Entanglement Entropy in $T\overline{T}$-deformed CFT

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Integrable currents

The $T\overline{T}$-deformed CFT belongs to a large class of integrable deformations Smirnov and Zomolodchikov 1608.05499

Define

$$T = -2\pi T_{zz}, \qquad \overline{T} = -2\pi T_{\bar{z}\bar{z}}, \qquad \Theta = 2\pi T_{z\bar{z}}$$

Assumption 1: Local translation and rotational symmetry

$$T^{\mu\nu} = T^{\nu\mu}, \qquad \partial_\mu T^{\mu\nu} = 0.$$

This leads to

$$\partial_{\bar{z}} T = \partial_z \Theta, \qquad \partial_z \overline{T} = \partial_{\bar{z}} \Theta.$$

These are the integrability conditions. Consequently

$$P_+ = \oint_C (T dz + \Theta d\bar{z}), \qquad P_- = \oint_C (\overline{T} d\bar{z} + \Theta dz),$$

are conserved currents.

$T\overline{T}$ deformations

Assumption 2: Global translation symmetry

$$\partial_z < \mathcal{O}_i(z) >= 0.$$

Assumption 3: Infinite separations along one direction

$$\lim_{t\to\infty} \langle \mathcal{O}_i(z+t)\mathcal{O}_j(z')\rangle = \langle \mathcal{O}_i\rangle\langle \mathcal{O}_j\rangle$$

Under these assumptions, one finds that

$$\lim_{z\to z'} (T(z)\overline{T}(z') - \Theta(z)\Theta(z')) = \mathcal{O}_{T\overline{T}}(z') + \text{derivative terms}.$$

where $\mathcal{O}_{T\overline{T}}$ is a local operator.
The so-called $T\overline{T}$ deformation is actually the one

$$T\overline{T}(z) \equiv \mathcal{O}_{T\overline{T}}(z)$$

In a CFT, there is $\Theta = 0$.

$T\overline{T}$-deformed theory

Now we consider a 2D CFT deformed by $T\overline{T}$ operator

$$S^{(\mu)} = S_{CFT} + \mu \int_{\mathcal{M}} \mathcal{O}_{T\overline{T}}.$$

The $T\overline{T}$ operator is an irrelevant operator, and the parameter μ is of the dimension $(\text{Length})^2$. The deformation may lead to a RG flow to the UV. However, it is not necessary to think in this way.

Simply speaking, the deformation defines a one-parameter family of QFTs.

Zamolodchikov equation

$$\langle T\overline{T}\rangle = \langle T\rangle\langle\overline{T}\rangle - \langle\Theta\rangle^2$$

Consider the function

$$\Xi(z,w) \equiv \langle T(z)\overline{T}(w) - \Theta(z)\Theta(w)\rangle.$$

In two different limits:

$$\begin{aligned}\lim_{w\to z}\Xi(z,w) &= \langle T\overline{T} - \Theta^2\rangle \\ \lim_{w\to\infty}\Xi(z,w) &= \langle T\rangle\langle\overline{T}\rangle - \langle\Theta\rangle^2\end{aligned}$$

As $\Xi(z,w)$ is a constant, we get the Zomolodchikov equation.

Spectrum

One remarkable fact is that the deformed theory has the same spectrum structure as the original one.
Consider a CFT on a cylinder: a spatial circle with a period L

$$
\begin{aligned}
P_n &= \frac{2\pi}{L} J_n, \qquad J_n \in Z, \\
E_n &= \frac{\mathcal{E}(\mu/L^2)}{L}.
\end{aligned}
$$

For a CFT,

$$
\begin{aligned}
E_n^{CFT} &= \frac{\mathcal{E}(\mu = 0)}{L} = \frac{2\pi}{L} M_n, \\
J_n &= \Delta_n - \overline{\Delta}_n.
\end{aligned}
$$

with

$$
M_n = \Delta_n + \overline{\Delta}_n - \frac{c}{12}.
$$

Spectrum II

Inserting a complete basis of states of CFT into the Zomolodchikov equation, we find that

$$4\frac{\partial E_n}{\partial \mu} + E_n \frac{\partial E_n}{\partial L} + \frac{P_n^2}{L} = 0,$$

which is the forced inviscid Burgers equation. The solution is

$$E_n(\mu, L)L = \mathcal{E}_n(\tilde{\mu}) = \frac{2\pi}{\tilde{\mu}}\left(1 - \sqrt{1 - 2\tilde{\mu}M_n + \tilde{\mu}^2 J_n^2}\right),$$

with

$$\tilde{\mu} = \frac{\pi\mu}{L^2}.$$

For each state in the original CFT, there is a corresponding state in the deformed CFT, with a modified spectrum.

$\tilde{\mu} < 0$ case

We have been working in the Euclidean theory. In the Minkowski theory, we need to

$$\tilde{\mu} \to -\tilde{\lambda}.$$

For a negative $\tilde{\mu}$ or a positive $\tilde{\lambda}$, the spectrum is now

$$E_n(\lambda, L)L = \mathcal{E}_n(\tilde{\mu}) = \frac{2\pi}{\tilde{\lambda}}\left(\sqrt{1 + 2\tilde{\lambda}M_n + \tilde{\lambda}^2 J_n^2} - 1\right),$$

It has a vacuum: $\Delta_n = \overline{\Delta}_n = 0, M_n = -\frac{c}{12}$.
For simplicity, we set $J_n = 0$,

$$\mathcal{E}(\tilde{\lambda}) = 2\pi\left(-\frac{1}{\tilde{\lambda}} + \sqrt{\frac{1}{\tilde{\lambda}^2} + \frac{2M}{\tilde{\lambda}}}\right)$$

There are large number of states with $|M| << 1/\tilde{\lambda}$, the energy is little modified by the perturbation $\mathcal{E} \simeq M$.
As $M\tilde{\lambda}$ becomes larger, the deviation increases.
If $|M\tilde{\lambda}| >> 1$, then

$$\mathcal{E}(\tilde{\lambda}) \simeq 2\pi\sqrt{\frac{2M}{\tilde{\lambda}}},$$

which is L-independent.

Entropy and LST

In a 2D CFT, the degeneracy of the highly excited states is captured by the Cardy entropy

$$S_C(M) = 2\pi\sqrt{\frac{c}{3}M}.$$

Correspondingly, we have

$$S_C(\mathcal{E}) = 2\pi\sqrt{\frac{c}{6}(2\mathcal{E} + \tilde{\lambda}\mathcal{E}^2)}$$

If $1 << \mathcal{E} << 1/\tilde{\lambda}$, we still have Cardy entropy.
However, if $\mathcal{E} >> 1/\tilde{\lambda}$, we have the Hagedorn behavior

$$S_H \simeq 2\pi\sqrt{\frac{\tilde{\lambda}c}{6}\mathcal{E}}.$$

This indicates the intriguing relation with the Little string theory. A. Giveon et.al. 1701.05576

Modular invariance

Consider the theory is defined on a torus with modular parameter τ. It was found that the partition function of the $T\overline{T}$-deformed theory obeys the modular property Datta and Jiang, 1806.07426, Aharony et.al. 1808.02492

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d} \mid \frac{\tilde{\lambda}}{|c\tau+d|^2}\right) = Z(\tau,\bar{\tau}|\tilde{\lambda}),$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z).$$

In other words $\tilde{\lambda}$ transforms as the modular form of weight $(-1,-1)$. With the boundary condition

$$Z(\tau,\bar{\tau}|\tilde{\lambda}=0) = Z_{CFT}(\tau,\bar{\tau})$$

This allows us to determine the spectrum completely to all orders of $\tilde{\lambda}$. No non-trivial non-perturbative effects.

$\tilde{\mu} > 0$ case

Let us focus on the case $\tilde{\mu} > 0$.

$$\mathcal{E}(\tilde{\mu}) = \frac{2\pi}{\tilde{\mu}} \left(1 - \sqrt{1 - 2\tilde{\mu} M_n + \tilde{\mu}^2 J_n^2}\right),$$

In this case, there is a shock singularity. Consider the case $J_n = 0$, then there must be a high energy cutoff

$$\tilde{\mu} M_n \leq \frac{1}{2}.$$

In other words, the spectrum must be truncated.
Open question: UV completion?

AdS_3 gravity in a finite region

Consider the holographic CFT which is dual to the AdS_3 gravity, under the $T\overline{T}$ deformation

$$S^{(\mu)} = S_{CFT} + \mu \int_{\mathcal{M}} \mathcal{O}_{T\overline{T}}.$$

The deformed CFT is conjectured to be dual to a gravity on a compact subregion of AdS_3 L. McGough, M.Mezei and H. Verlinde, 1611.03470

$$ds^2 = \frac{dr^2}{r^2} + r^2 g_{\alpha\beta} dx^\alpha dx^\beta, \qquad r < r_c,$$

with

$$r_c^2 = \frac{6 l^4}{\pi c \mu}.$$

One needs to impose the Dirichlet B.C.

$$\begin{aligned} ds^2|_{r=r_c} &= r_c^2 g_{\alpha\beta} dx^\alpha dx^\beta, \\ \phi_i|_{r=r_c} &= 0. \end{aligned}$$

Caution

A finite Dirichlet cutoff induces dramatic deformation of the gravitational theory.

The cutoff surface plays the role of a mirror. It induces negative image masses on the other side, which screen the gravitational force. This may lead to the violation of causality.
In other words, the theory cannot be UV completed as an ordinary QFT.

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There could be other issues on the Dirichlet cutoff:
Lack of strong ellipticity in Euclidean quantum gravity. Consequently, there exists infinite number of zero modes, and there exists no trace of the heat-kernels. I.G. Avramidi and G. Esposito 9708163
No well-defined perturbation theory. E. Witten 1805.11559

Pieces of Evidence

The are a few pieces of evidence to support Verlinde et.al.'s proposal in the large c limit: $c \to \infty, \quad \mu c$ fixed.

1. Deformation of the lightcone $\sim$ Superluminal propagating modes
   J. Cardy 1507.07266, D. Marolf and M. Rangamani, 1201.1233
2. Deformed energy spectrum $\sim$ The quasi-local energy of a BTZ BH of mass M and angular momentum J
   The truncation of the spectrum $\sim r_c \to r_H$.
3. Exact RG flow equation $\sim$ Hamilton-Jacobi equation

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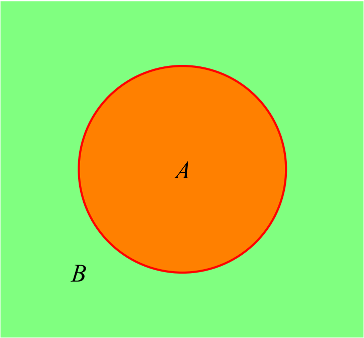
Here we would like to present another piece of evidence from the study of the entanglement entropy.

Entanglement entropy

Entanglement entropy is an important notion in quantum world. It has played an important role in understanding AdS/CFT in the past decade.



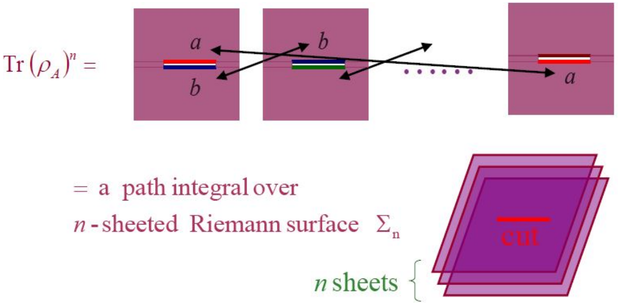
For A and its complement B

- $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- Reduced density matrix: $\rho_A = \mathrm{tr}_B \rho_{tot}$
- Entanglement entropy $S_A = -\mathrm{tr}_A \rho_A \ln \rho_A$
- Rényi entropy $S_A^{(n)} = -\frac{\ln \mathrm{tr}_A \rho_A^n}{n-1}$
- $S_A = \lim_{n\to 1} S_A^{(n)}$

Replica trick

In order to compute $\mathrm{tr}_A \rho_A^n$, one can use the replica trick J. Callan et.al. 9401072

For a 2D QFT, we have the picture Figures from T. Takayanagi's lecture



Replica trick II

The entanglement entropy is given by

$$S(A) = \lim_{n\to 1} S_n(A), \quad S_n(A) = \frac{1}{1-n} \log \frac{Z_n(A)}{Z(A)^n},$$

where $Z(A)$ is the partition function on $\mathcal{M}$, $Z_n(A)$ is the partition function on the manifold $\mathcal{M}^n(A)$ which is obtained by gluing n copies of $\mathcal{M}$ together along A.

For a 2D CFT, the Rényi entropy for one interval could be read P. Calabrese and J.L. Cardy 0405152

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right) \log \frac{\ell}{\epsilon},$$

and

$$S(A) = \frac{c}{3} \log \frac{\ell}{\epsilon},$$

Perturbative framework

For the deformed CFT,

$$S = S_{CFT} + \mu \int_{\mathcal{M}} T\overline{T},$$

let's compute the partition function perturbatively in μ.

$$\frac{Z_n(A)}{Z^n} = \left(\frac{\int_{\mathcal{M}^n} e^{-S_{CFT}}}{\left[\int_{\mathcal{M}} e^{-S_{CFT}}\right]^n} \right) \left(1 - n\mu \int_{\mathcal{M}} \left[\langle T\overline{T} \rangle_{\mathcal{M}^n} - \langle T\overline{T} \rangle_{\mathcal{M}} \right] + O(\mu^2) \right).$$

Then we can read the leading order correction to $S_n(A)$

$$\delta S_n(A) = \frac{-n\mu}{1-n} \int_{\mathcal{M}} \left[\langle T\overline{T} \rangle_{\mathcal{M}^n} - \langle T\overline{T} \rangle_{\mathcal{M}} \right].$$

Finite temperature case

The first case is a $2D$ deformed CFT at a finite temperature $1/\beta$. The spatial direction is not compactified and the manifold $\mathcal{M}$ on which the theory is defined is an infinitely long cylinder with circumference β.

$$S(A) = S_0(A) + \delta S(A).$$

where

$$\begin{aligned} S_0(A) &= \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon_0}\sinh\left(\frac{\pi\ell}{\beta}\right)\right), \\ \delta S(A) &= \frac{-\mu\pi^4 c^2 \ell \coth\left(\frac{\pi\ell}{\beta}\right)}{9\beta^3} \\ &\sim -(\tilde{\mu}c)\frac{\ell}{\beta}\coth\left(\frac{\pi\ell}{\beta}\right)c \end{aligned}$$

At low T, $\beta >> \ell$, $\delta S(A) \sim -(\tilde{\mu}c)c$.
At high T, $\beta << \ell$, $\delta S(A) \sim -\frac{\ell}{\beta}(\tilde{\mu}c)c$.

Rényi entropy

$$
\begin{aligned}
\delta S_n(A) \quad = \quad & -\frac{\pi^4 c^2 \ell \mu (n+1) \coth\left(\frac{\pi \ell}{\beta}\right)}{18 \beta^3 n} + \frac{\pi c^2 \mu (n-1)(n+1)^2}{576 n^3 \epsilon^2} \\
& -\frac{\pi^3 c^2 \mu (n-1)(n+1)^2 \left(\cosh\left(\frac{2\pi \ell}{\beta}\right) - 7\right) \operatorname{csch}^2\left(\frac{\pi \ell}{\beta}\right)}{864 \beta^2 n^3} \\
& +\frac{\pi^3 c^2 \mu (n-1)(n+1)^2 \coth^2\left(\frac{\pi \ell}{\beta}\right) \log\left(\frac{\beta \sinh\left(\frac{\pi \ell}{\beta}\right)}{2\pi \epsilon}\right)}{36 \beta^2 n^3}.
\end{aligned}
$$

When $n = 1$, only the first term survives, and it gives the leading order correction to the entanglement entropy. The second term diverges as $1/\epsilon^2$ and does not depend on β and ℓ. The third term does not depend on the cutoff ϵ and can have a finite contribution when $n \neq 1$. The last term has the form $\# \log\left(\frac{\beta \sinh\left(\frac{\pi \ell}{\beta}\right)}{2\pi \epsilon}\right)$, recalling that $\log\left(\frac{\beta \sinh\left(\frac{\pi \ell}{\beta}\right)}{2\pi \epsilon}\right)$ is the original entanglement entropy.

Finite size case

Another simple case is a $2D$ deformed CFT at zero temperature but with a finite size L. The spatial direction is now compactified, while the time direction is non-compact so the manifold $\mathcal{M}$ is still an infinitely long cylinder with circumference L.
Remarkably, the correction to the entanglement entropy is simply vanishing

$$\delta S(A) = 0.$$

The entanglement entropy is simply

$$S(A) = \frac{c}{3} \log \left(\frac{L}{\pi \epsilon_0} \sin \left(\frac{\pi \ell}{L} \right) \right),$$

with ϵ_0 the CFT cutoff.

Rényi entropy

$$
\begin{aligned}
\delta S_n(A) \quad = \quad & \frac{\pi c^2 \mu (n-1)(n+1)^2}{576 n^3 \epsilon^2} \\
& - \frac{\pi^3 c^2 \mu (n-1)(n+1)^2 \left(11 \cos\left(\frac{2\pi\ell}{L}\right) + 19\right) \csc^2\left(\frac{\pi\ell}{L}\right)}{864 L^2 n^3} \\
& + \frac{\pi^3 c^2 \mu (n-1)(n+1)^2 \cot^2\left(\frac{\pi\ell}{L}\right) \log\left(\frac{L \sin\left(\frac{\pi\ell}{L}\right)}{2\pi\epsilon}\right)}{36 L^2 n^3}.
\end{aligned}
$$

When $n = 1$, it is vanishing as we expect. Let us compare it with the finite T case: the quadratic divergent terms $(1/\epsilon^2)$ are the same, which is independent of the finite temperature or finite size; their logarithmic terms are the same under the identification $L \leftrightarrow i\beta$.

Remarks

Our results suggest that at the leading order of μ, the correction to the (Rényi) entanglement entropy takes a universal form, independent of the details of the CFT.

It holds for a generic CFT and no matter the parameter μ is positive or negative.

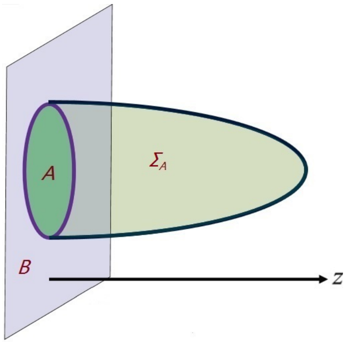
In the following, we would like to compare them with the holographic computation.

Holographic entanglement entropy

Ryu and Takayanagi(2006): Find a codimension two minimal surface Σ_A in the bulk that is homogeneous to A.

- The holographic entanglement entropy (for Einstein gravity)

$$S_A = \frac{\text{Area}(\Sigma_A)}{4G_N}$$



For the AdS_3 gravity, the minimal surface is simply a geodesic ending on the branch points.

HEE in the deformed theory

For the holographic CFT

$$c = \frac{3l}{2G},$$

one has to take the large c limit

$$c \to \infty, \quad \tilde{\mu} c \quad \text{fixed}$$

Considering the BTZ BH with radial truncation at $r = r_c$, the holographic EE is captured by the geodesic length via the RT formula

$$\frac{\lambda}{4G} \simeq S_0(\ell) - \frac{\pi^4 c^2 \mu \ell}{9\beta^3} \coth\left(\frac{\pi \ell}{\beta}\right)$$

Now the cutoff surface induce

$$\epsilon_0 = \frac{l^2}{r_c}.$$

The HEE is in exact agreement with the field theory result at finite T to the leading order of μ.

HEE in the deformed theory

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The HEE is in exact agreement with the field theory result at finite T to the leading order of μ.

Holographically, $\delta S(A) = 0$ in the finite size case could be understood as the redefinition of the cutoff.

More general picture

Actually, for a general state in CFT, the holographic EE always agrees with the field theory result.
The bulk action of the truncated geometry is

$$\begin{aligned} I &= I_{EH} + I_{GH} + I_{CT}, \\ &= -\frac{c}{96\pi}\int dzd\bar{z}(2a^2 + a^4 b^2 e^{-2\phi} - 8\partial\bar{\partial}\phi). \end{aligned}$$

where $a = \partial\phi$, with ϕ being the Liouville field and

$$b^2 = \frac{\mu c}{24\pi}.$$

In the Fefferman-Graham gauge with a proper regulator surface, the first term in the above action gives the usual holographic Rényi entropy. The last term is vanishing, while the middle term gives the correction to the HRE

$$I_2 = -\mu < T\bar{T} >_{CFT} .$$

Conclusions and Discussions

We computed the correction to the single-interval (Rényi) entanglement entropy due to the $T\overline{T}$ deformation.

The correction is of universal feature, independent of the details of the CFT.

For the holographic CFT, we compared the result with the dual gravity computation and found the consistent picture.

Open questions

The $T\overline{T}$-deformed holographic CFT opens a new window to study the AdS_3 gravity in a compact region.

Quantum gravity in a finite region? say, in de Sitter space?

Other kinds of boundary conditions at the finite cutoff surface?

Thanks for your attention!