Permutation in Bi-Adjoint Scalar

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based on work with Rijun Huang, Fei Teng, arXiv:1801.08965

Contents

I: Motivation

Scattering amplitude is one of most important concepts in QFT.

- It is the bridge connecting experiment data and theoretical prediction
- Its various properties contain also many important information about the theory, such as Lorentz symmetry, local interaction, unitarity etc.
- Recent study shows that scattering amplitude contains many interesting mathematical structures. In other words, scattering amplitudes provides another field where physics and mathematics are tightly connected.
- In this talk, we will use one of frameworks for on-shell amplitudes, i.e., the CHY-frame to discuss a relation between physics and mathematics structure

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In 2013, new formula for tree amplitudes of massless theories has been proposed by Cachazo, He and Yuan:

$$\mathcal{A}_n = \boxed{\int \frac{\left(\prod_{i=1}^{n} dz_i\right)}{d\omega} \Omega(\mathcal{E})} \mathcal{I},$$

[Freddy Cachazo, Song He, Ellis Ye Yuan , 2013, 2014]

In this frame:

- Each particle is represented by a puncture in Riemann sphere
- The expression holds for general D-dimension
- The box part is universal for all theories
- The CHY-integrand $\mathcal{I}$ determines the particular theory

For the universal part,

$$\Omega(\mathcal{E}) \equiv \prod_a{}' \delta(\mathcal{E}_a) = z_{ij} z_{jk} z_{ki} \prod_{a \neq i,j,k} \delta(\mathcal{E}_a)$$

provides the constraints:

- Scattering equations are defined

$$\mathcal{E}_a \equiv \sum_{b \neq a} \frac{2k_a \cdot k_b}{z_a - z_b} = 0, \qquad a = 1, 2, ..., n$$

- Only $(n-3)$ of them are independent by $SL(2,C)$ symmetry

$$\sum_a \mathcal{E}_a = 0, \quad \sum_a \mathcal{E}_a z_a = 0, \quad \sum_a \mathcal{E}_a z_a^2 = 0,$$

Universal part: $(n-3)$ integrations with $(n-3)$ delta-functions, so the integration becomes the sum over all solutions of scattering equations

$$\sum_{z\in \mathrm{Sol}} \frac{1}{\det'(\Phi)}\mathcal{I}(z)$$

where $\det'(\Phi)$ is the Jacobi coming from solving $\mathcal{E}_a$

$$\Phi_{ab}=\frac{\partial \mathcal{E}_a}{\partial z_b}=\begin{cases} \frac{s_{ab}}{z_{ab}^2} & a\neq b \\ -\sum_{c\neq a}\frac{s_{ac}}{z_{ac}^2} & a=b \end{cases},$$

In CHY-frame, different theories are defined by different CHY-integrand:

- For almost all theories, the integrand is given by factorization form, i.e.,

$$\mathcal{I} = \mathcal{I}_L \times \mathcal{I}_R$$

where each one at the right hand side has weight two.

- Let us define the closed cycle

$$(a_1 a_2 ... a_n) = (z_{a_1} - z_{a_2})(z_{a_2} - z_{a_3})...(z_{a_{n-1}} - z_{a_n})(z_{a_n} - z_{a_1})$$

then the color ordered Parker-Taylor factor is defined as

$$\mathcal{C}(\alpha) = \mathrm{PT}(\alpha) = \frac{1}{(\alpha(1)...\alpha(n))}$$

which has the weight two.

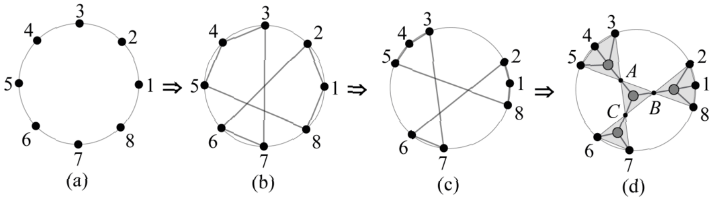
The Bi-Adjoint scalar theory is defined as

$$\mathcal{I} = \mathrm{PT}(\alpha)\mathrm{PT}(\beta)$$

- It is the simplest theory in CHY frame.
- It is also the basis for all other theories, since it provides the skeleton of Feynman diagrams. Any other theories, can be written as the linear combination of bi-adjoint scalars.
- For this simple case, there are two ways to read out analytic expression straightforward. So this theory is well understood.

First Method: Effective Feynman diagram

[Cachazo, He, Yuan , 2013]



Second Method: Integration rule

[Baadsgaard, Bjerrum-Bohr, Bourjaily and Damgaard, 2015]

- First there is a criteria for the appearance of pole $s_A = (\sum_{i \in A} k_i)^2$ for a subset $A \subset \{1, 2, ..., n\}$, i.e., the index

$$\chi(A) := \mathbb{L}[A] - 2(|A| - 1)$$

$\mathbb{L}[A]$ be the number (more accurately it is the difference of number between solid and dashed lines) of lines connecting these nodes inside A and $|A|$ is the number of nodes.

- It has nonzero contribution when and only when $\chi(A) \geq 0$ and the pole will be

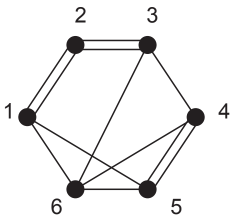
$$\frac{1}{s_A^{\chi(A)+1}}$$

The Reconstruction of cubic Feynman diagrams:

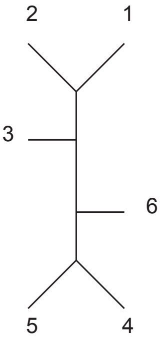
- Find all subsets A with $\chi(A) \geq 0$
- compatible condition for two subsets A_1, A_2: they are compatible if one subset is completely contained inside another subset or the intersection of two subsets is empty.
- Find all maximum compatible combinations, i.e., the combination of subsets with largest number such that each pair in the combination is compatible. For each maximum combination with m subsets, it gives nonzero contribution when and only when $m = n - 3$.

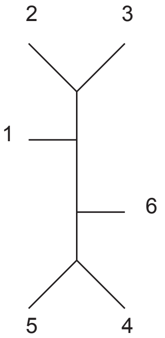
- Each combination giving nonzero contribution will correspond a (generalized) Feynman diagram with only cubic vertexes
- Now the key is how to read out expressions of Feynman diagrams?
- For simple pole, the rule is nothing, but the scalar propagator $\frac{1}{s_A}$!

Example of 6-point



$\{1,2\}, \quad \{2,3\}, \quad \{4,5\}, \quad \{1,2,3\}$





$\{1,2\} + \{4,5\} + \{1,2,3\}$

$$\frac{1}{s_{12}s_{123}s_{45}}$$

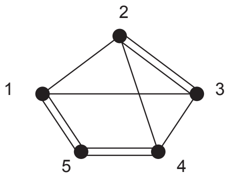
$\{2,3\} + \{4,5\} + \{1,2,3\}$

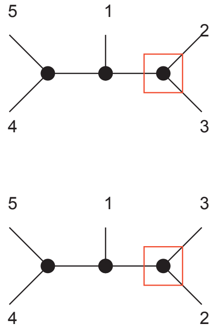
$$\frac{1}{s_{23}s_{123}s_{45}}$$

Above algorithm works well for $m[\alpha|\beta]$ with two PT-factors

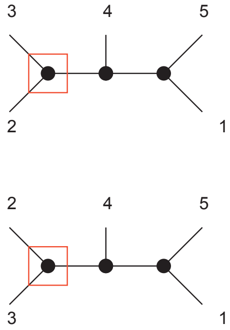
[Freddy Cachazo, Song He, Ellis Ye Yuan , 2013]

Example $\frac{1}{(12345)(13245)}$ with $(a_1 ... a_m) = z_{a_1 a_2} ... z_{a_m a_1}$





$\frac{1}{s_{23}s_{45}}$



$\frac{1}{s_{23}s_{51}}$

Given two PT-factors, it is very natural to related them to a permutation. Thus we wonder if we can understand all results in Bi-Adjoint scalar theory from this new point of view. This is our main motivation for such investigation!

II: The Set-up

- First by relabeling, we can always set one of PT-factors to the standard ordering, i.e.,

$$\mathrm{PT}(\alpha) \equiv \langle 12....n \rangle$$

- The second PT-factor can be arbitrary list among $n!$. However, since PT-factor is defined up to cyclic permutation and order reversing, so only $n!/(2n)$ are really different theories.

Thus each pair of $\mathrm{PT}(\alpha), \mathrm{PT}(\beta)$ has defined $(2n)$ permutations:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (1)(2)(34), \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (124)(3)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = (1423), \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = (132)(4)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = (1324), \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} = (143)(2)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} = (12)(3)(4), \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (1)(234)$$

These $2n$ permutations can also be generated as following:

- Define the cyclic generator $\mathbf{g}_c$

$$\mathbf{g}_c = (\beta_1\beta_2\cdots\beta_n)$$

- Define the reversing generator $\mathbf{g}_r$

$$\mathbf{g}_r = \begin{cases} (\beta_1\beta_n)(\beta_2\beta_{n-1})\cdots(\beta_{\frac{n}{2}}\beta_{\frac{n+2}{2}}) & \text{for even } n \\ (\beta_1\beta_n)(\beta_2\beta_{n-1})\cdots(\beta_{\frac{n-1}{2}}\beta_{\frac{n+3}{2}})(\beta_{\frac{n+1}{2}}) & \text{for odd } n \end{cases}$$

  $\mathbf{g}_c, \mathbf{g}_r$ generate the Dihedral group D_{2n}.

- For a permutation β, the equivalent class is thus given by,

$$\mathfrak{b}[\beta] = \left\{ \beta\,,\ \beta\,\mathbf{g}_c\,,\ldots,\ \beta\,\mathbf{g}_c^{n-1}\,, \right.$$
$$\left. \beta\,\mathbf{g}_r\,,\ \beta\,\mathbf{g}_r\,\mathbf{g}_c\,,\ \ldots,\ \beta\,\mathbf{g}_r\,\mathbf{g}_c^{n-1} \right\}.$$

Next we classify the cycle representation of a permutation by defining the *good cycle representations* as those satisfying the following criteria:

- the cycles in the considered cycle representation can be separated into at least two parts, while the union of cycles in each part is consecutive (later called *planar separation*).
- in case that the cycle representation *can only be* separated into two parts, then each part should contain at least two elements.
- Some examples:

$$\text{Good}: \quad (1)(2)(38)(4)(56)(7)\ ,(132)(4875)(6)$$
$$\text{Good}: \quad (15274)(3)(68)\ ,(176423)(58)\ ,(1764235)(5)\ ,$$

Moreover, for good cycle representation, we can further classified to following two types:

- If it contains at least three parts, we call it a *vertex type* (V-type) cycle representation. For example

$$(12)|(34)|(56)$$

- If it contains only two cyclic parts, we call it a *pole type* (P-type) cycle representation. For example

$$(12)|(35)(46)$$

Such a separation is called planar separation

III: First Main Result

Our first main result is the one-to-one mapping between the effective Feynman diagram and the cycle representation of permutations:

- At one side, each V-type cycle representation contain the vertex structure of the corresponding effective Feynman diagram. Thus combining them together, we can draw the effective Feynman diagram immediately just like the first method.
- At another side, given an effective Feynman diagram, we can construct the corresponding permutation class.

From permutation to Feynman diagram: consider the example $\mathrm{PT}(\beta) = \langle 12846573 \rangle$.

- Good V-type cycle representations:

$$(1)(2)(38)(4)(56)(7) \quad , \quad (12)(3)(47)(5)(6)(8)$$

- Drawing the planar separations. For (1)(2)(38)(4)(56)(7) allows two different planar separations,

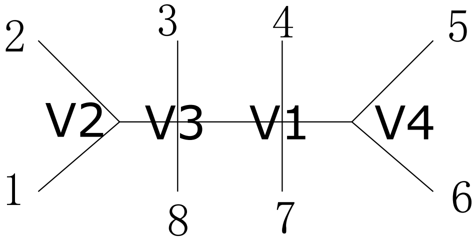
$$\begin{aligned} \text{4 parts } V_1 : &\quad (1)(2)(38)|(4)|(56)|(7)\,, \\ \text{3 parts } V_2 : &\quad (1)|(2)|(38)(4)(56)(7)\,. \end{aligned}$$

Similarly, the (12)(3)(47)(5)(6)(8) gives

$$\begin{aligned} \text{4 parts } V_3 : &\quad (12)|(3)|(47)(5)(6)|(8)\,, \\ \text{3 parts } V_4 : &\quad (8)(12)(3)(47)|(5)|(6)\,,\ . \end{aligned}$$

Combining four vertex structures, we get a single effective Feynman diagram, with the analytic result is

$$\frac{1}{s_{12}s_{56}s_{8123}}\left(\frac{1}{s_{812}}+\frac{1}{s_{123}}\right)\left(\frac{1}{s_{456}}+\frac{1}{s_{567}}\right).$$



A trivial consequence for cases with zero contributions:

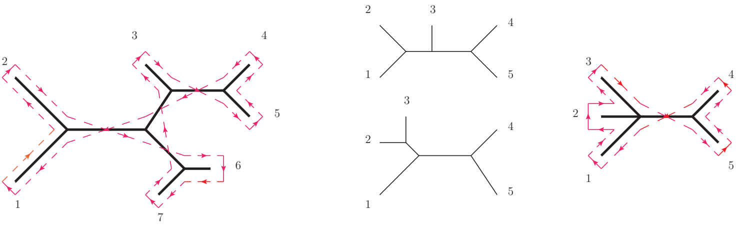
- There is no good V-type cycle representation.
- It has some good V-type cycle representations, but it does not satisfy the following necessary condition for existing the valid effective Feynman diagrams is

$$\sum_{m=3}^{n}(m-2)v_m = n-2\,.$$

where we use v_m to denote the number of m-point vertices For example, $\langle 124635\rangle$ has only one V-type $V_3 = (1)|(2)|(3465)$

From the diagram to permutation:

- There is one direct method to read out $\mathrm{PT}(\beta)$ by the zig-zag path: The left gives $\mathrm{PT}(\beta) = \langle 1267354 \rangle$ while the right gives $\mathrm{PT}(\beta) = \langle 12354 \rangle$



(a) (b)

Cycle representation by recursive construction. Let us focus on a given vertex in Feynman diagram, then there are two cycle representations:

- Planar ordered cycle representation: If a vertex is connected by k legs, we have cycle structure

$$V_P = (V_{O;P_1})(V_{O;P_2})...(V_{O;P_k})$$

where each part is given by order reversed cycle representation.

- Order reversed cycle representation of a given leg: Taking a leg, for example, P_k, then the cycle representation is given by multiplication of following two group elements

$$V_{O;P_k} = [(V_{P_1})(V_{P_2})...(V_{P_{k-1}})].[\mathbf{g}_r]$$

where $\mathbf{g}_r$ is defined as before.

- Now we can see the recursive construction between these two types of cycle representations.

First let us construct the V_O by following steps:

- Start from vertexes with only one propagator. For our example, we have V_2, V_4, thus we have

$$\begin{aligned} V_{2;O,P_{12}} &= [(1)(2)].[(12)] = (12), \\ V_{4;O,P_{56}} &= [(5)(6)].[(56)] = (56), \end{aligned}$$

- Then we consider the vertex with two propagators. Now we have the choice to do, i.e., which propagator has been selected to do the ordering reversing. For V_3, selecting the P_{8123} is simpler, and we have

$$V_{3;O,P_{8123}} = [(8)(12)(3)].[(38)(12)] = (38)(1)(2)$$

Similarly,

$$V_{1;O,P_{4567}} = [(4)(56)(7)].[(47)(56)] = (47)(5)(6)$$

- Having above result, we can calculate

$$V_{3;O,P_{12}} = [(3)(47)(5)(6)(8)].[(38)(47)(56)] = (38)(4)(7)(56)$$
$$V_{1;O,P_{56}} = [(7)(38)(1)(2)(4)].[(47)(38)(12)] = (47)(12)(3)(8)$$

- Having all pieces, we can read the final cycle representations:

$$V_1 : \quad (4)(7)(56)(38)(1)(2)$$
$$V_2 : \quad (1)(2)(38)(4)(7)(56)$$
$$V_3 : \quad (8)(12)(3)(47)(5)(6)$$
$$V_4 : \quad (5)(6)(47)(12)(3)(8)$$

We see that $V_3 = V_4$ and $V_1 = V_2$ are two V-type cycle representations we have found.

IV: Second Main Result

Our second main result is the relation among different pairs of PT-factors: contributions from one theory is contained in another theory. This "mother-daughter" relation is studied by two methods:

- Method One: order reversing acting on sub cycle representation
- Method Two: Multiplying the cross ratio factor

For the bi-adjoint theory, its result can be expressed by a geometric object "associahedron" with dimension $(n-3)$

[Arkani-Hamed, Bai, He, Yan, 2017]

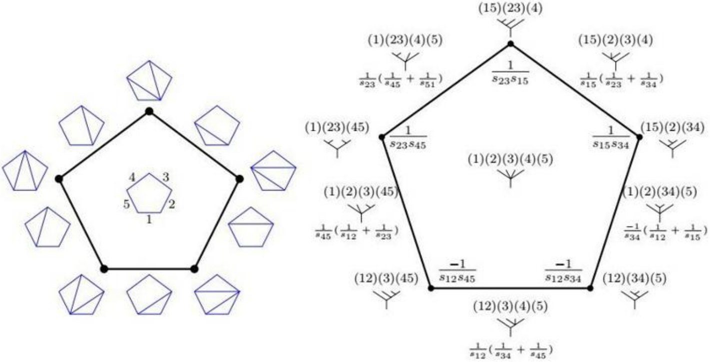


Figure 6: The associahedron for five point amplitudes and the PT-factors.

Method Two: Thus we see the relation is given by fixing or relaxing a given propagator. Now we see how to achieve this from cycle-representation:

- For good cycle-representation, we have

$$\beta = \beta_{\text{lower}}\beta_{\text{upper}}$$

  Notice that, the separation into two parts corresponding to pick up a pole.

- Taking either part, do the ordering reversing for this part, for example

$$\beta_{\text{upper}}^{\text{reversed}} = \beta_{\text{upper}}\beta_{r}$$

- Combing with untouched part, we get a new theory with the good cycle-representation.

There are two situations:

- If the pole is common for all contributions, we will get the mother of original theory.
- If the pole is not the common pole for all contributions, we will get the daughter of original theory, i.e., picking up only terms with this given pole.

Further remarks:

- First, not matter with good cycle-representation (having this pole) one pick, and which part (lower or upper) you do the order reversing, we will always get the same theory, but possible different good cycle representations.
- Secondly, the number of daughter and mothers can be read out from how many non-common or common poles from a given theory.

Example:

$$\begin{aligned}\text{PT}(\beta) &= \langle 12846573\rangle \\ &\Longrightarrow \frac{1}{s_{12}s_{56}s_{8123}}\left(\frac{1}{s_{812}}+\frac{1}{s_{123}}\right)\left(\frac{1}{s_{456}}+\frac{1}{s_{567}}\right).\end{aligned}$$

For daughter with the pole s_{123}:

- First fine the cycle representations (12)(3)(47)(5)(6)(8) and (132)(4875)(6) contain the pole.
- Secondly we do the manipulation

$$(12)(3)(47)(5)(6)(8) \Longrightarrow$$
$$\begin{cases} [(12)(3)]\cdot[(13)(2)](47)(5)(6)(8) &= (132)(47)(5)(6)(8) \\ (12)(3)[(47)(5)(6)(8)]\cdot[(48)(57)(6)] &= (12)(3)(4875)(6)\end{cases}$$

$$(132)(4875)(6) \Longrightarrow$$
$$\begin{cases} (132)]\cdot[(13)(2)](4875)(6) &= (12)(3)(4875)(6) \\ (132)[(4875)(6)]\cdot[(48)(57)(6)] &= (132)(47)(5)(6)(8)\end{cases}.$$

- Both results correspond to the PT-factor $\mathrm{PT}(\beta) = \langle 12756483 \rangle$, which is evaluated to,

$$\frac{1}{s_{12}s_{56}s_{8123}} \left(\frac{1}{s_{123}} \right) \left(\frac{1}{s_{456}} + \frac{1}{s_{567}} \right) .$$

Finding the mother with pole s_{12}:

- First, there are two cycle representations (1)(2)(38)(4)(56)(7) and (12)(3)(47)(5)(6)(8) with the pole
- Do similar thing

$$(1)(2)(38)(4)(56)(7) \Longrightarrow$$
$$\begin{cases} [(1)(2)]\cdot[(12)](38)(4)(56)(7) = (12)(38)(4)(56)(7) \\ (1)(2)[(38)(4)(56)(7)]\cdot[(38)(47)(56)] = (1)(2)(3)(47)(5)(6)(8) \end{cases}$$

$$(12)(3)(47)(5)(6)(8) \Longrightarrow$$
$$\begin{cases} [(12)]\cdot[(12)](3)(47)(5)(6)(8) = (1)(2)(3)(47)(5)(6)(8) \\ (12)[(3)(47)(5)(6)(8)]\cdot[(38)(47)(56)] = (12)(38)(4)(56)(7) \end{cases},$$

- All gives $\mathrm{PT}(\beta) = \langle 12375648 \rangle$

Method Three: Using the cross ratio selecting factor

$$f^{\text{select}}[a,b,c,d] := \frac{[ab][cd]}{[ac][bd]}\,, \qquad [ab] := \sigma_{ab}\,.$$

To find the daughter theory:

- First, both PT-factor must contain the sequence of pole, for example, for pole s_{2345}, following sequences are right: $(2435),(2345),(3542)$....
- Using $f^{\text{select}}[a,b,c,d]$ multiply either PT-factor, we get the daughter, where a,b,c,d are the letter in the sequence

$$..., a, \{\text{pole}\}, \text{d}, ..., \qquad \{\text{pole}\} = \text{b}, ..., \text{c}$$

To find the mother theory, we do similar thing.

V: Final Remarks

Symmetric discussion:

- For bi-adjoint theory, there are two PT-factors. Thus permutation group S_n acting on them natrually.
- If S_n acts on both PT-factors same time, we will get same theory. Using this symmetry, we can fix one PT-factor to be the standard ordering $\langle 12...n \rangle$.
- For the standard ordering, there is an invariant subgroup, dihedral D_{2n} generated by cyclic Z_n and ordering reversing Z_2. Such D_{2n} acts on another PT-factor, will related different theories. Such action generates various orbits. Thus we can category different theories by these orbits.

- Bi-adjoint theories are the basis for all other theories in CHY frame. Thus it is naturally to ask if there is similar symmetric understanding for them?
- Results given in this talk are observed from various examples. Why it is true and what is the deep relation are very interesting problems to investigate.
- Bi-adjoint theories are related to associahedron. Thus the symmetric action should be applicable from this point of view.

Thanks for your attention !!!