions, the transition matrix depends on 6 real numbers that are only computed numerically before. It is conjectured and checked that 2 of these numbers are related to the L-function values of the modular form. (Klemm, Scheidegger, Zagier, to appear)

- Relations to higher genus topological strings?

Summary

- Some interesting connections



Thank You