Gravity from squaring Yang-Mills

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based on

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1.1 Aims

- MATHEMATICS:

  $(DIVISION\ ALGEBRAS)^2 = MAGIC\ SQUARE\ OF\ LIE\ ALGEBRAS$

- PHYSICS:

  $(YANG - MILLS)^2 = MAGIC\ SQUARE\ OF\ SUPERGRAVITIES$

- RESULT :

  $MATHEMATICS\ MAGIC\ SQUARE = PHYSICS\ MAGIC\ SQUARE$

1.2 Division algebras

- Mathematicians deal with four kinds of numbers, called Divison Algebras.
- The Octonions occupy a privileged position :

Name	Symbol	Imaginary parts
Reals	$\mathbb{R}$	0
Complexes	$\mathbb{C}$	1
Quaternions	$\mathbb{H}$	3
Octonions	$\mathbb{O}$	7

Table : Division Algebras

1.3 Lie algebras

- They provide an intuitive basis for the exceptional Lie algebras:

Classical algebras		Rank	Dimension
A_n	$SU(n+1)$	n	$(n+1)^2-1$
B_n	$SO(2n+1)$	n	$n(2n+1)$
C_n	$Sp(2n)$	n	$n(2n+1)$
D_n	$SO(2n)$	n	$n(2n-1)$
Exceptional algebras			
E_6		6	78
E_7		7	133
E_8		8	248
F_4		4	52
G_2		2	14

Table : Classical and exceptional Lie alebras

1.4 Magic square

- Freudenthal-Rozenfeld-Tits magic square

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 + A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Table : Magic square

- Despite much effort, however, it is fair to say that the ultimate physical significance of octonions and the magic square remains an enigma.

1.5 Comment

- There are different magic squares depending on the choice of real forms. Ours should not to be confused with other versions that have appeared in the so-called "magic" supergravities in $D = 4, 5, 6$ [Gunaydin, Sierra and Townsend 1983]
- The specific square of real forms we derive from the triality construction was first obtained in [Cacciatori-Cerchiai-Marrani:2012] using a "Lorentizian Jordan algebra" adaptation of the Tits formula [Tits:1962]

1.6 Earlier work involving octonions

- For earlier work on octonions in high-energy physics, see, for example,
  [Kugo and Townsend: 1983
  Gunaydin, Sierra and Townsend: 1983
  MJD: 1986
  Fairlie and Manogue 1987
  Berkovits: 1993
  Evans: 1994
  Baez: 2001
  Barton and Sudbery: 2003
  Baez: 2009
  MJD et al: 2013]
- Despite much effort, however, it is fair to say that the ultimate physical significance of octonions and the magic square remains an enigma.

1.7 Squaring Yang-Mills

- In apparently different developments, a recurring theme in attempts to understand the quantum theory of gravity is the idea of "Gravity as the square of Yang-Mills".
- This idea of tensoring left (L) and right (R) multiplets appears in many different (but sometimes overlapping) guises. For example:
- KLT relations in string theory [Kawai et al:1985]
- $D = 10$ dimensional Type IIA and IIB supergravity (SG) multiplets from $D = 10$ super Yang-Mills (SYM) multiplets [Green et al:1987]
- Asymmetric orbifold contructions [Sen and Vafa:1995]
- Gravity anomalies from gauge anomalies [Antoniadis et al:1992]
- (Super) gravity scattering amplitudes from those of (super) Yang-Mills [Bern et al:2008; Bianchi, Elvang, Freedman:2008] in various dimensions
- [Thanks to Zvi for his encouragement.]

1.8 Squaring D=10 Yang Mills to get TypeIIA supergravity

In D=10 Type IIA comes from the product of two Yang-Mills multiplets of opposite chirality with SO(8) reps.

	8_v	8_s
8_v	$1 + 28 + 35_v$	$8_c + 56_c$
8_c	$8_s + 56_s$	$8_v + 56_v$

Table : SO(8) reps of D=10 Type IIA supergravity

	A_M	χ^s
A_N	$\phi + B_{MN} + g_{MN}$	${\psi^c}_N + \chi^c$
χ^c	${\psi^s}_M + \chi^s$	$A_P + A_{PQR}$

Table : Fields of D=10 Type IIA supergravity

1.9 Squaring D=10 Yang Mills to get TypeIIB supergravity

In D=10 Type IIB comes from the product of two Yang-Mills multiplets of same chirality with SO(8) reps.

	8_v	8_c
8_v	$1+28+35_v$	8_s+56_s
8_c	8_s+56_s	$1+28+35_c$

Table : SO(8) reps of D=10 Type IIB supergravity

	A_M	χ^c
A_N	$\phi+B_{MN}+g_{MN}$	${\psi^s}_N+\chi^s$
χ^c	${\psi^s}_M+\chi^s$	$A+A_{PQ}+A^+_{PQRS}$

Table : Fields of D=10 Type IIB sugravity

1.10 Summary Yang-Mills

- We give a unified description of
  $D = 3$ Yang-Mills with $\mathcal{N} = 1, 2, 4, 8$
  $D = 4$ Yang-Mills with $\mathcal{N} = 1, 2, 4$
  $D = 6$ Yang-Mills with $\mathcal{N} = 1, 2$
  $D = 10$ Yang-Mills with $\mathcal{N} = 1$
  in terms of a pair of division algebras $(\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}})$, $n = D - 2$
- We present a master Lagrangian, defined over $\mathbb{A}_{n\mathcal{N}}$-valued fields, which encapsulates all cases.
- The overall (spacetime plus internal) on-shell symmetries are given by the corresponding *triality* algebras.
- We use imaginary $\mathbb{A}_{n\mathcal{N}}$-valued auxiliary fields to close the non-maximal supersymmetry algebra off-shell. The failure to close off-shell for maximally supersymmetric theories is attributed directly to the non-associativity of the octonions.

1.11 Summary Gravity Magic Square in $D = 3$

- Tensoring left and right multiplets yields
- $D = 3$ A magic 4×4 square $\mathbb{R}\mathbb{R}$, $\mathbb{C}\mathbb{R}$, $\mathbb{C}\mathbb{C}$, $\mathbb{H}\mathbb{R}$, $\mathbb{H}\mathbb{C}$, $\mathbb{H}\mathbb{H}$, $\mathbb{O}\mathbb{R}$, $\mathbb{O}\mathbb{C}$, $\mathbb{O}\mathbb{H}$, $\mathbb{O}\mathbb{O}$ description of supergravities with $\mathcal{N} = 2, 3, 4, 5, 6, 8, 9, 10, 12, 16$.

- The U-duality groups are precisely those of the Freudenthal Magic Square!

1.12 Summary Gravity Magic Pyramid

BUT there is also a more familiar $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ description of spacetime in D = 3, 4, 6, 10. Tensoring left and right yields:

- $D = 3$ A magic 4×4 square $\mathbb{RR}$, $\mathbb{CR}$, $\mathbb{CC}$, $\mathbb{HR}$, $\mathbb{HC}$, $\mathbb{HH}$, $\mathbb{OR}$, $\mathbb{OC}$, $\mathbb{OH}$, $\mathbb{OO}$ description of supergravities with $\mathcal{N} = 2, 3, 4, 5, 6, 8, 9, 10, 12, 16$.
  $D = 4$ A 3×3 square $\mathbb{RR}$, $\mathbb{CR}$, $\mathbb{CC}$, $\mathbb{HR}$, $\mathbb{HC}$, $\mathbb{HH}$ description of supergravities with $\mathcal{N} = 2, 3, 4, 5, 6, 8$.
  $D = 6$ A 2×2 square $\mathbb{RR}$, $\mathbb{CR}$, $\mathbb{CC}$ description of supergravities with $\mathcal{N} = 2, 3, 4$.
  $D = 10$ A 1×1 square $\mathbb{RR}$ description of supergravities with $\mathcal{N} = 2$.
- Together these form *The Magic Pyramid*.
- The corresponding U-duality groups are given by a new algebraic structure, the magic pyramid formula, which may be regarded as being defined over three division algebras, one for spacetime and each of the left/right Yang-Mills multiplets.

1.13 Summary Gravity: Conformal Magic Pyramid

- We also construct a *conformal* magic pyramid by tensoring conformal supermultiplets in D = 3, 4, 6.
- The missing entry in D = 10 is suggestive of an exotic theory with G/H duality structure $F_{4(4)}/Sp(3) \times Sp(1)$.

2.1 Octonions

- An element $x \in \mathbb{O}$ may be written $x = x^a e_a$, where $a = 0, \ldots, 7$, $x^a \in \mathbb{R}$ and $\{e_a\}$ is a basis with one real $e_0 = 1$ and seven $e_i, i = 1, \ldots, 7$ imaginary elements. The octonionic conjugation is denoted by e_a^*, where $e_0^* = e_0$ and $e_i^* = -e_i$.
- The octonionic multiplication rule is,

$$e_a e_b = (\delta_{a0}\delta_{bc} + \delta_{0b}\delta_{ac} - \delta_{ab}\delta_{0c} + C_{abc})\, e_c\,,$$

where C_{abc} is totally antisymmetric such that $C_{0bc} = 0$. The non-zero C_{ijk} are the octonionic structure constants:

$$C_{ijk} = \varepsilon_{ijk} \text{ if } ijk \in \mathbf{L}$$

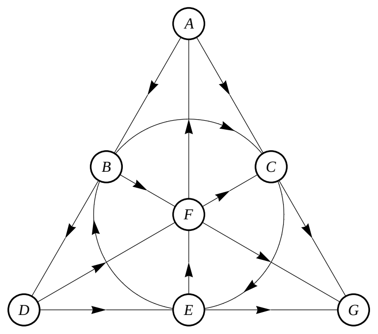
with $\mathbf{L}$ the set of oriented lines of the Fano plane. See [Baez:2001].

$$\mathbf{L} = \{124, 235, 346, 457, 561, 672, 713\}.$$

2.2 Fano plane

The Fano plane has seven points and seven lines (the circle counts as a line) with three points on every line and three lines through every point.

Fano plane



2.3 Gino Fano

Gino Fano (5 January 1871 to 8 November 1952) was an Italian mathematician. He was born in Mantua and died in Verona.
Fano worked on projective and algebraic geometry; the Fano plane, Fano fibration, Fano surface, and Fano varieties are named for him.
Ugo Fano and Robert Fano were his sons.

2.4 Cayley-Dickson

- $$O = O^0 e_0 + O^1 e_1 + O^2 e_2 + O^3 e_3 + O^4 e_4 + O^5 e_5 + O^6 e_6 + O^7 e_7$$
  $$= H(0) + e_3 H(1)$$

- $$H(0) = O^0 e_0 + O^1 e_1 + O^2 e_2 + O^4 e_4 \quad H(1) = O^3 e_0 - O^7 e_1 - O^5 e_2 + O^6 e_4$$
  $$H(0) = C(00) + e_2 C(10) \quad H(1) = C(01) + e_2 C(11)$$

- $$C(00) = O^0 e_0 + O^1 e_1 \quad C(01) = O^3 e_0 - O^7 e_1$$
  $$C(10) = O^2 e_0 - O^4 e_1 \quad C(11) = -O^5 e_0 - O^6 e_1$$
  $$C(00) = R(000) + e_1 R(100) \quad C(01) = R(001) + e_1 R(101)$$
  $$C(10) = R(010) + e_1 (110) \quad C(11) = R(011) + e_1 R(111)$$

- $$R(000) = O^0 \;\; R(100) = O^1 \quad R(001) = O^3 \;\; R(101) = -O^7$$
  $$R(010) = O^2 \;\; R(110) = -O^4 \quad R(011) = -O^5 \;\; R(111) = -O^6$$

2.5 Division algebras

- Division: ax+b=0 has a unique solution
- Associative: a(bc)=(ab)c
- Commutative: ab=ba

A	construction	division?	associative?	commutative?	ordered?
R	R	yes	yes	yes	yes
C	$R + e_1 R$	yes	yes	yes	no
H	$C + e_2 C$	yes	yes	no	no
O	$H + e_3 H$	yes	no	no	no
S	$O + e_4 O$	no	no	no	no

2.6 Fano quadrangles

- It will also be useful to define Q_{ijkl}, which is equal to 1 (-1) when $ijkl$ is an even (odd) permutation of an element of $\mathbf{Q}$, the set of oriented quadrangles in the Fano plane:

$$\mathbf{Q} = \{3567, 4671, 5712, 6123, 7234, 1345, 2456\},$$

and equal to zero otherwise. Equivalently, we can define Q_{ijkl} by

$$Q_{ijkl} = -\frac{1}{3!} C_{mnp} \varepsilon_{mnpijkl}.$$

2.7 The Associator

- The octonions have a trilinear map called the associator given by :

$$[x, y, z] = (xy)z - x(yz)$$

which measures the failure of associativity.

- In the same way that the multiplication of the octonionic bases was realised using the lines of the Fano plane, the associator of three octonionic bases can be realised using its quadrangles **Q** :

$$[e_a, e_b, e_c] = 2Q_{abcd}e_d$$

The object Q_{abcd} is totally antisymmetric with $Q_{0ijk} = C_{ijk}$.

$$Q_{abcd} = \frac{1}{4!}\epsilon_{abcdefgh}Q^{efgh}$$

2.8 Norm-preserving algebras

- To understand the symmetries of the magic square and its relation to YM we shall need in particular two algebras defined on $\mathbb{A}$.
- First, the algebra $\mathfrak{norm}(\mathbb{A})$ that preserves the norm

$$\langle x|y\rangle := \frac{1}{2}(x\overline{y} + y\overline{x}) = x^a y^b \delta_{ab}$$

$$\mathfrak{norm}(\mathbb{R}) = 0$$
$$\mathfrak{norm}(\mathbb{C}) = \mathfrak{so}(2)$$
$$\mathfrak{norm}(\mathbb{H}) = \mathfrak{so}(4)$$
$$\mathfrak{norm}(\mathbb{O}) = \mathfrak{so}(8)$$

2.9 Triality Algebra

- Second, the triality algebra $\mathfrak{tri}(\mathbb{A})$

$$\mathfrak{tri}(\mathbb{A}) \equiv \{(A, B, C)|A(xy) = B(x)y + xC(y)\}, \quad A, B, C \in \mathfrak{so}(n), \quad x, y \in \mathbb{A}.$$

$$\begin{aligned}
\mathfrak{tri}(\mathbb{R}) &= 0 \\
\mathfrak{tri}(\mathbb{C}) &= \mathfrak{so}(2) + \mathfrak{so}(2) \\
\mathfrak{tri}(\mathbb{H}) &= \mathfrak{so}(3) + \mathfrak{so}(3) + \mathfrak{so}(3) \\
\mathfrak{tri}(\mathbb{O}) &= \mathfrak{so}(8)
\end{aligned}$$

[Barton and Sudbery:2003]:

2.10 Pauli matrices

In $D = 4$ the Pauli matrices $\{\bar{\sigma}_\mu\}$ are used as a basis for Hermitian matrices, so that we can write $X = X^\mu \bar{\sigma}_\mu$. This suggests a generalised set of Pauli matrices for $\mu = 0, 1, \ldots, (n+1)$.

$$\bar{\sigma}_\mu = \sigma^\mu = (+1, \sigma_{a+1}, \sigma_{n+1}),$$
$$\sigma_\mu = \bar{\sigma}^\mu = (-1, \sigma_{a+1}, \sigma_{n+1}),$$

where

$$\sigma_{a+1} = \begin{pmatrix} 0 & e_a^* \\ e_a & 0 \end{pmatrix}, \qquad \sigma_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The notation is chosen so that in $D = 4$ (where $n = 2$ and $\mathbb{A} = \mathbb{C}$) the matrices reduce to the usual Pauli set:

$$\sigma_1 = \begin{pmatrix} 0 & e_0^* \\ e_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & e_1^* \\ e_1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2.11 More Pauli matrices

It is easy to see that the generalised Pauli matrices satisfy the required algebra:

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2\eta^{\mu\nu}1,$$
$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2\eta^{\mu\nu}1.$$

As a result, they can be used to construct Lorentz generators of the spinor and conjugate spinor representations, which we will see explicitly in the following subsection. Note that in $D = 3$, the matrices

$$\gamma^\mu \equiv \sigma^\mu\epsilon$$

with

$$\varepsilon \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

satisfy the Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}1,$$

2.12 Spinors

- When $\mathbb{A} = \mathbb{R}$, $S_+ \sim S_-$, Ψ is the Majorana spinor in $D = 3$
- When $\mathbb{A} = \mathbb{C}$, S_+ and S_- , Ψ and χ are the Weyl spinors in $D = 4$
- When $\mathbb{A} = \mathbb{H}$, S_+ and S_- , Ψ and χ are the Symplectic-Weyl spinors in $D = 6$
- When $\mathbb{A} = \mathbb{O}$, S_+ and S_- , Ψ and χ are the Majorana-Weyl spinors in $D = 10$.

  We seek a generalisation of the equations:

$$\begin{aligned} \delta\Psi &= \frac{1}{4}\lambda^{\mu\nu}\sigma_{\mu\nu}\Psi = \frac{1}{4}\lambda^{\mu\nu}\sigma_{\mu}\bar{\sigma}_{\nu}\Psi, \\ \delta\chi &= \frac{1}{4}\lambda^{\mu\nu}\bar{\sigma}_{\mu\nu}\chi = \frac{1}{4}\lambda^{\mu\nu}\bar{\sigma}_{\mu}\sigma_{\nu}\chi. \end{aligned} \tag{1}$$

  Note that for $\mathbb{A} = \mathbb{O}$, the choice of association $(\sigma_{[\mu}\sigma_{\nu]})\Psi$ is wrong, this gives only 31 independent generators when we expect $45 = \dim[\mathrm{SO}(1,9)]$.
- The 14 generators we are missing are the generators of G_2, the automorphism group of the octonions. It is fairly straightforward to check that the correct answer is given by $\sigma_{[\mu}(\sigma_{\nu]}\Psi)$.

3.1 $D = 3, \mathcal{N} = 8$ Yang-Mills

- The $D = 3$, $\mathcal{N} = 8$ super YM action is given by

$$S = \int d^3x \left(-\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} - \frac{1}{2} D_\mu \phi^A_i D^\mu \phi^A_i + i\bar{\lambda}^A_a \gamma^\mu D_\mu \lambda^A_a \right.$$
$$\left. -\frac{1}{4} g^2 f_{BC}{}^A f_{DE}{}^A \phi^B_i \phi^D_i \phi^C_j \phi^E_j - g f_{BC}{}^A \phi^B_i \bar{\lambda}^{Aa} \Gamma^i_{ab} \lambda^{Cb} \right),$$

  where the Dirac matrices Γ^i_{ab}, $i = 1, \ldots, 7$, $a, b = 0, \ldots, 7$, belong to the SO(7) Clifford algebra.

- The key observation is that Γ^i_{ab} can be represented by the octonionic structure constants,

$$\Gamma^i_{ab} = i(\delta_{bi}\delta_{a0} - \delta_{b0}\delta_{ai} + C_{iab}),$$

  which allows us to rewrite the action over octonionic fields

3.2 $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \rightarrow \mathcal{N} = 1, 2, 4, 8$ supersymmetries

- If we replace $\mathbb{O}$ with a general division algebra $\mathbb{A}$, the result is $\mathcal{N} = 1, 2, 4, 8$ over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$:

$$S = \int d^3x \left(-\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} - \frac{1}{2} D_\mu \phi^{*A} D^\mu \phi^A + i\bar{\lambda}^A \gamma^\mu D_\mu \lambda^A \right.$$
$$-\frac{1}{4} g^2 f_{BC}{}^A f_{DE}{}^A \langle \phi^B | \phi^D \rangle \langle \phi^C | \phi^E \rangle$$
$$\left. +\frac{i}{2} g f_{BC}{}^A \left((\bar{\lambda}^A \phi^B) \lambda^C - \bar{\lambda}^A (\phi^{*B} \lambda^C) \right) \right),$$

where $\phi = \phi^i e_i$ is an Im$\mathbb{A}$-valued scalar field, $\lambda = \lambda^a e_a$ is an $\mathbb{A}$-valued two-component spinor and $\bar{\lambda} = \bar{\lambda}^a e^*_a$.

- Note, since λ^a is anti-commuting we are dealing with the *algebra of octonions defined over the Grassmanns*.

3.3 Transformation rules

- The supersymmetry transformations in this language are given by

$$
\begin{aligned}
\delta\lambda^A &= \frac{1}{2}(F^{A\mu\nu} + \varepsilon^{\mu\nu\rho} D_\rho \phi^A)\sigma_{\mu\nu}\epsilon + \frac{1}{2} g f_{BC}{}^A \phi_i^B \phi_j^C \sigma_{ij}\epsilon, \\
\delta A_\mu^A &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu \lambda^A - \bar{\lambda}^A \gamma_\mu \epsilon), \\
\delta\phi^A &= \frac{i}{2} e_i [(\bar{\epsilon} e_i)\lambda^A - \bar{\lambda}^A (e_i \epsilon)],
\end{aligned} \tag{2}
$$

where $\sigma_{\mu\nu}$ are the generators of $SL(2,\mathbb{R}) \cong SO(1,2)$. The σ_{ij} generate $SO(\mathrm{Im}\mathbb{A})$ and are proportional to the identity as 2×2 matrices, but act as operators on $\mathbb{A}$ itself.

4.1 Magic square of non-compact groups G

- For the purposes of squaring YM a manifestly $\mathbb{A}_L \leftrightarrow \mathbb{A}_R$ symmetric formulation of the square is required. This is achieved by adapting the *triality algebra* construction introduced by Barton and Sudbery [Barton and Sudbery:2003, MJD et al:2013].
- Our definition is given by

$$\mathfrak{g}_3(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) + \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \times \mathbb{A}_R).$$

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SL(2,\mathbb{R})$	$SU(2,1)$	$USp(4,2)$	$F_{4(-20)}$
$\mathbb{C}$	$SU(2,1)$	$SU(2,1)\times SU(2,1)$	$SU(4,2)$	$E_{6(-14)}$
$\mathbb{H}$	$USp(4,2)$	$SU(4,2)$	$SO(8,4)$	$E_{7(-5)}$
$\mathbb{O}$	$F_{4(-20)}$	$E_{6(-14)}$	$E_{7(-5)}$	$E_{8(8)}$

Table : Magic square

- eg

$$E_{8(8)} = SO(8) + SO(8) + 3(\mathbb{O}\times\mathbb{O})$$

$$248 = 28 + 28 + (8_v, 8_v) + (8_s, 8_s) + (8_c, 8_c)$$

4.2 Comment

- There are different magic squares depending on the choice of real forms. Ours should not to be confused with other versions that have appeared in the so-called "magic" supergravities in $D = 4, 5, 6$ [Gunaydin, Sierra and Townsend 1983]
- The specific square of real forms we derive from the triality construction was first obtained in [Cacciatori-Cerchiai-Marrani:2012] using a "Lorentizian Jordan algebra" adaptation of the Tits formula [Tits:1962]

4.3 Magic square of maximal compact subalgebras H

- We shall also need a magic square of the maximal compact subalgebras. This is given by the *reduced* triality construction,

$$\mathfrak{g}_1(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) + \mathfrak{tri}(\mathbb{A}_R) + (\mathbb{A}_L \times \mathbb{A}_R),$$

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	SO(2)	SO(3) $\times$ SO(2)	SO(5) $\times$ SO(3)	SO(9)
$\mathbb{C}$	SO(3) $\times$ SO(2)	[SO(3) $\times$ SO(2)]^2	SO(6) $\times$ SO(3)	SO(10) $\times$ SO(2)
$\mathbb{H}$	SO(5) $\times$ SO(3)	SO(6) $\times$ SO(3)	SO(8) $\times$ SO(4)	SO(12) $\times$ SO(3)
$\mathbb{O}$	SO(9)	SO(10) $\times$ SO(2)	SO(12) $\times$ SO(3)	SO(16)

Table : Magic square of maximal compact subalgebras.

5.1 Squaring $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ Yang-Mills

- Having cast the magic square in terms of a manifestly $\mathbb{A}_L \leftrightarrow \mathbb{A}_R$ symmetric triality algebra construction, and having written $\mathcal{N} = 1, 2, 4, 8$ YM in terms of fields valued in $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ we shall now obtain the magic square of supergravities by "squaring" $\mathcal{N} = 1, 2, 4, 8$ YM.
- In the supersymmetric context it is not difficult to see that the amount of supersymmetry is given by

$$[\mathcal{N}_L \text{ SYM}] \times [\mathcal{N}_R \text{ SYM}] \to [\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R \text{ SG}],$$

- It is harder to see how the other gravitational symmetries arise from those of Yang-Mills. For example $D = 4, N = 8$ supergravity has a global non-compact symmetry (U-duality) $E_{7(7)}$ and a local compact symmetry $SU(8)$, whereas $D = 4, N = 4$ super Yang-Mills has global $SU(4)$ R-symmetry.

5.2 U-dualities of supergravity

$\mathcal{N}$	D	scalars	vectors	G	H
2	10A	1	1	$SO(1,1,\mathbb{R})$	–
2	10B	2	0	$SL(2,\mathbb{R})$	$SO(2,\mathbb{R})$
4	6	25	16	$SO(5,5,\mathbb{R})$	$SO(5,\mathbb{R}) \times SO(5,\mathbb{R})$
8	4	70	28	$E_{7(7)}(\mathbb{R})$	$SU(8)$
16	3	128	-	$E_{8(8)}(\mathbb{R})$	$SO(16,\mathbb{R})$

Table : The symmetry groups (G) of the low energy supergravity theories with 32 supercharges in different dimensions (D) and their maximal compact subgroups (H). In each D one may truncate to lower $\mathcal{N}$ to get smaller G and H

The scalars belong to the space G/H

5.3 Squaring $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ Yang-Mills in $D = 3$

Taking a left SYM multiplet

$$\{A_\mu(L) \in \text{Re}\mathbb{A}_L, \quad \phi(L) \in \text{Im}\mathbb{A}_L, \quad \lambda(L) \in \mathbb{A}_L\}$$

and tensoring with a right multiplet

$$\{A_\mu(R) \in \text{Re}\mathbb{A}_R, \quad \phi(R) \in \text{Im}\mathbb{A}_R, \quad \lambda(R) \in \mathbb{A}_R\}$$

we obtain the field content of a supergravity theory valued in both $\mathbb{A}_L$ and $\mathbb{A}_R$:

$\mathbb{A}_L/\mathbb{A}_R$	$A_\mu(R) \in \text{Re}\mathbb{A}_R$	$\phi(R) \in \text{Im}\mathbb{A}_R$	$\lambda(R) \in \mathbb{A}_R$
$A_\mu(L) \in \text{Re}\mathbb{A}_L$	$g_{\mu\nu} + \varphi \in \text{Re}\mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\varphi \in \text{Re}\mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\Psi_\mu + \chi \in \text{Re}\mathbb{A}_L \otimes \mathbb{A}_R$
$\phi(L) \in \text{Im}\mathbb{A}_L$	$\varphi \in \text{Im}\mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\varphi \in \text{Im}\mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\chi \in \text{Im}\mathbb{A}_L \otimes \mathbb{A}_R$
$\lambda(L) \in \mathbb{A}_L$	$\Psi_\mu + \chi \in \mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\chi \in \mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\varphi \in \mathbb{A}_L \otimes \mathbb{A}_R$

Grouping spacetime fields of the same type we find,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}, \quad \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}$$

5.4 Grouping together

- Grouping spacetime fields of the same type we find,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi, \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}. \tag{3}$$

- Note we have dualised all resulting p-forms, in particular vectors to scalars. The $\mathbb{R}$-valued graviton and $\mathbb{A}_L \oplus \mathbb{A}_R$-valued gravitino carry no degrees of freedom. The $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$-valued scalar and Majorana spinor each have $2(\dim \mathbb{A}_L \times \dim \mathbb{A}_R)$ degrees of freedom.
- Fortunately, $\mathbb{A}_L \oplus \mathbb{A}_R$ and $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ are precisely the representation spaces of the vector and (conjugate) spinor. For example, in the maximal case of $\mathbb{A}_L, \mathbb{A}_R = \mathbb{O}$, we have the $\mathbf{16}, \mathbf{128}$ and $\mathbf{128}'$ of SO(16).

5.5 U-dualities from division algebras

- U-dualities G are realised non-linearly on the scalars, which parametrise the symmetric spaces G/H.
- This can be understood using the remarkable identity relating the projective planes over $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ to G/H,

$$(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2 \cong G/H.$$

  The scalar fields may be regarded as points in division-algebraic projective planes [Baez:2001, Freudenthal:1964, Landsberg2001].
- The tangent space at any point of $(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2$ is just $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$, the required representation space of H
- Example: recall the *Cayley plane* $\mathbb{O}\mathbb{P}^2$, with isometry group $F_{4(-52)}$, is a classic example:

$$F_{4(-52)}/\text{Spin}(9) \cong (\mathbb{R} \otimes \mathbb{O})\mathbb{P}^2 = \mathbb{O}\mathbb{P}^2$$

  The tangent space at any point of $\mathbb{O}\mathbb{P}^2$ is $\mathbb{O}^2$, the spinor of Spin(9) as required

5.6 Final result

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathcal{N}=2, f=4$ $G=\mathrm{SL}(2,\mathbb{R})$, dim 3 $H=\mathrm{SO}(2)$, dim 1	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36
$\mathbb{C}$	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=4, f=16$ $G=\mathrm{SU}(2,1)^2$, dim 16 $H=\mathrm{SU}(2)^2\times\mathrm{SO}(2)^2$, dim 8	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46
$\mathbb{H}$	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=8, f=64$ $G=\mathrm{SO}(8,4)$, dim 66 $H=\mathrm{SO}(8)\times\mathrm{SO}(4)$, dim 34	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69
$\mathbb{O}$	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69	$\mathcal{N}=16, f=256$ $G=E_{8(8)}$, dim 248 $H=\mathrm{SO}(16)$, dim 120

- The $\mathcal{N}>8$ supergravities in $D=3$ are unique, all fields belonging to the gravity multiplet, while those with $\mathcal{N}\leq 8$ may be coupled to k additional matter multiplets [Marcus:1983, deWit:1992]. The real miracle is that tensoring left and right YM multiplets yields the field content of $\mathcal{N}=2,3,4,5,6,8$ supergravity with $k=1,1,2,1,2,4$: just the right matter content to produce the U-duality groups appearing in the magic square.

5.7 Conclusion

- In both cases the field content is such that the U-dualities exactly match the groups of of the magic square:

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SL(2,\mathbb{R})$	$SU(2,1)$	$USp(4,2)$	$F_{4(-20)}$
$\mathbb{C}$	$SU(2,1)$	$SU(2,1)\times SU(2,1)$	$SU(4,2)$	$E_{6(-14)}$
$\mathbb{H}$	$USp(4,2)$	$SU(4,2)$	$SO(8,4)$	$E_{7(-5)}$
$\mathbb{O}$	$F_{4(-20)}$	$E_{6(-14)}$	$E_{7(-5)}$	$E_{8(8)}$

Table : Magic square

- This $D = 3$ square is the base of a Magic Pyramid

6.1 Spacetime Fields in $D = n + 2$

The division algebras can be used to describe field theory in Minkowski space using the Lie algebra isomorphism

$$\mathfrak{so}(1, 1+n) \sim \mathfrak{sl}(2, \mathbb{A}). \qquad (4)$$

Field Symbol	Representation	Rep. Symbol	Group
$\Psi_{\mathbb{A}}$	Spinor	S_+	$\mathrm{SO}(1, n+1)$
$\mathcal{X}_{\mathbb{A}}$	Conjugate Spinor	S_-	$\mathrm{SO}(1, n+1)$
$A_{\mathbb{A}}$	Vector	V	$\mathrm{SO}(1, n+1)$
$\psi_{\mathbb{A}}$	Spinor	s	$\mathrm{SO}(n)$
$\chi_{\mathbb{A}}$	Conjugate Spinor	c	$\mathrm{SO}(n)$
$a_{\mathbb{A}}$	Vector	v	$\mathrm{SO}(n)$

Table : A summary of the fields and notation used in $D = n + 2$

$D \setminus \mathcal{N}$	1	2	4	8
10	$\mathrm{SO}(8)_{ST}$			
6	$\mathrm{Sp}(1)^2_{ST} \times \mathrm{Sp}(1)_I$	$\mathrm{Sp}(1)^2_{ST} \times \mathrm{Sp}(1)^2_I$		
4	$\mathrm{U}(1)_{ST} \times \mathrm{U}(1)_I$	$\mathrm{U}(1)_{ST} \times \mathrm{U}(2)_I$	$\mathrm{U}(1)_{ST} \times \mathrm{SU}(4)_I$	
3	1	$\mathrm{SO}(2)_I$	$\mathrm{SO}(4)_I$	$\mathrm{SO}(7)_I$

Table : Spacetime little groups and internal symmetry groups

6.4 Super Yang-Mills in $D = 3, 4, 6, 10$

$D/\mathcal{N}$	1	2	4	8
10	$\mathrm{SO}(8)_{ST}$			
6	$\mathrm{SO}(4)_{ST} \times \mathrm{Sp}(1)_R$	$\mathrm{SO}(4)_{ST} \times \big(\mathrm{Sp}(1) \times \mathrm{Sp}(1)\big)_R$		
4	$\mathrm{SO}(2)_{ST} \times \mathrm{U}(1)_R$	$\mathrm{SO}(2)_{ST} \times \big(\mathrm{SU}(2) \times \mathrm{U}(1)^2\big)_R$	$\mathrm{SO}(2)_{ST} \times \mathrm{SU}(4)_R$	
3	1	$\mathrm{SO}(2)_R$	$\mathrm{SO}(4)_R$	$\mathrm{SO}(8)_R$

Table : Space-time Little groups and R-symmetry groups

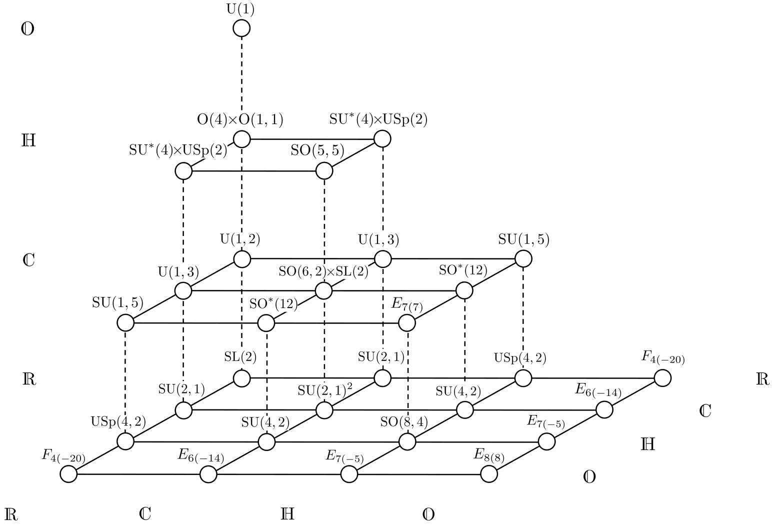
- $\mathrm{SL}(2,\mathbb{A}) = SO(2,1), SO(3,1), SO(5,1), SO(9,1)$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

6.5 Components in $D = 3, 4, 6, 10$

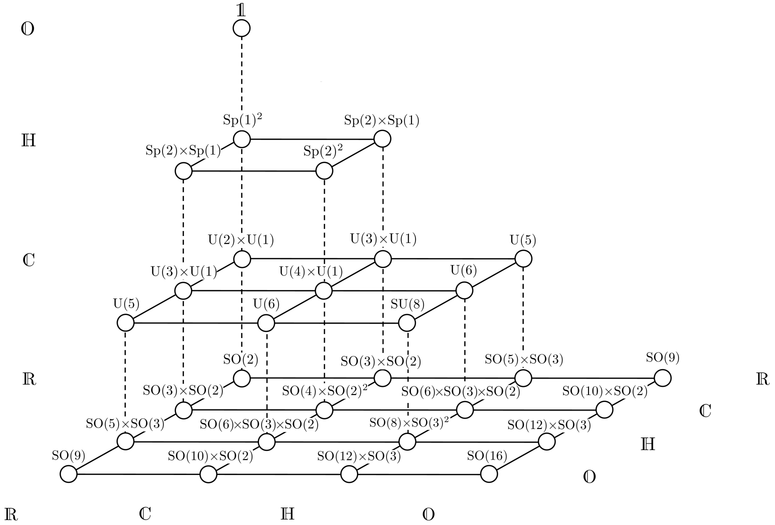
Dim.	Field	No. of comps.	Little field	No. of comps.
$10 = 8 + 2$	$\Psi_{\mathbb{O}}$	$16 = 2 \times 8$	$\psi_{\mathbb{O}}$	8
	$X_{\mathbb{O}}$	$16 = 2 \times 8$	$\chi_{\mathbb{O}}$	8
	$A_{\mathbb{O}}$	10	$a_{\mathbb{O}}$	8
$6 = 4 + 2$	$\Psi_{\mathbb{H}}$	$8 = 2 \times 4$	$\psi_{\mathbb{H}}$	4
	$X_{\mathbb{H}}$	$8 = 2 \times 4$	$\chi_{\mathbb{H}}$	4
	$A_{\mathbb{H}}$	6	$a_{\mathbb{H}}$	4
$4 = 2 + 2$	$\Psi_{\mathbb{C}}$	$4 = 2 \times 2$	$\psi_{\mathbb{C}}$	2
	$X_{\mathbb{C}}$	$4 = 2 \times 2$	$\chi_{\mathbb{C}}$	2
	$A_{\mathbb{C}}$	4	$a_{\mathbb{C}}$	2
$3 = 2 + 1$	$\Psi_{\mathbb{R}}$	$2 = 2 \times 1$	$\psi_{\mathbb{R}}$	1
	$A_{\mathbb{R}}$	3	$a_{\mathbb{R}}$	1

Table : Space-time fields

6.7 Magic Pyramid: G symmetries



6.8 Magic Pyramid: H symmetries



6.9 Conformal Magic Pyramid: G symmetries