Friedrichs model and an application in hadron spectrum

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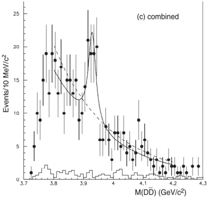
Introduction

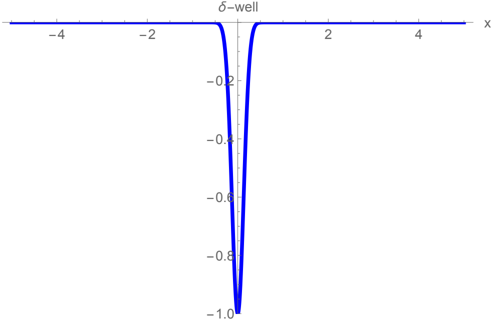
In phenomenology study:

- Bound state: stable states.
- In particle physics, a peak in he cross section is always related to a resonance: Unstable



- Unstable particles are everywhere in physics. How to describe the unstable particle *nonperturbatively* in quantum theory is a problem.
- Another kind of state: Virtual state

A quantum mechanics example: Virtual state



$$V(x) = -\gamma\ \delta(x), \qquad \gamma > 0$$

There is a bound state with even parity:

$$\psi(x) = \frac{1}{\sqrt{L}} e^{-|x|/L}\ , \quad L = \hbar^2/m\gamma$$

$$E = -\ \frac{m\gamma^2}{2\hbar^2}$$

Scattering in the Delta potential

$$\psi(x) = \begin{cases} e^{ikx} + R\ e^{-ikx} & x < 0\ ; \\ T\ e^{ikx} & x > 0\ . \end{cases}$$

$$R = \frac{im\gamma/k\hbar^2}{1 - im\gamma/k\hbar^2}\ , \qquad T = \frac{1}{1 - im\gamma/k\hbar^2}$$

- There is a pole in R and T, which corresponds to the bound state before.

  $$k = i\frac{m\gamma}{\hbar^2}\ , \quad E = \frac{\hbar^2 k^2}{2m} = -\frac{m\gamma^2}{2\hbar^2}$$

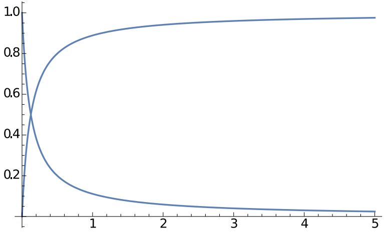
  Thus we can read out the bound state information from the scattering amplitude.
- In the k plane, it is on the positive imaginary axis.
- The energy plane is also continued to a two-sheeted complex plane with a cut at $(0, +\infty)$, since $E \sim k^2$. The pole lies on the negative real axis of the first sheet.

Transmission coefficient and reflection coefficient

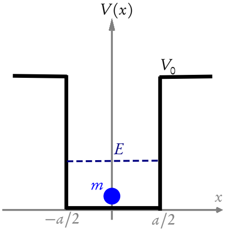
$$\mathscr{R} = \frac{m\gamma^2/(2\hbar^2 E)}{1 + m\gamma^2/(2\hbar^2 E)}, \qquad \mathscr{T} = \frac{1}{1 + m\gamma^2/(2\hbar^2 E)}$$

The same for positive or negative γ.

- For $\gamma < 0$, there is also a pole of R and T, $k = i\frac{m\gamma}{\hbar^2}$, *virtual state* or *anti-bound state*. No well defined normalization.
- It lies on the real axis of the second sheet of the energy plane.



Square well



Consider potential

$$V(x) = \begin{cases} 0, & -a/2 \leq x \leq a/2 \\ V_0, & |x| > a/2 \end{cases}$$

- Boundstates: ($\kappa = \sqrt{2m(V_0 - E)}/\hbar$, $k = \sqrt{2mE}/\hbar$)

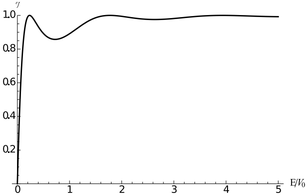
$$k \tan(ka/2) = \kappa, \quad \text{Even parity}$$
$$-k \cot(ka/2) = \kappa, \quad \text{Odd parity}$$

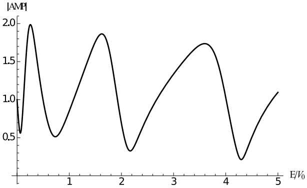
- Resonance: $(k \to i\kappa,\ \kappa \to ik)$

$$T\ e^{ika} = -\frac{2ik/\kappa}{[1-(k/\kappa)^2]\sinh(\kappa a) - 2\frac{k}{\kappa}\cosh(\kappa a)}$$

The pole position satisfies the same equation as the eigenvalue for bound state.

- There are also complex solutions: Resonances responsible for the resonant energy level and peaks in the |scattering amplitude|.





$E = 0.197976 + 0.153714i, 1.76434 + 0.314351i, 3.92488 + 0.428143i, \ldots$

The simplest Friedrichs model [Friedrichs, Commun. Pure Appl. Math., 1(1948), 361]

- The system couples one bare state $|1\rangle$ and a continuum state $|\omega\rangle$, $(\omega > 0)$, which are eigenstates of the free Hamiltonian

$$H_0|1\rangle = \omega_0|1\rangle, \qquad H_0|\omega\rangle = \omega|\omega\rangle.$$

- Orthonormal condition: $\langle 1|1\rangle = 1$, $\langle 1|\omega\rangle = 0$, and $\langle\omega|\omega'\rangle = \delta(\omega - \omega')$
  Completeness: $|1\rangle\langle 1| + \int_0^\infty d\omega|\omega\rangle\langle\omega| = 1$
- The free Hamiltonian can be expressed as:

$$H_0 = \omega_0|1\rangle\langle 1| + \int_0^\infty \omega|\omega\rangle\langle\omega|\mathrm{d}\omega$$

- Interaction: $\langle\omega|V|1\rangle = \lambda f(\omega)$, $\langle\omega'|V|\omega\rangle = \langle 1|V|1\rangle = 0$.

$$V = \lambda\int_0^\infty [f(\omega)|\omega\rangle\langle 1| + f^*(\omega)|1\rangle\langle\omega|]\mathrm{d}\omega$$

Eigenvalue equation:

$$H|\Psi(x)\rangle = (H_0 + V)|\Psi\rangle = x|\Psi(x)\rangle.$$

Solutions:

- Continuum: Eigenvalue $x > 0$, real Solution: define $\eta^{\pm}(x) = x - \omega_0 - \lambda^2 \int_0^{\infty} \frac{f(\omega) f^*(\omega)}{x - \omega \pm i\epsilon} \mathrm{d}\omega$

$$|\Psi_{\pm}(x)\rangle \quad = \quad |x\rangle + \lambda \frac{f^*(x)}{\eta^{\pm}(x)} \Big[|1\rangle + \lambda \int_0^{\infty} \frac{f(\omega)}{x - \omega \pm i\epsilon} |\omega\rangle \mathrm{d}\omega \Big]$$

- S-matrix:

$$S(E, E') = \delta(E - E') \Big(1 - 2\pi i \frac{\lambda f(E) f^*(E)}{\eta^{+}(E)} \Big).$$

- Discrete states:The zero point of $\eta(x)$ corresponds to eigenvalues of the full Hamiltonian — discrete states.

Discrete state solutions:Bound states

$$\eta^I(x) = x - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega)f^*(\omega)}{x-\omega}\mathrm{d}\omega = 0$$
$$\eta^{II}(x) = \eta^I(z) - 2i\pi G(z), \quad G \equiv \lambda^2 f(x)f^*(x)$$

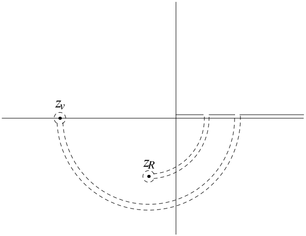
- Bound states: solutions on the first sheet real axis below the threshold.
$$|z_B\rangle = N_B\Big(|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z_B-\omega}|\omega\rangle \mathrm{d}\omega\Big)$$
where $N_B = (\eta'(z_B))^{-1/2} = (1+\lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B-\omega)^2})^{-1/2}$, such that $\langle z_B|z_B\rangle = 1$.
- Elementariness: $Z = N_B^2$;
Compositeness: $X = N_B^2 \lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B-\omega)^2}$.
- Eg. If $\omega_0 < 0$, there could be a bound state. In the weak coupling limit, it $\rightarrow |1\rangle$,
- Eg. there could also be dynamically generated bound state in the strong coupling.

Discrete state solutions:Virtual states

- Virtual states: Solutions on the second sheet real axis below the threshold.

$$|z_v^\pm\rangle = N_v^\pm\Big(|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{[z_v-\omega]_\pm}|\omega\rangle \mathrm{d}\omega\Big), \quad \langle\tilde{z}_v^\pm| = \langle z_v^\mp|,$$

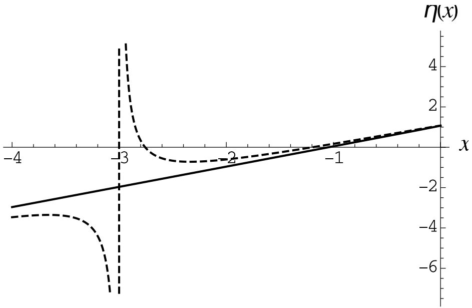
  where
  $N_v^- = N_v^{+*} = (\eta'^+(z_v))^{-1/2} = (1+\lambda^2\int d\omega \frac{|f(\omega)|^2}{[(z_v-\omega)_+]^2})^{-1/2}$,
  such that $\langle\tilde{z}_v^\pm|z_v^\pm\rangle = 1$.
- When $\omega < 0$, a bound state generated from $|1\rangle$ is always accompanied with a virtual state in weak coupling.



Discrete state solutions: Virtual states

- Virtual states from the singularity of the form factor, analytically continued $G(\omega) = |f(\omega)|^2$:

$$\eta^I = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega} \mathrm{d}\omega$$
$$\eta^{II}(\omega) = \eta^I(\omega) + 2\pi i \lambda^2 \, G^{II}(\omega) = \eta^I(\omega) - 2\lambda^2 \pi i \, G(\omega),$$

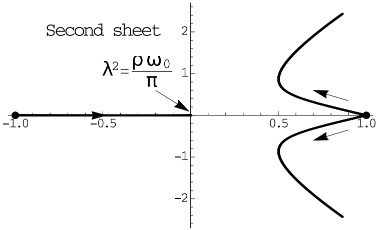


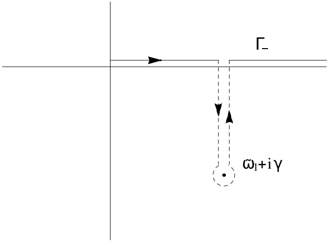
- Virtual state generated from the bare states: $\omega_0 < 0$

Discrete state solutions: Resonance

- Resonant states: $\omega_0 >$ threshold, the discrete state becomes a pair of solutions z_R, z_R^*, on the second sheet of the complex plane. $\hat{H}|z_R\rangle = z_R|z_R\rangle$

$$|z_R\rangle = N_R\Big(|1\rangle + \lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R-\omega]_+}|\omega\rangle\Big),$$
$$|z_R^*\rangle = N_R^*\Big(|1\rangle + \lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R^*-\omega]_-}|\omega\rangle\Big),$$





Discrete state solutions: Resonance

Resonant states:

- Normalization: $\langle z_R|z_R\rangle = 0$, naïve argument, $z_R^* \neq z_R$,

$$\langle z_R|\hat{H}|z_R\rangle = z_R\langle z_R|z_R\rangle = z_R^*\langle z_R|z_R\rangle = 0$$

  $|z_R\rangle$ is not in the Hilbert space — need rigged Hilbert space description.

- Left eigenstates:$\langle\tilde{z}_R|\hat{H} = \langle\tilde{z}_R|z_R$

$$\langle\tilde{z}_R| = N_R\Big(\langle 1| + \lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R-\omega]_+}\langle\omega|\Big),$$
$$\langle\tilde{z}_R^*| = N_R^*\Big(\langle 1| + \lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R^*-\omega]_-}\langle\omega|\Big).$$

  N_R is a complex normalization parameter,
  $N_R = (\eta'^+(z_R))^{-1/2} = (1+\lambda^2\int d\omega \frac{|f(\omega)|^2}{[(z_R-\omega)_+]^2})^{-1/2}$ such that
  $\langle\tilde{z}_R|z_R\rangle = 1$

Discrete state solutions: Accidental higher order poles

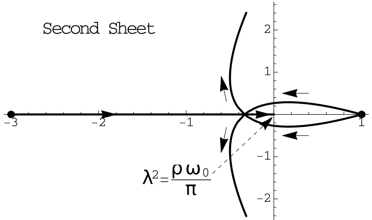
Eg. a triple pole

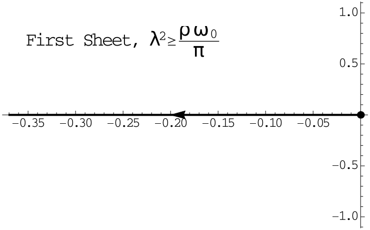
$$|z_v^{\pm}\rangle = N_v^{\pm}\Big(|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{[z_v-\omega]_\pm}|\omega\rangle \mathrm{d}\omega\Big), \quad \langle\tilde{z}_v^{\pm}| = \langle z_v^{\mp}|,$$

$$|z_{v2}^{\pm}\rangle = -N_{v2}\lambda\int_0^\infty \frac{f(\omega)}{([z_v-\omega]_\pm)^2}|\omega\rangle \mathrm{d}\omega, \quad \langle\tilde{z}_{v2}^{\pm}| = \langle z_{v2}^{\mp}|,$$

$$|z_{v3}^{\pm}\rangle = N_{v3}\lambda\int_0^\infty \frac{f(\omega)}{([z_v-\omega]_\pm)^3}|\omega\rangle \mathrm{d}\omega\,, \quad \langle\tilde{z}_{v3}^{\pm}| = \langle z_{v3}^{\mp}|\,.$$

Normalization: $\langle\tilde{z}_{v3}^{\pm}|z_v^{\pm}\rangle = 1$ and $\langle\tilde{z}_{v2}^{\pm}|z_{v2}^{\pm}\rangle = 1$.
Then $N_v = N_{v_2} = N_{v3} = (6/\eta''')^{1/2}$.





Completeness relation

- In general, the resonance state and the virtual states do not enter the completeness relation. If there is only continuum eigenstates:

$$\mathbf{1} = \int_0^\infty d\omega |\Psi_+(\omega)\rangle\langle\Psi_+(\omega)|.$$

With one bound state $|E_B\rangle$ eigenstate:

$$\mathbf{1} = |E_B\rangle\langle E_B| + \int_0^\infty d\omega |\Psi_+(\omega)\rangle\langle\Psi_+(\omega)|.$$

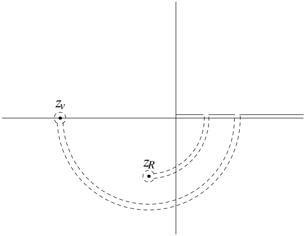
Completeness relation

To treat the resonances and virtual states the same as the bound state and Continuum state:

- Petrosky, Prigogine, Tasaki: in solving the large Poincaré problem, propose a definition of continuum state. $|\Psi_+(x)\rangle$ as a distribution, includes the integral contour information

$$\frac{1}{\eta_d^+(x)} \equiv \frac{1}{\eta^+(x)} \frac{x-\tilde{\omega}_1+i\gamma}{[x-\tilde{\omega}_1+i\gamma]_+},$$

$$|\Psi_\pm(x)\rangle = |x\rangle + \lambda \frac{f(x)}{\eta_d^\pm(x)}\left[|1\rangle + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{x-\omega\pm i\epsilon}|\omega\rangle\right].$$



Completeness relation

- The left state is not modified:

$$\langle\tilde{\Psi}_{\pm}(x)| = \langle x| + \lambda \frac{f(x)}{\eta^{\mp}(x)} \left[\langle 1| + \lambda \int_0^{\infty} \mathrm{d}\omega \frac{f(\omega)}{x - \omega \mp i\epsilon} \langle\omega| \right].$$

- Using these continuum states, the completeness relation reads,

$$\mathbf{1} = \int_0^{\infty} d\omega |\Psi_{+}(\omega)\rangle\langle\tilde{\Psi}_{+}(\omega)| + |z_R\rangle\langle\tilde{z}_R|.$$

The resonant state enters the completeness relation.

Completeness relation: higher-order pole

When there is an nth-order pole, n degenerate states:

$$|z^{(1)}\rangle = N\Big(|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{[z-\omega]_+}|\omega\rangle \mathrm{d}\omega\Big)\,,$$

$$\langle\tilde{z}^{(1)}| = N\Big(\langle 1| + \lambda\int_0^\infty \frac{f(\omega)}{[z-\omega]_+}\langle\omega| \mathrm{d}\omega\Big)\,,$$

$$|z^{(n)}\rangle = N(-1)^{n-1}\lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{([z-\omega]_+)^n}|\omega\rangle\,, \quad \text{for } n\geq 2\,,$$

$$\langle\tilde{z}^{(n)}| = N(-1)^{n-1}\lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{([z-\omega]_+)^n}\langle\omega|\,, \quad \text{for } n\geq 2\,,$$

$N = (\frac{n!}{\eta^{(n)}(z)})^{1/2} = \Big((-)^{n-1}\frac{\lambda^2}{n!}\int d\omega \frac{|f(\omega)|^2}{([z-\omega]_+)^{n+1}}\Big)^{-1/2}$ is chosen such that $\langle\tilde{z}^{(r)}|z^{(n-r+1)}\rangle = 1$. the completeness relation can also be deduced

$$\mathbf{1} = \int_0^\infty d\omega |\Psi_+(\omega)\rangle\langle\tilde{\Psi}_+(\omega)| + \sum_{r=1}^{n} |z^{(r)}\rangle\langle\tilde{z}^{(n-r+1)}|.$$

Coupled channel Friedrichs model

Hamiltonian: $H = H_0 + V$

$$
\begin{aligned}
H &= \omega_0|1\rangle\langle 1| + \int_{a_1}^{\infty} \mathrm{d}\omega\omega|\omega\rangle_{11}\langle\omega| + \int_{a_2}^{\infty} \mathrm{d}\omega\omega|\omega\rangle_{22}\langle\omega| \\
&+ \lambda_1 \int_{a_1}^{\infty} \mathrm{d}\omega[f_1(\omega)|\omega\rangle_1\langle 1| + f_1^*(\omega)|1\rangle\ {}_1\langle\omega|] \\
&+ \lambda_2 \int_{a_2}^{\infty} \mathrm{d}\omega[f_2(\omega)|\omega\rangle_2\langle 1| + f_2^*(\omega)|1\rangle\ {}_2\langle\omega|]
\end{aligned}
$$

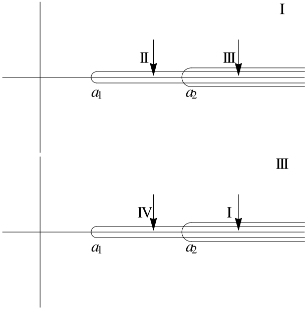
Solution:
Continuous states,

$$
|\Psi_{i\pm}(x)\rangle = |x\rangle_i + \frac{\lambda_i f_i^*(x)}{\eta^{\pm}(x)}\left[|1\rangle + \sum_{j=1,2}\lambda_j \int_{a_j}^{\infty} \mathrm{d}\omega \frac{f_j(\omega)}{x-\omega\pm i\epsilon}|\omega\rangle_j\right].
$$

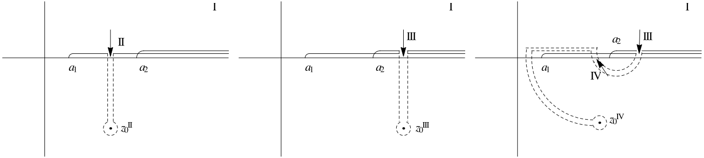
where $\eta^{\pm}(x) = x - \omega_0 - \lambda_1^2 \int_{a_1}^{\infty} \frac{G_1(\omega)}{x-\omega\pm i\epsilon}\mathrm{d}\omega - \lambda_2^2 \int_{a_2}^{\infty} \frac{G_2(\omega)}{x-\omega\pm i\epsilon}\mathrm{d}\omega\,.$
Orthonormal condition: $\langle\Psi_i(x')|\Psi_j(x)\rangle = \delta_{ij}\delta(x'-x)$.

Discrete states are determined by $\eta(z) = 0$, analytically continued to different Riemann sheets.

$$\eta(x) = x - \omega_0 - \lambda_1^2 \int_{a_1}^{\infty} \frac{G_1(\omega)}{x - \omega \pm i\epsilon} \mathrm{d}\omega - \lambda_2^2 \int_{a_2}^{\infty} \frac{G_2(\omega)}{x - \omega \pm i\epsilon} \mathrm{d}\omega \,.$$



Wave function



$$
\begin{aligned}
|z_0^I\rangle =& N^I\Big(|1\rangle + \lambda_1 \int_{a_1}^{\infty} \mathrm{d}\omega \frac{f_1(\omega)}{z_0^I - \omega}|\omega\rangle_1 + \lambda_2 \int_{a_2}^{\infty} \mathrm{d}\omega \frac{f_2(\omega)}{z_0^I - \omega}|\omega\rangle_2\Big) , \\
|z_0^{II}\rangle =& N^{II}\Big(|1\rangle + \lambda_1 \int_{a_1}^{\infty} \mathrm{d}\omega \frac{f_1(\omega)}{[z_0^{II} - \omega]_+}|\omega\rangle_1 + \lambda_2 \int_{a_2}^{\infty} \mathrm{d}\omega \frac{f_2(\omega)}{z_0^{II} - \omega}|\omega\rangle_2\Big) , \\
|z_0^{III}\rangle =& N^{III}\Big(|1\rangle + \lambda_1 \int_{a_1}^{\infty} \mathrm{d}\omega \frac{f_1(\omega)}{[z_0^{III} - \omega]_+}|\omega\rangle_1 + \lambda_2 \int_{a_2}^{\infty} \mathrm{d}\omega \frac{f_2(\omega)}{[z_0^{III} - \omega]_+}|\omega\rangle_2\Big) , \\
|z_0^{IV}\rangle =& N^{IV}\Big(|1\rangle + \lambda_1 \int_{a_1}^{\infty} \mathrm{d}\omega \frac{f_1(\omega)}{z_0^{IV} - \omega}|\omega\rangle_1 + \lambda_2 \int_{a_2}^{\infty} \mathrm{d}\omega \frac{f_2(\omega)}{[z_0^{IV} - \omega]_+}|\omega\rangle_2\Big) ,
\end{aligned}
$$

Completeness relation

Continuum state:

$$|\Psi^d_{i\pm}(x)\rangle = |x\rangle_i + \frac{\lambda_i f_i^*(x)}{\eta_d^{\pm}(x)}\left[|1\rangle + \sum_{j=1,2}\lambda_j \int_{a_j}^{\infty} \mathrm{d}\omega \frac{f_j(\omega)}{x-\omega\pm i\epsilon}|\omega\rangle_j\right]$$

$$\langle\tilde{\Psi}_{i\pm}(x)| = {}_i\langle x| + \frac{\lambda_i f_i(x)}{\eta^{\mp}(x)}\left[\langle 1| + \sum_{j=1,2}\lambda_j \int_{a_j}^{\infty} \mathrm{d}\omega \frac{f_j^*(\omega)}{x-\omega\mp i\epsilon}\ {}_j\langle\omega|\right]$$

$$\eta_d^{\pm}(\omega) \equiv \eta^{\pm}(\omega) \prod_{J=II,III,IV}\prod_{i=1}^{N_J} \frac{\omega - z_i^J}{[\omega - z_i^J]_{\pm}}\,.$$

$$\sum_{i=1,2}\int_{a_i}^{\infty} dx|\Psi_i^d(x)\rangle\langle\tilde{\Psi}_i(x)| + \sum_{J,i}|z_{0,i}^J\rangle\langle\tilde{z}_{0,i}^J| = \mathbf{1}$$

Generalization

To use this model in the real world

- Partial wave decomposition.
- Include more discrete states.
- Include interaction among continuum.

Non-relativistic partial wave decomposition

A general Hamiltonian, in the c.m. frame, a discrete particle with spin l, $|0; l l_z\rangle$ interacting with a two-particle continuum state with total spin S, $|\vec{p}; S S_z\rangle$

$$H = H_0 + V$$

$$H_0 = M_0 \sum_M |0; JM\rangle\langle 0; JM| + \sum_{S_z} \int \mathrm{d}^3 p\, \omega |\vec{p}; S S_z\rangle\langle \vec{p}; S S_z| \,,$$

$$\langle 0; JM|\, V |\vec{p}; S S_z\rangle = \sum_{lm} i^l \tilde{g}_l(p^2)\, C^{JM}_{lm, S S_z}\, Y_l^{m*}(\hat{p})$$

l is the orbital angular momentum. The continuum momentum eigenstate can be expanded using the angular momentum eigenstates,

$$|\vec{p}; S S_z\rangle = \sum_{lm} i^l\, Y_l^{m*}(\hat{p}) |p; lm, S S_z\rangle = \sum_{JM, lm} i^l\, Y_l^{m*}(\hat{p})\, C^{JM}_{lm, S S_z} |p; JM; lS\rangle$$

Non-relativistic partial wave decomposition

In angular momentum eigenstates

$$H_0 = M_0|0;JM\rangle\langle 0;JM| + \sum_l \int p^2 \mathrm{d}p\, \omega |p;JM;lS\rangle\langle p;JM;lS|\,,$$

$$H_{01} = \sum_l \int \mu p \mathrm{d}\omega \tilde{g}_l(p^2)|0;JM\rangle\langle p;JM;lS| + h.c.$$

We suppose the system conserves J and M, and restrict to such subspace. Redefine

$$|\omega,l\rangle = \sqrt{\mu p}|p;JM;lS\rangle\,, \quad |0\rangle = |0;JM\rangle\,, \quad g_l(\omega) = \sqrt{\mu p}\tilde{g}_l(p^2)\,,$$

$$H = M_0|0\rangle\langle 0| + \sum_l \int \mathrm{d}\omega\, \omega|\omega,l\rangle\langle\omega,l| + \sum_l \int d\omega\, g_l(\omega)|0\rangle\langle\omega,l| + h.c.$$

The simplest Friedrichs model.

Including direct continuum interaction

Including two kinds of continuum, spins S_1 and S_2

$$\langle \vec{p}' S_2 S_{2z} | V | \vec{p} S_1 S_{1z} \rangle$$
$$= \sum_{JM l_1' m_1' l_2' m_2'} (-i)^{l_2'} Y_{l_2'}^{m_2'}(\hat{p}') C^{JM*}_{l_2' m_2', S_2 S_{2z}} i^{l_1'} Y_{l_1'}^{m_1'*}(\hat{p}) C^{JM}_{l_1' m_1', S_1 S_{1z}} \tilde{f}^{JM}_{l_2' S_2, l_1' S_1}(p'^2, p^2)$$

$$\tilde{f}^{JM}_{l_2' S_2, l_1' S_1}(p'^2, p^2) = \langle p' JM; l_2' S_2 | V | p, JM; l_1' S_1 \rangle$$

Redefine:

$$|\omega, l_i; i\rangle = \sqrt{\mu_i p} |p; JM; l_i S_i\rangle \,, \text{ for } i = 1, 2 \,,$$
$$f^{(21)}_{l_2 l_1}(\omega', \omega) = \sqrt{\mu p \mu' p'} \tilde{f}^{JM}_{l_2 S_2, l_1 S_1}(p'^2, p^2) \,.$$

Suppose conservation of the total angular momentum JM. Restrict to subspace with fixed JM. The interaction Hamiltonian then becomes

$$H^{JM}_{21} = \sum_{l_2, l_1} \int \mathrm{d}\omega' \, \mathrm{d}\omega \, f^{(21)}_{l_2 l_1}(\omega', \omega) |\omega', l_2; 2\rangle \langle \omega, l_1; 1| + h.c.$$

D DISCRETE STATES AND C CONTINUUM STATES

We can include D discrete states and C continuum states, and restrict the discussion in a fixed JM channel. Relabel the states with different (l_i, i) using consecutive $1, 2, 3 \ldots$.

$$H = \sum_{i=1}^{D} M_i |i\rangle\langle i| + \sum_{i=1}^{C} \int_{M_{i,th}}^{\infty} d\omega_i \omega_i |\omega_i; i\rangle\langle\omega_i; i|$$
$$+ \sum_{i_2, i_1} \int_{M_{i_1,th}} \mathrm{d}\omega' \int_{M_{i_2,th}} \mathrm{d}\omega\, f_{i_2,i_1}(\omega', \omega) |\omega'; i_2\rangle\langle\omega; i_1|$$
$$+ \sum_{i=1}^{D} \sum_{j=1}^{C} \int_{M_{j,th}} d\omega\, g_{i,j}(\omega) |i\rangle\langle\omega; j| + h.c.$$

We get the most general Friedrichs-like model.

Solution: A special kind of form factor

For general form factors, the generlized Friedrichs model is not solvable. If the form factor is factorizable like in [E. Hernández et.al, PRC29(1984),722;Aceti et.al., PRD86,(2012),014012;Sekihara, PTEP(2015)063D04]:

$$g_{ij}(\omega) = u_{ij}^* f_j^*(\omega)\,, f_{j'j}(\omega',\omega) = v_{j'j} f_{j'}(\omega') f_j^*(\omega)$$

$$\begin{aligned} H = &\sum_{i=1}^{D} M_i |i\rangle\langle i| + \sum_{i=1}^{C} \int_{a_i}^{\infty} \mathrm{d}\omega\, \omega |\omega; i\rangle\langle\omega; i| \\ &+ \sum_{i,j=1}^{C} v_{ij} \Big(\int_{a_i}^{\infty} \mathrm{d}\omega f_i(\omega) |\omega; i\rangle \Big) \Big(\int_{a_j}^{\infty} \mathrm{d}\omega f_j^*(\omega) \langle\omega; j| \Big) \\ + &\sum_{j=1}^{D} \sum_{i=1}^{C} \Big[u_{ji}^* |j\rangle \Big(\int_{a_i}^{\infty} \mathrm{d}\omega f_i^*(\omega) \langle\omega; i| \Big) + u_{ji} \Big(\int_{a_i}^{\infty} \mathrm{d}\omega f_i(\omega) |\omega; i\rangle \Big) \langle j| \Big] \end{aligned}$$

then it is solvable. v_{ij} hermitian.

Discrete Solutions:

Define $G_j(E) \equiv \int_{a_j} \mathrm{d}\omega \frac{|f_j(\omega)|^2}{\omega - E}, V_{ij} \equiv v_{ij} - \sum_{l=1}^{D} \frac{u_{lj}^* u_{li}}{M_l - E}$,
$M_{ji} \equiv \delta_{ji} + G_j(E)\, V_{ji}$, for $i = 1, \cdots, C$.

- Discrete state: pole positions are determined by $\det M(E) = 0$, discrete eigenvalues $\tilde{E}_k$.

$$|\Psi(\tilde{E}_k)\rangle = \sum_{i=1}^{C} A_i(\tilde{E}_k)\Big(- \sum_{l=1}^{D} \frac{u_{li}^*}{M_l - \tilde{E}_k} |l\rangle + \sum_{j=1}^{C} G_i^{-1}(\tilde{E}_k) \int_{a_i} \mathrm{d}\omega \frac{f_i(\omega)}{\omega - \tilde{E}_k} |\omega; i\rangle \Big) .$$

  A_i are vectors satisfying $\sum_j M_{ij}(\tilde{E}_k) A_j(\tilde{E}_k) = 0$.

- $M_{ij}^*(E) = M_{ji}(E^*)$, solutions should be symmetric w.r.t the real axis.

Discrete Solutions:

- If the solution is on the real axis below the threshold E_B, it is a bound state. Normalization: $\sum Z_l + \sum_i X_i = 1$

$$Z_l = |\alpha_l(E_B)|^2 = \frac{\sum_{ij} A_i u_{li}^* u_{lj} A_j^*}{(M_l - E_B)^2},$$

$$X_i = |\psi_i(E_B)|^2 = G_i'(E_B) \sum_{jj'} A_j A_{j'}^* V_{ij'}^* V_{ij}$$

Z_l: elementariness, X_i compositeness.

- If the solutions are on the complex plane: Resonances. $\tilde{E}_k$ complex, on unphysical sheets. We have the left eigenstate $\langle \tilde{\Psi}(\tilde{E}_k)|$ with the same eigenvalue $\tilde{E}_k$. Normalization:

$$\langle \tilde{\Psi}(\tilde{E}_k)|\Psi(\tilde{E}_k)\rangle = \sum Z_l + \sum_i X_i = 1.$$

Z_l and X_i become complex.

Continuum solutions:

- Solution with energy eigenvalue $E > a_k$

$$|\Psi_\pm^{(k)}(E)\rangle = \gamma_k(E)\Big[|E;k\rangle - f_k^*(E)\sum_{j=1}^{C}(M_\pm^{-1})_{jk} \times \Big(-\sum_{i=1}^{C} V_{ij}\int \mathrm{d}\omega \frac{f_i(\omega)}{E-\omega\pm i\epsilon}|\omega;i\rangle + \sum_{l=1}^{D}\frac{u_{lj}^*}{M_l - E}|l\rangle\Big)\Big]$$

- $|\Psi_+(E)\rangle$ is the in-state, and $|\Psi_-(E)\rangle$ is the out-state.
- S matrix:

$$S_{k',k}(E',E) = \langle\Psi_-^{(k')}(E')|\Psi_+^{(k)}(E)\rangle = \delta(E'-E) - 2\pi i\delta(E'-E)f_{k'}(E)f_k^*(E)(V^{-1}+G_+)^{-1}_{k'k}$$

Eg.: Dynamically generated states

- No discrete bare states $\rightarrow$ dynamically generated discrete state — Bound state (molecular state), resonances, or virtual state.
- Hamiltonian:

$$H = \int_a \mathrm{d}\omega\, \omega |\omega\rangle\langle\omega| \pm \lambda^2 \int_a \mathrm{d}\omega \int_a \mathrm{d}\omega' f(\omega) f^*(\omega') |\omega\rangle\langle\omega'| \quad (1)$$

- Form factor
  $f(\omega) = (\omega - a)^{(l+1/2)/2} \exp\{-(\omega - a)/(2\Lambda)\}$.
- Discrete state pole position:

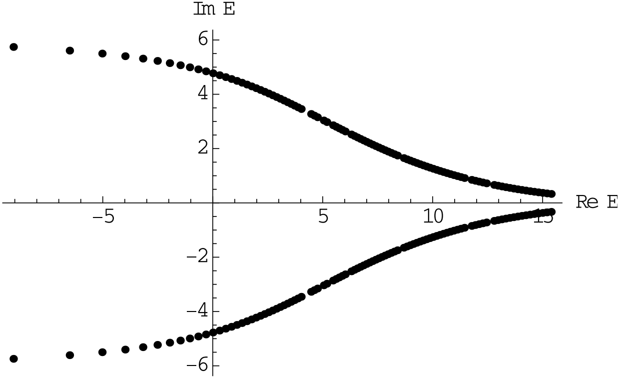
$$\mathbf{M}_\pm(E) = \det\ M_\pm = 1 \pm \lambda^2 G(E) = 1 \pm \lambda^2 \int_a \mathrm{d}\omega \frac{|f(\omega)|^2}{\omega - E} = 0$$

- $\pm$ sign: attractive or repulsive in the long range.

Eg: S-wave dynamically generated states

S-wave: repulsive potential.

- No solution on the first sheet for $\mathbf{M}_+ = 0$: No bound state.
- There are a pair of resonance poles on the second sheet:
- As the coupling is turning down, the poles move to the negative infinity.

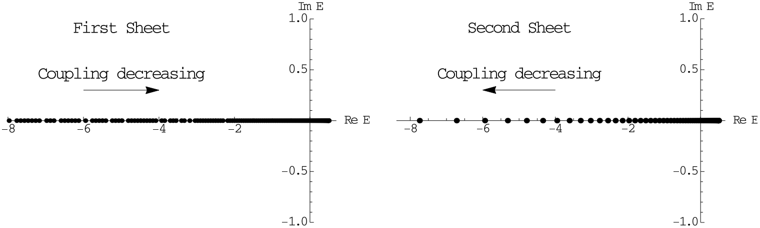


Threshold $a = 0.5$, $\Lambda = 2$.

Eg: S-wave dynamically generated states

S-wave: attractive potential.

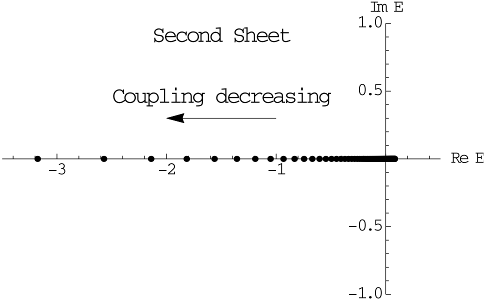
- Strong coupling: a solution on the first Riemann sheet — bound state.
- Weak coupling : The bound state moves to the second sheet — virtual state.
- As the coupling is turning off, the virtual pole moves to the negative infinity.



Eg: P-wave dynamically generated states

P-wave: repulsive potential.

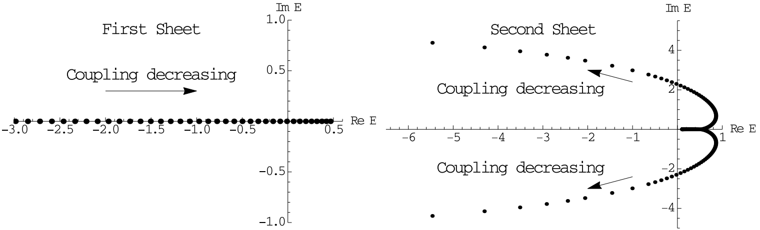
- No bound state.
- There is a virtual state on the second sheet.
- As the coupling decreases, it moves to the negative infinity.
- Strong coupling limit of the pole position: $G_P^{II}(E) = 0$.



Eg: P-wave dynamically generated states

P-wave: attractive potential.

- Strong coupling: A bound state and a virtual state. Virtual state has a limiting point in the strong coupling limit, determined by $G_P^{II}(E) = 0$.
- Weak coupling: The bound state and the virtual state move to the threshold and to the second sheet — a pair of resonance poles.
- Bound state and an accompanied virtual state: Threshold effect.



Difference between the dynamically generated states by attractive potential and the states from the bare state:

- S-wave: The dynamically generated bound state do not need to have acompanied virtual state.

  Higher partial wave, no such difference: The dynamically generated states if appear from the theshold, it must acompanied with a virtual state.
- In weak coupling limit: The dynamically generated states do not go to bare states (but go to the complex plane in our example).

Summary:

- Friedrichs model in single channel and coupled channels: exactly solvable model.
- Wave function for bound state, virtual state, and Resonances can be solved in terms of the bare states.
- Dynamically generated poles and poles generated from bare states: different properties.
- Probability explanation: Compositeness and elementariness. Good for bound state, not for resonances and virtual state.
- Strong-weak duality?
  The perturbative quantum field theory can not use a fundamental field to describe the S-wave dynamically generated states having no accompanied virtual states.

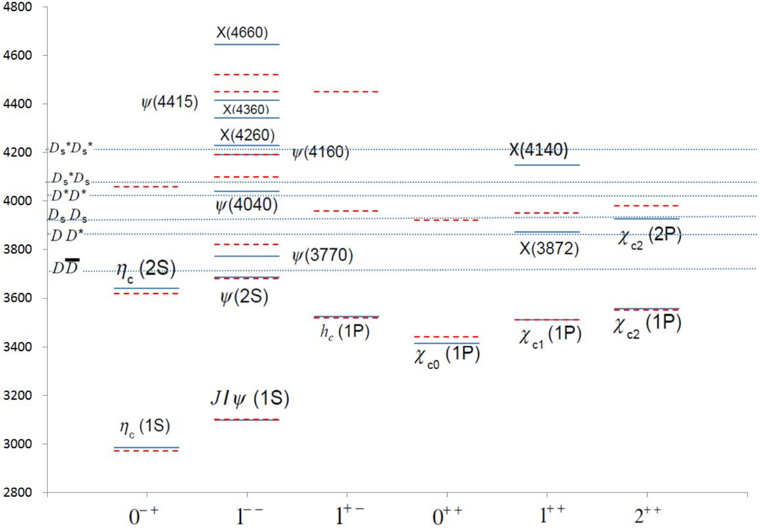
Friedrichs model in P-wave first excited charmonium

Introduction

Recent years, more and more new X, Y, Z states were found which can not be satisfactorily explained by the naive quark model.

- The quark model such as Godfrey-Isgur can describe the hadron spectrum below the open flavor thresholds. Such as J/ψ, $h_c(1P)$, $\chi_{c0,1,2}(1P)$.
- But for resonances above the open flavor thresholds, general descrepancies are found between the quark model and the experimental observation.
- There are also charged exotic states which do not present in the quark model: $Z_c(3900)$, $Z_c(4020)$

Charmonium-like states



red ones are GI's prediction.

GI's potential model

Godfrey-Isgur's relativized quark potential model, [PRD32,189,(1985)]

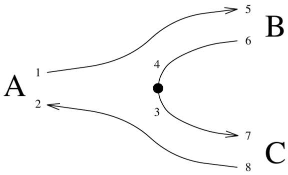
- A relativized quark potential model:

$$H_{GI} = H_0 + V = H_0 + \tilde{H}_{ij}^{\rm conf} + \tilde{H}_{ij}^{\rm hyp} + \tilde{H}_{ij}^{\rm so} + H_A$$

- By diagnalizing the meson-meson matrix $\langle \mathrm{Meson}_i | H | \mathrm{Meson}_j \rangle$, the mass spectra are obtained.
- However, the hadron loop effects in the propagator from the interactions are not included in the spectra. This is the main reason for the discrepancy of the GI's results and the experimentsal ones above the open flavor thresholds.
- We will use Friedrichs model, combined with QPC (3P0) model, to incorporate this effect into the spectra.
- We then analyze the spectra of $2P$ charmonium-like states using this method.

Friedrichs Model

- If one has the interaction between the discrete states (charmonium) and the continuum states f_i (open flavor threshold), then the eigenstates for the full Hamiltonian are obtained— the masses, widths and wave functions for the states.
- The interactions can be estimated using differnent models: we will use the QPC (3P0) model.



QPC MODEL

- The meson coupling $A \to BC$ can be defined using the transition matrix element

$$\langle BC|T|A\rangle = \delta^3(\vec{P}_f - \vec{P}_i)M^{ABC}$$

where the transition operator T is the one in the QPC model

$$T = -3\gamma \sum_m \langle 1m1-m|00\rangle \int d^3\vec{p_3}\, d^3\vec{p_4}\, \delta^3(\vec{p_3}+\vec{p_4}) \times \mathcal{Y}_1^m\left(\frac{\vec{p_3}-\vec{p_4}}{2}\right)\chi^{34}_{1-m}\phi^{34}_0\omega^{34}_0 b^\dagger_3(\vec{p_3})\, d^\dagger_4(\vec{p_4}).$$

γ parameterize the strength of creating a quark-antiquark pair from the vacuum which is fixed to be a typical standard value [Godfrey & Isgur, PRD32,189(1985)].

- The wave function of A, B, C are GI's results.

After we have the interaction between the discrete states and the continuum states, we can use the solution to the Friedrichs model to find out the different eigenstates of the full Hamiltonian.

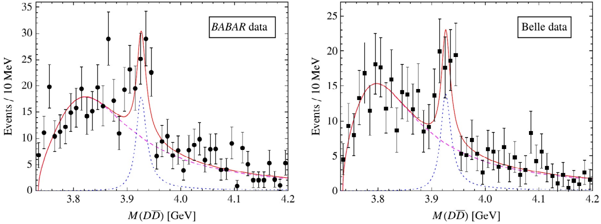
$2P$ Charmonium-like states

$L=1,\ S=1,\ J^{PC}=0,1,2^{++}$,$\chi_{c0,1,2}$; $L=1,\ S=0,\ J^{PC}=1^{+-}$, $h_c(2P)$.

Current status:

- 2^3P_2 is well established: $X(3930)$, [Belle,PRL96,082003;BaBar,PRD81,092003]
- 2^3P_1 channel: $X(3872)$ [Belle,PRL91,262001;] molecular state or $c\bar{c}$? Mixture of molecule and $c\bar{c}$, and which is the dominant component?
- If $X(3872)$ is a dynamically generated molecular state, where is the charmonium $\chi_{c1}(2P)$ state.
- 2^1P_1 channel: The $h_c(2P)$ state still has not been seen by experiments.

- 2^3P_0 channel: $X(3915)$ [Belle,PRL104,092001;BaBar,PRD86,072002] used to be assigned to χ_{c0}. However there are some arguement against this assignment [Guo,Meissner,PRD,86,091501; Olsen,PRD91,057501].



  The real χ_{c0} state may lie far below, around 3860, and may be a wide one.
- A reanalysis of the BABAR and Belle Data shows that the 2^{++} assignment to $X(3915)$ can not be excluded and it may possibly be the same tensor state as $X(3930)$ [PRL115,022001].
- In 2017, Belle's also found a possible alternative $\chi_{c0}(2P)$ candidate around 3860MeV ($M = 3862^{+66}_{-45}$, $\Gamma = 201^{+242}_{-149}$). [PRD95,112003].

Our scheme

- Using the masses and wave functions from the Godfrey-Isgur model for the bare states in the Friedrichs model and the QPC model.
- Bare discretes states: $\chi_{c0}(2P)$ at 3917 MeV, $\chi_{c1}(2P)$ at 3953 MeV, $\chi_{c2}(2P)$ at 3979 MeV. $h_c(2P)$ at 3956MeV.
- Continuum states: $D\bar{D}$, $D\bar{D}^*$, $D^*\bar{D}^*$ threshold, upto D-wave.

  Coupled channels:
  $\chi_{c0}(2P)$: $D\bar{D}$ (S-wave), $D^*\bar{D}^*$ (S-wave, D-wave).
  $\chi_{c1}(2P)$: $D\bar{D}^*$(S,D-wave), $D^*\bar{D}^*$ (D-wave)
  $\chi_{c2}(2P)$: $D\bar{D}$ (D-wave), $D\bar{D}^*$ (D-wave), $D^*\bar{D}^*$ (S,D-wave)
  $h_c(2P)$: to $D\bar{D}^*$ (S,D-wave), $D^*\bar{D}^*$ (S,D-wave).
- Parameterize the interaction between the bare states and the continuum using the QPC model.

Our scheme

Try to understand the mass spectrum and width using our method

- Solution of $\eta(z) = 0$ gives the mass and width of the resonances $z_R = M + i\Gamma/2$, or masses of the bound states.
- We also give a Breit-Wigner mass for narrow resonances to compare with the experimental results, i.e. from Real part of $\eta(z) = 0$,

$$M_{BW} - \omega_0 - \lambda^2 \sum_n \mathcal{P} \int_{\omega_{th,n}}^{\infty} \frac{\sum_{SL} |f_{SL}^n(\omega)|^2}{M_{BW} - \omega} \mathrm{d}\omega = 0,$$

$$\Gamma_{BW}^n = 2\pi \sum_{S,L} |f_{SL}^n(M_{BW})|^2,$$

Numerical results

Table: Comparison of the experimental masses and the total widths (in MeV) [PDG2016] with our results.

$n^{2s+1}L_J$	M_{expt}	Γ_{expt}	M_{BW}	Γ_{BW}	pole	GI
2^3P_2	3927.2 ± 2.6	24 ± 6	3910	12	3908-5i	3979
2^3P_1	3942 ± 9	37^{+27}_{-17}			3917-45i	3953
	3871.69 ± 0.17	< 1.2	3871	0	3871-0i	
2^3P_0	3862^{+66}_{-45}	201^{+242}_{-149}	3860	25	3861-11i	3917
2^1P_1			3890	26	3890-22i	3956

Numerical results

- The there is a narrow 2^3P_2 state which can be assigned to the well-established χ_{c2}.
- The 2^3P_0 state is found to be around 3860 MeV, which is consistent with mass of the experimental reanalysis of the Belle data. Our result prefers an unexpected narrow width, whereas the experimental result has a large uncertainty 201^{+242}_{-149} MeV.

  There are also other predictions with small width, [Barnes et.al., PRD72,054026;Eichten et.al, PRD69,094019]

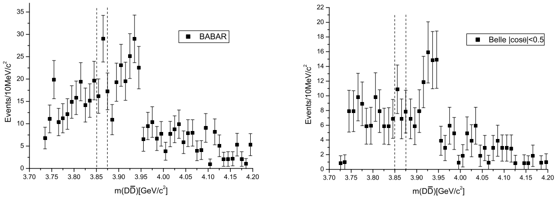


Figure: $\gamma\gamma \to D\bar{D}$ from BABAR [PRD81,092003] and Belle [RPL,96,082003]. The two dashed lines are set at $m(D\bar{D}) = 3850$ MeV and 3875 MeV.

Numerical results

(2^3P_1) : $X(3872)$ & χ_{c1}

- A dynamically generated bound state , located just at 3871 is naturally assigned to $X(3872)$.
- The $X(3872)$ is not originated from the bare state χ_{c1}, but from the interaction between the bare state and the continuum.
- If we reduce the γ parameter, the $X(3872)$ pole will move to the second sheet becoming a virtual state.
- The bare state pole is shifted to about 3917 MeV with a large width — may be related to $X(3940)$ observed by the experiment.
- $X(3872)$:

$$\frac{\text{elementariness}}{\text{compositeness}} = 1 : 2.7.$$

A large portion of continuum state $D\bar{D}^*$ — more molecular component than the $c\bar{c}$ component.

Numerical results

2^1P_1: a prediction of h_c,

- Mass: shifted from GI's result 3956 to 3890.
- $J^{PC} = 1^{+-}$: need a negetive C-parity channel to look for it, such as $\eta_c\gamma$, $J/\psi\eta$.

Summary

- We study the first excited P-wave charmonium-like states in a Friedrichs-like model method. This method can also be generalized to study other states above the open flavor states.
- The $X(3872)$ is naturally dynamically generated by the interaction of χ_{c1} state and the continuum states. The continuum components constitute a larger portion of the $X(3872)$ than the $c\bar{c}$.
- With the compositeness and the elementariness, the further properties of $X(3872)$ can also be studied which is a work in progress.
- In our scheme, the χ_{c0} state is a narrow one around 3860.
- We also give a prediction of the position and width of the $h_c(2P)$.

Thank you !