From kinetic theory
to dissipative fluid dynamics

Dirk H. Rischke

Institut für Theoretische Physik and
Frankfurt Institute for Advanced Studies

Johann Wolfgang Goethe-Universität
Frankfurt am Main

with:
Barbara Betz, Daniel Henkel

thanks to:
Ulrich W. Heinz, Giorgio Torrieri, Urs A. Wiedemann

Preliminaries (I)

Tensor decomposition of net charge current and energy-momentum tensor:

1. Net charge current: $\boxed{N^\mu = n\,u^\mu + \nu^\mu}$

u^μ **fluid 4-velocity,** $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

$g_{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$ **(West coast!!) metric tensor,**

$n \equiv u^\mu N_\mu$ **net charge density in fluid rest frame**

$\nu^\mu \equiv \Delta^{\mu\nu} N_\nu$ **diffusion current (flow of net charge relative to u^μ),** $\nu^\mu u_\mu = 0$

$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ **projector onto 3-space orthogonal to u^μ,** $\Delta^{\mu\nu} u_\nu = 0$

2. Energy-momentum tensor: $\boxed{T^{\mu\nu} = \epsilon\,u^\mu u^\nu - (p+\Pi)\,\Delta^{\mu\nu} + 2\,q^{(\mu} u^{\nu)} + \pi^{\mu\nu}}$

$\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ **energy density in fluid rest frame**

p **pressure in fluid rest frame**

Π **bulk viscous pressure,** $p + \Pi \equiv -\frac{1}{3}\Delta^{\mu\nu} T_{\mu\nu}$

$q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ **heat flux current (flow of energy relative to u^μ),** $q^\mu u_\mu = 0$

$\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ **shear stress tensor,** $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0$, $\pi^\mu_{\ \mu} = 0$

$a^{(\mu\nu)} \equiv \frac{1}{2}(a^{\mu\nu} + a^{\nu\mu})$ **symmetrized tensor**

$a^{<\mu\nu>} \equiv \left(\Delta_\alpha^{\ (\mu}\Delta^{\nu)}_{\ \beta} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right) a^{\alpha\beta}$ **symmetrized, traceless spatial projection**

Preliminaries (II)

Fluid dynamical equations:

1. **Net charge conservation:**

$$\boxed{\partial_\mu N^\mu = 0} \quad \Longleftrightarrow \quad \boxed{\dot{n} + n\,\theta + \partial \cdot \nu = 0}$$

$$\dot{a} \equiv u^\mu \partial_\mu a \quad \text{convective (comoving) derivative}$$
$$\text{(time derivative in fluid rest frame, } \dot{a}_{\mathrm{RF}} \equiv \partial_t a)$$

$$\theta \equiv \partial_\mu u^\mu \quad \text{expansion scalar}$$

2. **Energy-momentum conservation:**

$$\boxed{\partial_\mu T^{\mu\nu} = 0} \quad \Longleftrightarrow \quad \text{energy conservation:}$$

$$\boxed{u_\nu\,\partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\,\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0}$$

acceleration equation:

$$\boxed{\begin{aligned} &\Delta^{\mu\nu}\,\partial^\lambda T_{\nu\lambda} = 0 \quad \Longleftrightarrow \\ &(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p+\Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q^\nu\partial_\nu u^\mu - \Delta^{\mu\nu}\,\partial^\lambda \pi_{\nu\lambda} \end{aligned}}$$

$$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu \quad \text{3-gradient,}$$
$$\text{(spatial gradient in fluid rest frame, } u^\mu_{\mathrm{RF}} \equiv (1,0,0,0))$$

Preliminaries (III)

Problem:

5 equations, but 15 unknowns (for given u^μ): ϵ, p, n, Π, ν^μ (3), q^μ (3), $\pi^{\mu\nu}$ (5)

Solution:

1. **clever choice of frame (Landau, Eckart,...): eliminate ν^μ or q^μ**

   $\implies$ **does not help! Promotes u^μ to dynamical variable!**

2. **ideal fluid limit: all dissipative terms vanish, $\Pi = \nu^\mu = q^\mu = \pi^{\mu\nu} = 0$**

   $\implies$ **6 unknowns: ϵ, p, n, u^μ (3)   (not quite there yet...)**

   $\implies$ **fluid is in local thermodynamical equilibrium**

   $\implies$ **provide equation of state (EOS) $p(\epsilon, n)$ to close system of equations**

3. **provide additional equations for dissipative quantities**

   $\implies$ **dissipative relativistic fluid dynamics**

   (a) **First-order theories: e.g. generalization of Navier-Stokes (NS) equations to the relativistic case (Landau, Lifshitz)**

   (b) **Second-order theories: e.g. Israel-Stewart (IS) equations**

Preliminaries (IV)

Navier-Stokes (NS) equations:

1. bulk viscous pressure: $\boxed{\Pi_{\mathrm{NS}} = -\zeta\,\theta}$

ζ bulk viscosity

2. heat flux current: $\boxed{q^{\mu}_{\mathrm{NS}} = \dfrac{\kappa}{\beta}\,\dfrac{n}{\beta(\epsilon + p)}\,\boldsymbol{\nabla}^{\mu}\alpha}$

$\beta \equiv 1/T$ inverse temperature,

$\alpha \equiv \beta\,\mu,$ μ chemical potential,

κ thermal conductivity

3. shear stress tensor: $\boxed{\pi^{\mu\nu}_{\mathrm{NS}} = 2\,\eta\,\sigma^{\mu\nu}}$

η shear viscosity,

$\sigma^{\mu\nu} = \boldsymbol{\nabla}^{<\mu}u^{\nu>}$ shear tensor

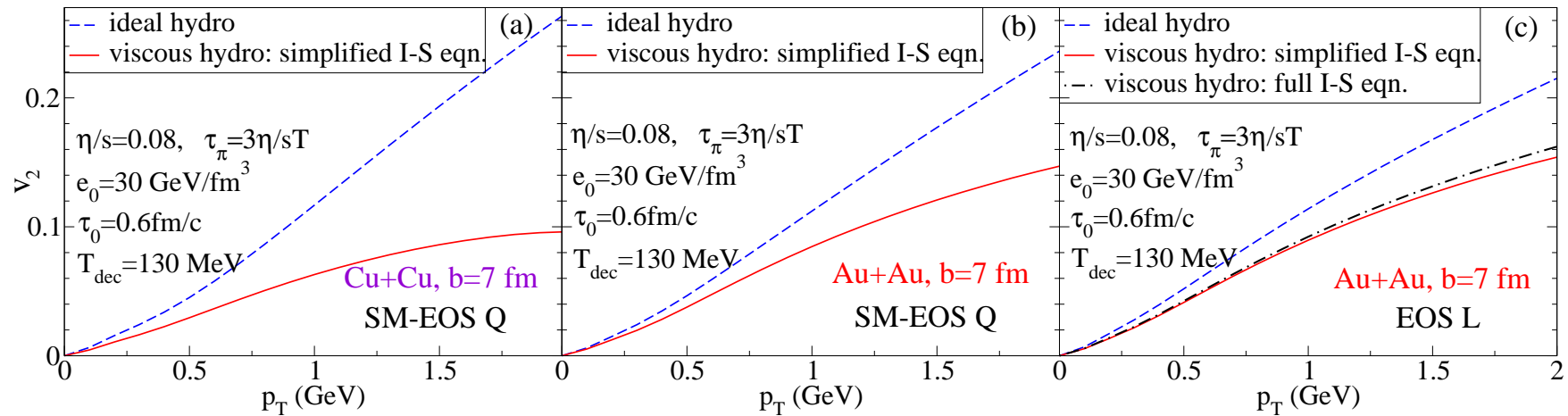$\Longrightarrow$ algebraic expressions in terms of thermodynamic and fluid variables

$\Longrightarrow$ simple... but: unstable and acausal equations of motion!!

Motivation (I)
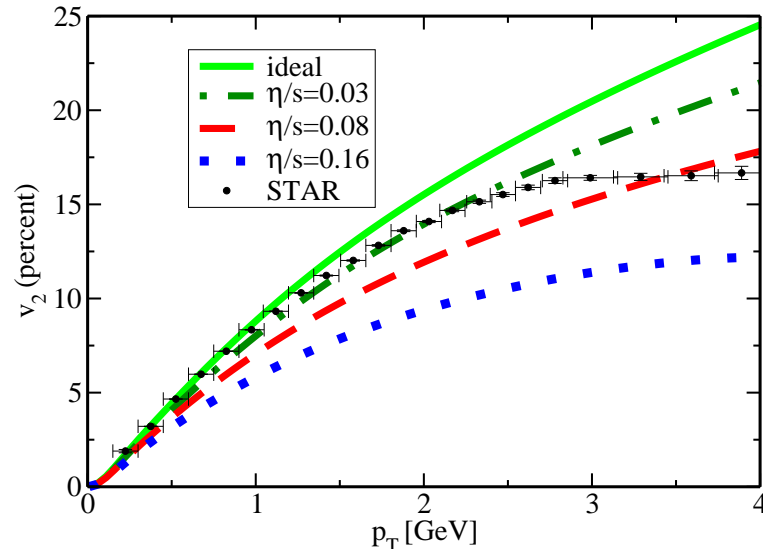
P. Romatschke, U. Romatschke, PRL 99 (2007) 172301

Au+Au @ $\sqrt{s} = 200$ AGeV

charged particles, min. bias

H. Song, U.W. Heinz, PRC 78 (2008) 024902

Motivation (II)

Israel-Stewart (IS) equations: second-order, dissipative relativistic fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

"Simplified" IS equations: e.g. shear stress tensor

$$\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS}$$

$\Longrightarrow$ dynamical (instead of algebraic) equations for dissipative terms!

$\Longrightarrow$ $\pi^{\mu\nu}$ relaxes to its NS value $\pi^{\mu\nu}_{\rm NS}$ on the time scale τ_π

$\Longrightarrow$ stable and causal fluid dynamical equations of motion!

"Full" IS equations:

$$\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS} - \frac{\eta}{2\beta}\, \pi^{\mu\nu}\, \partial_\lambda \left(\frac{\tau_\pi}{\eta}\, \beta\, u^\lambda \right) + 2\,\tau_\pi\, \pi_\lambda{}^{<\mu}\omega^{\nu>\lambda}$$

$$\omega^{\mu\nu} \equiv \frac{1}{2}\, \Delta^{\mu\alpha}\Delta^{\nu\beta}\, (\partial_\alpha u_\beta - \partial_\beta u_\alpha) \quad \text{vorticity}$$

Motivation (III)

Note: $-\dfrac{\eta}{2\beta}\,\pi^{\mu\nu}\,\partial_\lambda\left(\dfrac{\tau_\pi}{\eta}\,\beta\,u^\lambda\right) = -\dfrac{\tau_\pi}{2}\,\pi^{\mu\nu}\,\theta - \dfrac{\eta}{2\beta}\,\pi^{\mu\nu}\left(\dfrac{\tau_\pi}{\eta}\,\beta\right)^{\cdot} \simeq -2\,\eta\,\hat{\delta}_2\,\pi^{\mu\nu}\,\theta$

Proof: define $\varphi(\alpha,\beta) \equiv \dfrac{\tau_\pi}{\eta}\,\beta$

net charge conservation: $\quad \dot{n} \equiv \dfrac{\partial n}{\partial \alpha}\,\dot{\alpha} + \dfrac{\partial n}{\partial \beta}\,\dot{\beta} \simeq -n\,\theta$

energy conservation: $\quad \dot{\epsilon} \equiv \dfrac{\partial \epsilon}{\partial \alpha}\,\dot{\alpha} + \dfrac{\partial \epsilon}{\partial \beta}\,\dot{\beta} \simeq -(\epsilon + p)\,\theta$

(Note: ideal fluid limit O.K., as $\dot{\varphi}$ is multiplied with small quantity $\pi^{\mu\nu}$)

$\Longrightarrow$ linear system of eqs. for $\dot{\alpha}$ and $\dot{\beta}$ $\quad \Longrightarrow \quad \dot{\alpha} = A(\alpha,\beta)\,\theta,\ \dot{\beta} = B(\alpha,\beta)\,\theta$

$\Longrightarrow \dot{\varphi} \equiv \dfrac{\partial \varphi}{\partial \alpha}\,\dot{\alpha} + \dfrac{\partial \varphi}{\partial \beta}\,\dot{\beta} = \left[\dfrac{\partial \varphi}{\partial \alpha}\,A(\alpha,\beta) + \dfrac{\partial \varphi}{\partial \beta}\,B(\alpha,\beta)\right]\theta$

$\Longrightarrow \hat{\delta}_2 \equiv \dfrac{1}{4\,\beta}\left[\varphi(\alpha,\beta) + \dfrac{\partial \varphi}{\partial \alpha}\,A(\alpha,\beta) + \dfrac{\partial \varphi}{\partial \beta}\,B(\alpha,\beta)\right],\ $ q.e.d.

Motivation (IV)

$\Longrightarrow$ Difference between "simplified" and "full" IS equations:

the latter include higher-order terms?

For instance, if $\dfrac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$, $\tau_\pi \theta \sim \delta \ll 1$ $\Longrightarrow$ $\tau_\pi \theta \dfrac{\pi^{\mu\nu}}{\epsilon} \sim \delta^2$

$\Longrightarrow$ Goals:

1. What are the correct equations of motion for the dissipative quantities?

   $\Longrightarrow$ develop consistent power counting scheme

2. Generalization to $\mu \neq 0$ (relevant for FAIR physics!)

   $\Longrightarrow$ include heat flux q^μ

3. Generalization to non-conformal fluids (relevant near T_c!)

   $\Longrightarrow$ include bulk viscous pressure Π

Results (I)

Power counting:

3 length scales: 2 microscopic, 1 macroscopic

- **thermal wavelength** $\lambda_{\text{th}} \sim \beta \equiv 1/T$

- **mean free path** $\ell_{\text{mfp}} \sim (\langle\sigma\rangle n)^{-1}$

  $\langle\sigma\rangle$ **averaged cross section,** $n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$

- **length scale over which macroscopic fluid fields vary** L_{hydro}, $\partial_\mu \sim L_{\text{hydro}}^{-1}$

Note: **since** $\eta \sim (\langle\sigma\rangle\lambda_{\text{th}})^{-1} \implies$

$$\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{1}{\langle\sigma\rangle n}\frac{1}{\lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle\sigma\rangle\lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle\sigma\rangle\lambda_{\text{th}}} \sim \frac{\eta}{s}$$

s **entropy density,** $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$

$\implies \dfrac{\eta}{s}$ **solely determined by the 2 microscopic length scales!**

Note: similar argument holds for $\dfrac{\zeta}{s}$, $\dfrac{\kappa}{\beta s}$

Results (II)

3 regimes:

- **dilute gas limit** $\quad \dfrac{\ell_{\mathrm{mfp}}}{\lambda_{\mathrm{th}}} \sim \dfrac{\eta}{s} \gg 1 \iff \langle\sigma\rangle \ll \lambda_{\mathrm{th}}^2 \implies$ **weak-coupling limit**

- **viscous fluids** $\quad \dfrac{\ell_{\mathrm{mfp}}}{\lambda_{\mathrm{th}}} \sim \dfrac{\eta}{s} \sim 1 \iff \langle\sigma\rangle \sim \lambda_{\mathrm{th}}^2$

  interactions happen on the scale $\lambda_{\mathrm{th}} \implies$ moderate coupling

- **ideal fluid limit** $\quad \dfrac{\ell_{\mathrm{mfp}}}{\lambda_{\mathrm{th}}} \sim \dfrac{\eta}{s} \ll 1 \iff \langle\sigma\rangle \gg \lambda_{\mathrm{th}}^2 \implies$ **strong-coupling limit**

gradient (derivative) expansion: $\qquad \boxed{\ell_{\mathrm{mfp}}\,\partial_\mu \sim \dfrac{\ell_{\mathrm{mfp}}}{L_{\mathrm{hydro}}} \sim \delta \ll 1}$

$\implies$ **equivalent to $k\,\ell_{\mathrm{mfp}} \ll 1$, $\quad k$ typical momentum scale**

     R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, arXiv:0712.2451

$\implies$ **separation of macroscopic fluid dynamics (large scale $\sim L_{\mathrm{hydro}}$)**

     from microscopic particle dynamics (small scale $\sim \ell_{\mathrm{mfp}}$)

Results (III)

Primary quantities: ϵ, p, n, s $\Longleftrightarrow$ **Dissipative** quantities: Π, q^μ, $\pi^{\mu\nu}$

$$\text{If} \quad \ell_{\mathrm{mfp}}\,\partial_\mu \sim \delta \ll 1 \text{ , then} \quad \frac{\Pi}{\epsilon} \sim \frac{q^\mu}{\epsilon} \sim \frac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$$

Dissipative quantities are small compared to **primary** quantities
$\Longrightarrow$ small deviations from local thermodynamical equilibrium!

Note: statement independent of value of $\dfrac{\zeta}{s}$, $\dfrac{\kappa}{\beta\,s}$, $\dfrac{\eta}{s}$!

Proof: Gibbs relation: $\epsilon + p = Ts + \mu n \implies \beta\,\epsilon \sim s$!

Estimate dissipative terms by their **Navier-Stokes** values:

$$\Pi \sim \Pi_{\mathrm{NS}} = -\zeta\,\theta\,, \quad q^\mu \sim q^\mu_{\mathrm{NS}} = \frac{\kappa}{\beta}\,\frac{n}{\beta(\epsilon+p)}\,\nabla^\mu\alpha\,, \quad \pi^{\mu\nu} \sim \pi^{\mu\nu}_{\mathrm{NS}} = 2\,\eta\,\sigma^{\mu\nu}$$

$$\Longrightarrow \quad \frac{\Pi}{\epsilon} \sim -\frac{\zeta}{\beta\,\epsilon}\,\beta\,\theta \sim -\frac{\zeta}{s}\,\frac{\beta}{\lambda_{\mathrm{th}}}\,\frac{\lambda_{\mathrm{th}}}{\ell_{\mathrm{mfp}}}\,\ell_{\mathrm{mfp}}\,\theta \sim \ell_{\mathrm{mfp}}\,\partial_\mu u^\mu \sim \delta\,,$$

$$\frac{q^\mu}{\epsilon} \sim \frac{\kappa}{\beta}\,\frac{1}{\beta\,\epsilon}\frac{n}{\beta(\epsilon+p)}\,\beta\,\nabla^\mu\alpha \sim \frac{\kappa}{\beta\,s}\,\frac{\beta}{\lambda_{\mathrm{th}}}\,\frac{\lambda_{\mathrm{th}}}{\ell_{\mathrm{mfp}}}\,\ell_{\mathrm{mfp}}\,\nabla^\mu\alpha \sim \ell_{\mathrm{mfp}}\,\nabla^\mu\alpha \sim \delta\,,$$

$$\frac{\pi^{\mu\nu}}{\epsilon} \sim 2\,\frac{\eta}{\beta\,\epsilon}\,\beta\,\sigma^{\mu\nu} \sim 2\,\frac{\eta}{s}\,\frac{\beta}{\lambda_{\mathrm{th}}}\,\frac{\lambda_{\mathrm{th}}}{\ell_{\mathrm{mfp}}}\,\ell_{\mathrm{mfp}}\,\sigma^{\mu\nu} \sim \ell_{\mathrm{mfp}}\,\nabla^{<\mu}u^{\nu>} \sim \delta\,, \quad \text{q.e.d.}$$

Results (IV)

IS equations (derived by Grad's 14 moment method, cf. back-up slides):

$$\tau_\Pi \dot{\Pi} + \Pi = \Pi_{\text{NS}} + \tau_{\Pi q}\, q \cdot \dot{u} - \ell_{\Pi q}\, \partial \cdot q - \zeta\, \hat{\delta}_0\, \Pi\, \theta$$

$$+ \lambda_{\Pi q}\, q \cdot \nabla \alpha + \lambda_{\Pi \pi}\, \pi^{\mu\nu} \sigma_{\mu\nu}$$

$$\tau_q\, \Delta^{\mu\nu} \dot{q}_\nu + q^\mu = q^\mu_{\text{NS}} - \tau_{q\Pi}\, \Pi\, \dot{u}^\mu - \tau_{q\pi}\, \pi^{\mu\nu} \dot{u}_\nu$$

$$+ \ell_{q\Pi}\, \nabla^\mu \Pi - \ell_{q\pi}\, \Delta^{\mu\nu}\, \partial^\lambda \pi_{\nu\lambda} + \tau_q\, \omega^{\mu\nu}\, q_\nu - \frac{\kappa}{\beta}\, \hat{\delta}_1\, q^\mu\, \theta$$

$$- \lambda_{qq}\, \sigma^{\mu\nu}\, q_\nu + \lambda_{q\Pi}\, \Pi\, \nabla^\mu \alpha + \lambda_{q\pi}\, \pi^{\mu\nu}\, \nabla_\nu \alpha$$

$$\tau_\pi\, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\text{NS}} + 2\, \tau_{\pi q}\, q^{<\mu} \dot{u}^{\nu>}$$

$$+ 2\, \ell_{\pi q}\, \nabla^{<\mu} q^{\nu>} + 2\, \tau_\pi\, \pi_\lambda{}^{<\mu} \omega^{\nu>\lambda} - 2\, \eta\, \hat{\delta}_2\, \pi^{\mu\nu}\, \theta$$

$$- 2\, \tau_\pi\, \pi_\lambda{}^{<\mu} \sigma^{\nu>\lambda} - 2\, \lambda_{\pi q}\, q^{<\mu} \nabla^{\nu>} \alpha + 2\, \lambda_{\pi\Pi}\, \Pi\, \sigma^{\mu\nu}$$

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

A. Muronga, PRC 76 (2007) 014909

A. Muronga, PRC 76 (2007) 014909

this work; B. Betz, D. Henkel, DHR, in preparation

Results (V)

Remarks:

1. **Derivation is based on kinetic theory (Boltzmann equation)**

2. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, arXiv:0712.2451:
   second-order term $\sim \frac{\lambda_1}{\eta^2} \pi_\lambda^{<\mu} \pi^{\nu>\lambda}$ is contained in these equations:
   since $\pi^{\mu\nu} \simeq \pi^{\mu\nu}_{\text{NS}} = 2\,\eta\,\sigma^{\mu\nu} \implies \frac{\lambda_1}{\eta^2} \pi_\lambda^{<\mu} \pi^{\nu>\lambda} \simeq 2\frac{\lambda_1}{\eta} \pi_\lambda^{<\mu} \sigma^{\nu>\lambda} \implies \lambda_1 \equiv \tau_\pi\,\eta$
   (second-order terms from collision integral may change this result!)

3. Coefficients $\hat{\delta}_0$, $\hat{\delta}_1$, $\hat{\delta}_2$ are (complicated) functions of α, β

4. Viscosities and thermal conductivity ζ, η, κ, relaxation times τ_Π, τ_q, τ_π,
   coefficients $\tau_{\Pi q}$, $\tau_{q\Pi}$, $\tau_{q\pi}$, $\tau_{\pi q}$, $\ell_{\Pi q}$, $\ell_{q\Pi}$, $\ell_{q\pi}$, $\ell_{\pi q}$, $\lambda_{\Pi q}$, $\lambda_{\Pi \pi}$, λ_{qq}, $\lambda_{q\Pi}$, $\lambda_{q\pi}$,
   $\lambda_{\pi q}$, $\lambda_{\pi \Pi}$ are (complicated) functions of α, β, divided by
   tensor coefficients of second moment of collision integral, $C^{\mu\nu}[f]$:
   $\sim \chi_i(\alpha, \beta)/\langle\sigma\rangle \to 0$ as cross section $\sigma \to \infty$ ("strong coupling limit"!)
   $\implies \Pi = q^\mu = \pi^{\mu\nu} \to 0$ ideal fluid limit!

5. IS equations are formally independent of calculational frame (Eckart, Landau,...), but ...

Results (VI)

6. **Values of coefficients are frame dependent! We have analyzed:**

(a) **Eckart (N) or (net) charge frame:** $\boxed{\nu^\mu = 0\,,\ \epsilon = \epsilon_0\,,\ n = n_0}$

$\epsilon_0\,,\ n_0$: **energy density and charge density in local thermodyn. equilibrium**

(b) **Landau (E) or energy frame:** $\boxed{q^\mu = 0\,,\ \epsilon = \epsilon_0\,,\ n = n_0}$

Note: in IS equations $\quad q^\mu \equiv -\dfrac{\epsilon + p}{n}\,\nu^\mu$

(c) **Tsumura-Kunihiro-Ohnishi (TKO) frame:** $\boxed{\nu^\mu = 0\,,\ \epsilon = \epsilon_0 - 3\,\Pi\,,\ n = n_0}$

We have checked agreement with the results of IS for most coefficients computed by IS...

7. **R.h.s.: all terms except NS terms are of second order, $\sim \delta^2$**

$\Longrightarrow$ $t < \tau_\Pi \sim \tau_q \sim \tau_\pi$: **dissipative terms relax towards their NS values,**

$t > \tau_\Pi \sim \tau_q \sim \tau_\pi$: **last terms on r.h.s. and NS terms on l.h.s. largely cancel, second-order terms govern evolution!**

Conclusions and open problems

1. Derived Israel-Stewart (IS) equations from kinetic theory via Grad's 14-moment method $\implies$ new second-order terms!

2. Results consistent with

    R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, arXiv:0712.2451

3. Coefficients of terms in IS equations are frame dependent
    $\implies$ have (not yet completely) been computed in various frames
    (Eckart, Landau, TKO)

4. Generalization to a system of various particle species (done: quarks, antiquarks, gluons), various conserved charges

5. Numerical implementation

Derivation: Grad's 14-moment method (I)

1. history: derivation of IS equations from kinetic theory

- H. Grad, Commun. Pure Appl. Math. 2 (1949) 381
- J.M. Stewart, Proc. Roy. Soc. London Ser. A 357 (1977) 59
- W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341
- DHR, PhD thesis, 1992 (unpublished)
- R. Baier, P. Romatschke, U.A. Wiedemann, PRC 73 (2006) 064903
- A. Muronga, PRC 76 (2007) 014909, 014910

2. single-particle distribution function in local thermodynamical equilibrium:

$$f_0 = (e^{-y_0} + a)^{-1}\,, \quad y_0 = \alpha_0 - \beta_0\, K \cdot u$$

$a = \pm 1, 0$ for fermions/bosons, Boltzmann particles

local: $\alpha_0 \equiv \alpha_0(X)\,,\ \beta_0 \equiv \beta_0(X)\,,\ u^\mu \equiv u^\mu(X)\,,\ X^\mu = (t, \vec{x})$

single-particle distribution function in local non-equilibrium:

$$f = (e^{-y} + a)^{-1}\,, \quad y = \alpha - \beta K \cdot u - K \cdot v + K^\mu K^\nu w_{\mu\nu}$$

$v \cdot u = 0\,,\ w^\mu{}_\mu = 0$

Derivation: Grad's 14-moment method (II)

3. assume small deviations from equilibrium:

$$y - y_0 = \alpha - \alpha_0 - (\beta - \beta_0)\, K \cdot u - K \cdot v + K^\mu K^\nu w_{\mu\nu} \sim \delta \ll 1$$

$$\Longleftrightarrow \quad \alpha - \alpha_0 \sim \beta - \beta_0 \sim v^\mu \sim w^{\mu\nu} \sim \delta$$

$\Longrightarrow$ expand f around f_0 to linear order in $y - y_0$:

$$f \simeq f_0 + f_0(1 - a\, f_0)(y - y_0) + O(\delta^2)$$

4. compute N^μ and $T^{\mu\nu}$ as moments of f:

$$N^\mu = \int \mathrm{d}\tilde{K}\, K^\mu\, f = I_0^\mu + (\alpha - \alpha_0)\, J_0^\mu - (\beta - \beta_0)\, J_0^{\mu\nu} u_\nu - J_0^{\mu\nu} v_\nu + J_0^{\mu\nu\lambda} w_{\nu\lambda}$$

$$T^{\mu\nu} = \int \mathrm{d}\tilde{K}\, K^\mu K^\nu\, f = I_0^{\mu\nu} + (\alpha - \alpha_0)\, J_0^{\mu\nu} - (\beta - \beta_0)\, J_0^{\mu\nu\lambda} u_\lambda - J_0^{\mu\nu\lambda} v_\lambda + J_0^{\mu\nu\lambda\rho} w_{\lambda\rho}$$

where $\int \mathrm{d}\tilde{K} \equiv \frac{g}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{k}}{E}$, $E = \sqrt{k^2 + m^2}$, g no. of internal d.o.f.'s, and

$$I_0^{\alpha_1 \cdots \alpha_n} \equiv \int \mathrm{d}\tilde{K}\, K^{\alpha_1} \cdots K^{\alpha_n}\, f_0\,,$$

$$J_0^{\alpha_1 \cdots \alpha_n} \equiv \int \mathrm{d}\tilde{K}\, K^{\alpha_1} \cdots K^{\alpha_n}\, f_0(1 - af_0)$$

Note: $I_0^\mu \equiv N_0^\mu$ net charge current in local thermodyn. equilibrium

$I_0^{\mu\nu} \equiv T_0^{\mu\nu}$ energy-momentum tensor in local thermodyn. equilibrium

Derivation: Grad's 14-moment method (III)

5. tensor decomposition of $I_0^{\alpha_1 \cdots \alpha_n}$ up to $n = 3$ and of $J_0^{\alpha_1 \cdots \alpha_n}$ up to $n = 5$:

$$I_0^\mu = I_{10}\, u^\mu \,, \qquad\qquad\qquad J_0^\mu = J_{10}\, u^\mu \,,$$

$$I_0^{\mu\nu} = I_{20}\, u^\mu u^\nu + I_{21}\, \Delta^{\mu\nu} \,, \qquad J_0^{\mu\nu} = J_{20}\, u^\mu u^\nu + J_{21}\, \Delta^{\mu\nu} \,,$$

$$I_0^{\mu\nu\lambda} = I_{30}\, u^\mu u^\nu u^\lambda + 3\, I_{31}\, u^{(\mu} \Delta^{\nu\lambda)} \qquad J_0^{\mu\nu\lambda} = J_{30}\, u^\mu u^\nu u^\lambda + 3\, J_{31}\, u^{(\mu} \Delta^{\nu\lambda)}$$

$$J_0^{\mu\nu\lambda\rho} = J_{40}\, u^\mu u^\nu u^\lambda u^\rho + 6\, J_{41}\, u^{(\mu} u^\nu \Delta^{\lambda\rho)} + 3\, J_{42}\, \Delta^{\mu(\nu} \Delta^{\lambda\rho)}$$

$$J_0^{\mu\nu\lambda\rho\sigma} = J_{50}\, u^\mu u^\nu u^\lambda u^\rho u^\sigma + 10\, J_{51}\, u^{(\mu} u^\nu u^\lambda \Delta^{\rho\sigma)} + 15\, J_{52}\, u^{(\mu} \Delta^{\nu\lambda} \Delta^{\rho\sigma)}$$

Note: $I_{10} \equiv n_0 \,,\ I_{20} \equiv \epsilon_0 \,,\ I_{21} \equiv -p_0$

In general: $\quad I_{nq} \equiv \dfrac{1}{(2q+1)!!} \int \mathrm{d}\tilde{K}\, E^{n-2q}(-k^2)^q\, f_0$

$$J_{nq} \equiv \dfrac{1}{(2q+1)!!} \int \mathrm{d}\tilde{K}\, E^{n-2q}(-k^2)^q\, f_0(1 - a f_0)$$

6. insert tensor decomposition for $I_0^{\alpha_1 \cdots \alpha_n}$, $J_0^{\alpha_1 \cdots \alpha_n}$, as well as

$$w_{\mu\nu} = w\left(u_\mu u_\nu - \tfrac{1}{3}\, \Delta_{\mu\nu}\right) + 2\, w_{(\mu} u_{\nu)} + \tilde{w}_{\mu\nu}$$

$$w_\mu u^\mu = \tilde{w}_{\mu\nu} u^\nu = \tilde{w}_{\mu\nu} u^\mu = 0 \,; \quad \tilde{w}_{<\mu\nu>} = \tilde{w}_{\mu\nu}$$

into expansion for N^μ, $T^{\mu\nu}$, cf. 4.

Derivation: Grad's 14-moment method (IV)

7. choose a specific frame, i.e., impose matching conditions (cf. Results (V), 6.)

$$\Longrightarrow \quad \boxed{\alpha - \alpha_0 = w\,\chi_\alpha(\alpha, \beta)\,, \ \ \beta - \beta_0 = w\,\chi_\beta(\alpha, \beta)\,, \ \ v^\mu = w^\mu \chi_v(\alpha, \beta)}$$

8. re-insert into tensor decomposition for N^μ, $T^{\mu\nu}$

$$\Longrightarrow \quad \boxed{w = \Pi\,\chi_\Pi(\alpha, \beta)\,, \ \ w^\mu = q^\mu\,\chi_q(\alpha, \beta)\,, \ \ \tilde{w}^{\mu\nu} = \pi^{\mu\nu}\,\chi_\pi(\alpha, \beta)}$$

9. compute third moment of f:

$$S^{\mu\nu\lambda} = \int \mathrm{d}\tilde{K}\, K^\mu K^\nu K^\lambda\, f$$
$$= S_1\, u^\mu u^\nu u^\lambda + 3\, S_2\, u^{(\mu}\Delta^{\nu\lambda)} + 3\,\psi_1\, q^{(\mu}u^\nu u^{\lambda)} + 3\,\psi_2\, q^{(\mu}\Delta^{\nu\lambda)} + 3\,\psi_3\, \pi^{(\mu\nu}u^{\lambda)}$$

$$S_1 = I_{30} + \psi_4\,\Pi\,, \ \ S_2 = I_{31} - \tfrac{1}{3}\,\psi_4\,\Pi\,, \ \ \psi_i = \psi_i(\alpha, \beta)\,, \ \ i = 1, \ldots, 4$$

10. compute second moment of collision integral $\mathcal{C}[f]$:

$$\boxed{C^{\nu\lambda} = \int \mathrm{d}\tilde{K}\, K^\nu K^\lambda\, \mathcal{C}[f] = A\,\Pi\left(u^\nu u^\lambda - \tfrac{1}{3}\Delta^{\nu\lambda}\right) + B\,\pi^{\nu\lambda} + 2\,C\, u^{(\nu}q^{\lambda)}}$$

$$A\,, \ B\,, \ C \sim \langle\sigma\rangle$$

Derivation: Grad's 14-moment method (V)

11. compute second moment of Boltzmann equation, $K \cdot \partial f = \mathcal{C}[f]$:

$\Longrightarrow$ $\boxed{\partial_\mu S^{\mu\nu\lambda} = C^{\nu\lambda}}$ or, tensor-decomposed:

$$A \, \Pi - \psi_4 \, \dot{\Pi} = -\beta \, J_{41} \, \theta + \dot{I}_{30} + \dot{\psi}_4 \, \Pi + \tfrac{5}{3} \, \psi_4 \, \Pi \, \theta$$
$$+ \, \psi_1 \, \partial \cdot q - q \cdot \nabla \psi_1 - 2 \, \psi_1 \, q \cdot \dot{u} - 2 \, \psi_3 \, \pi^{\mu\nu} \sigma_{\mu\nu}$$

$$C \, q^\mu - \psi_1 \, \Delta^{\mu\nu} q_\nu = -\beta \, J_{41} \, \dot{u}^\mu + \nabla^\mu J_{31} - \tfrac{1}{3} \, \psi_4 \, \nabla^\mu \Pi - \tfrac{1}{3} \, \Pi \, \nabla^\mu \psi_4 + \tfrac{5}{3} \, \psi_4 \, \Pi \, \dot{u}^\mu$$
$$+ \, (\psi_1 - 2 \, \psi_2) \, q_\nu \sigma^{\mu\nu} - \psi_1 \, q_\nu \omega^{\mu\nu} + q^\mu \left[\dot{\psi}_1 + \tfrac{1}{3} \, (4 \, \psi_1 - 5 \, \psi_2) \, \theta \right]$$
$$+ \, \psi_3 \, \Delta^{\mu\nu} \partial^\lambda \pi_{\nu\lambda} + \pi^{\mu\nu} \, (\nabla_\nu \psi_3 - \psi_3 \, \dot{u}_\nu)$$

$$B \, \pi^{\mu\nu} - \psi_3 \, \dot{\pi}^{<\mu\nu>} = 2 \, I_{31} \, \sigma^{\mu\nu} - \tfrac{2}{3} \, \psi_4 \, \Pi \, \sigma^{\mu\nu}$$
$$+ \, 2 \, \psi_2 \, \nabla^{<\mu} q^{\nu>} + 2 \, q^{<\mu} \nabla^{\nu>} \psi_2 + 2 \, (\psi_1 - \psi_2) \, q^{<\mu} \dot{u}^{\nu>}$$
$$+ \, 2 \, \psi_3 \, \pi_\lambda{}^{<\mu} \sigma^{\nu>\lambda} - 2 \, \psi_3 \, \pi_\lambda{}^{<\mu} \omega^{\nu>\lambda} + \pi^{\mu\nu} \left(\dot{\psi}_3 + \tfrac{5}{3} \, \psi_3 \, \theta \right)$$

12. substitute $\dot{I}_{30}$, $-\beta \, J_{41} \, \dot{u}^\mu$, $\nabla^\mu I_{31}$, $\dot{\psi}_i$, $\nabla^\mu \psi_i$, $i = 1, \ldots, 4$

with the help of **energy conservation** and **acceleration equation**

$\Longrightarrow$ **IS equations with explicit (frame-dependent) expressions for coefficients**