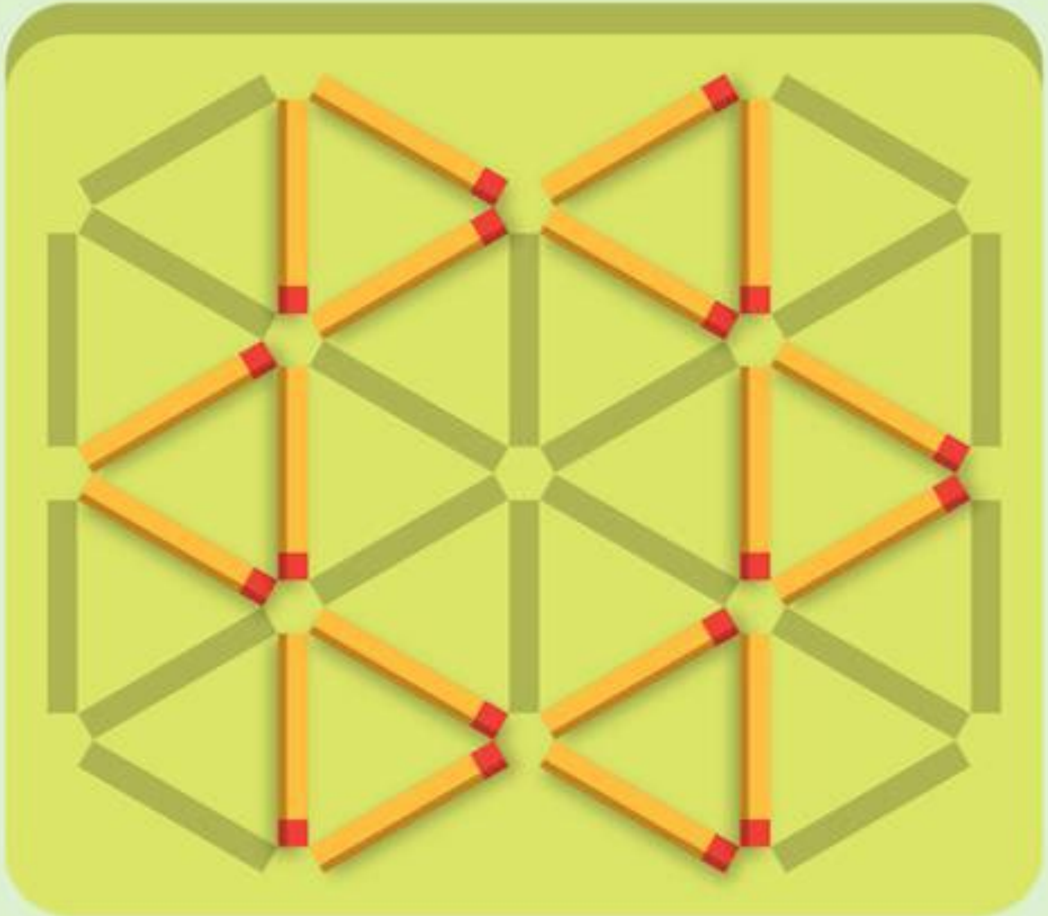
L3-96

0

5

MENU

MOVE 2 MATCHES TO GET 6 TRIANGLES



Scattering Amplitude in CHY-Formulation

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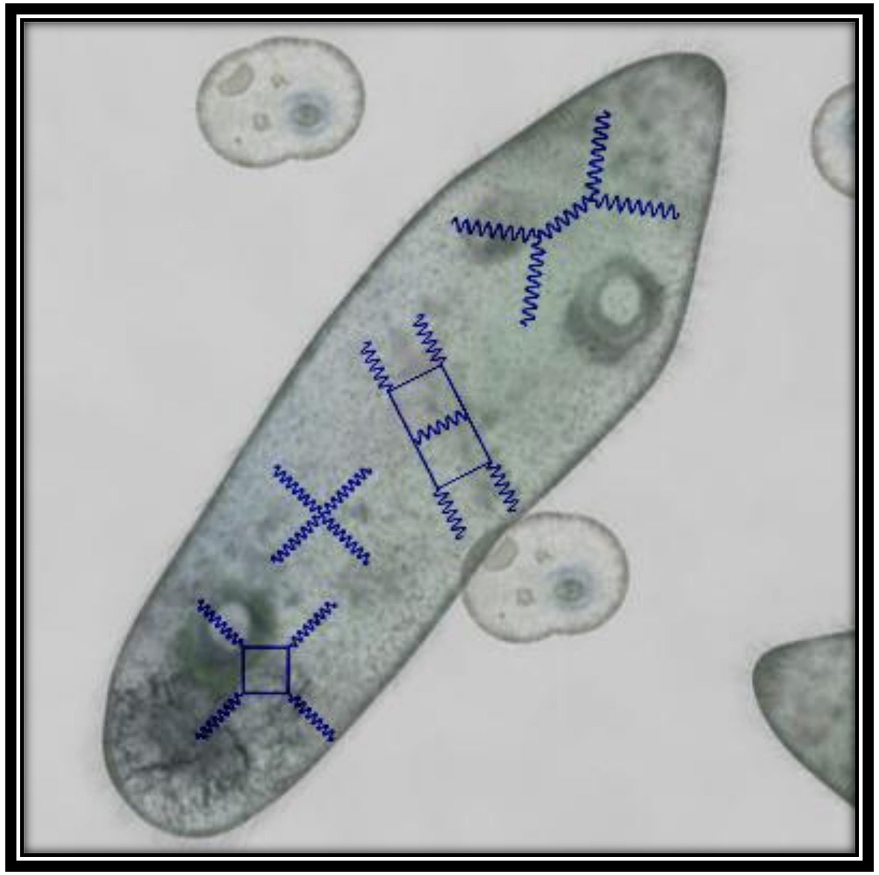
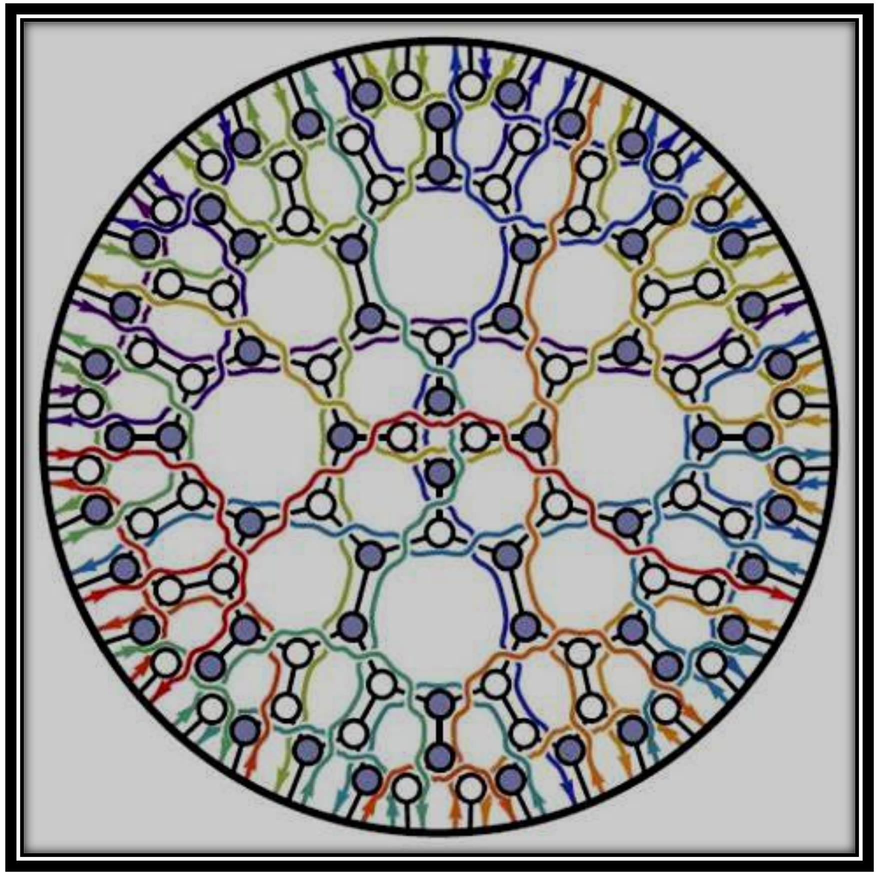
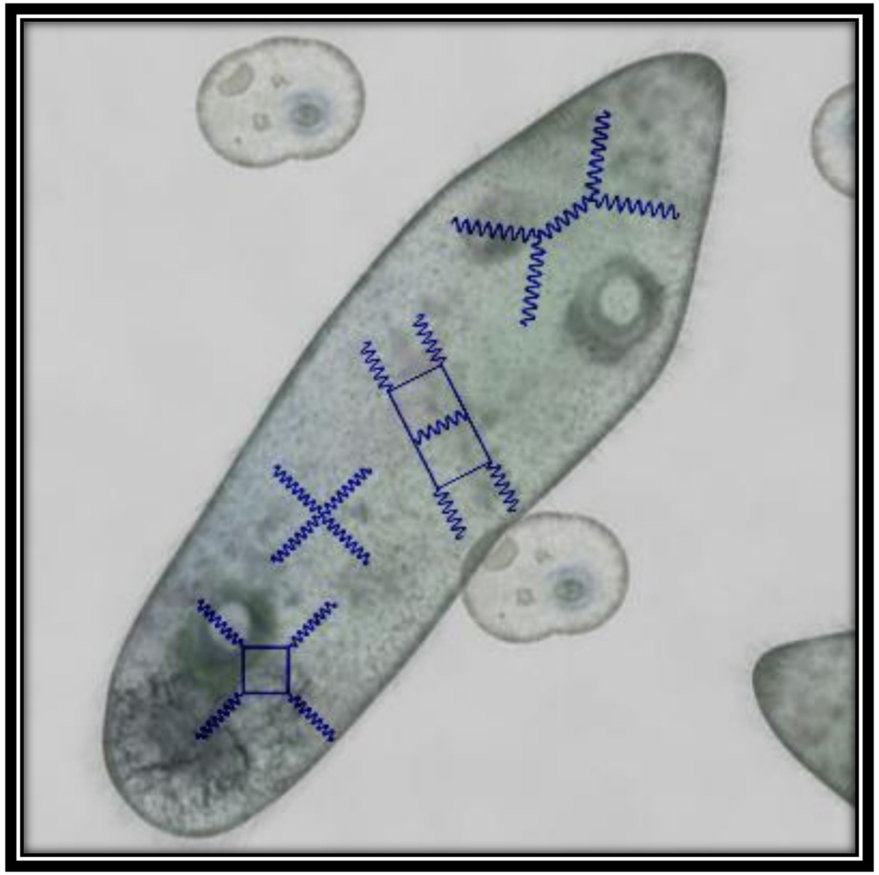
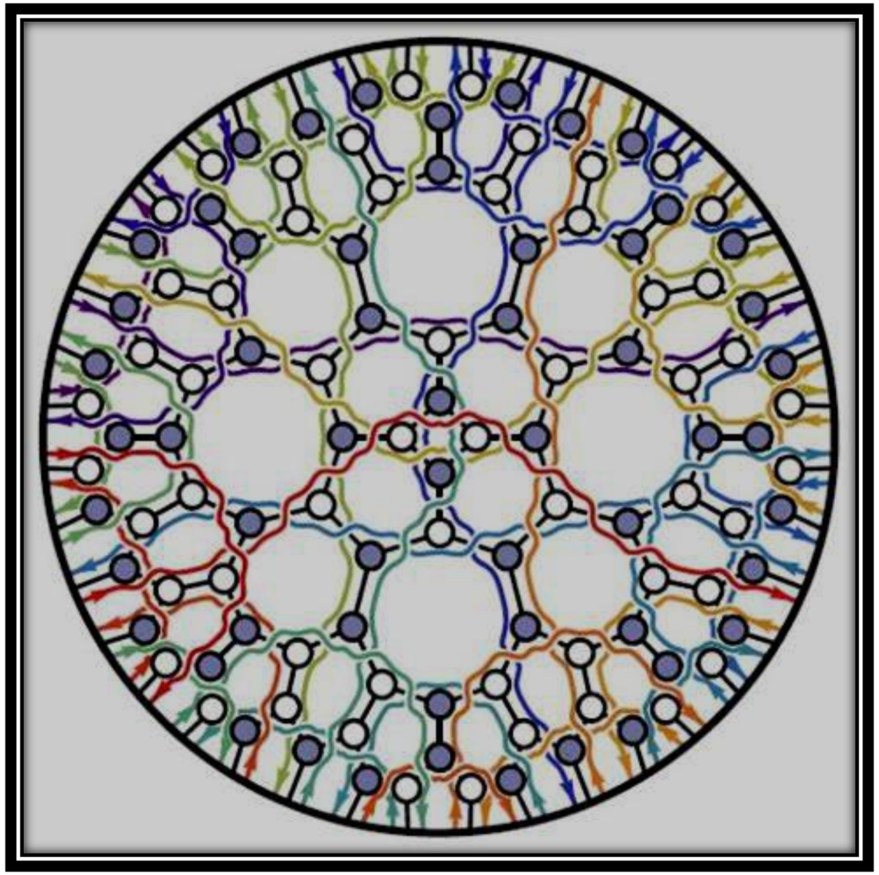
2019-09-08

Feynman Diagrams and on-shell methods





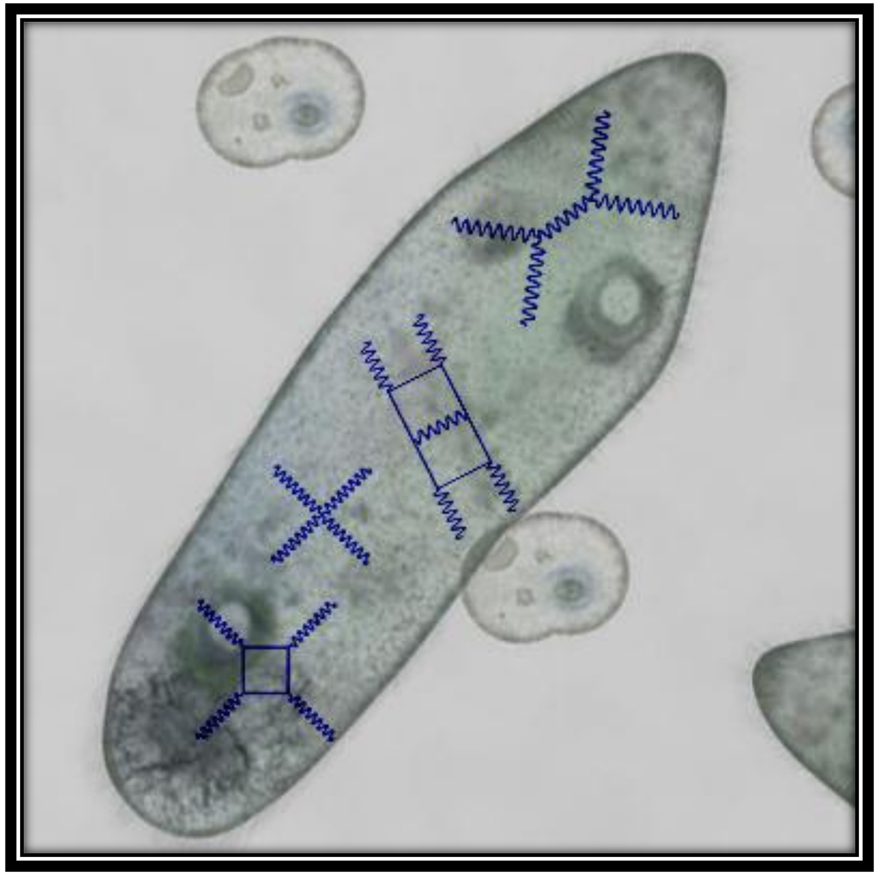


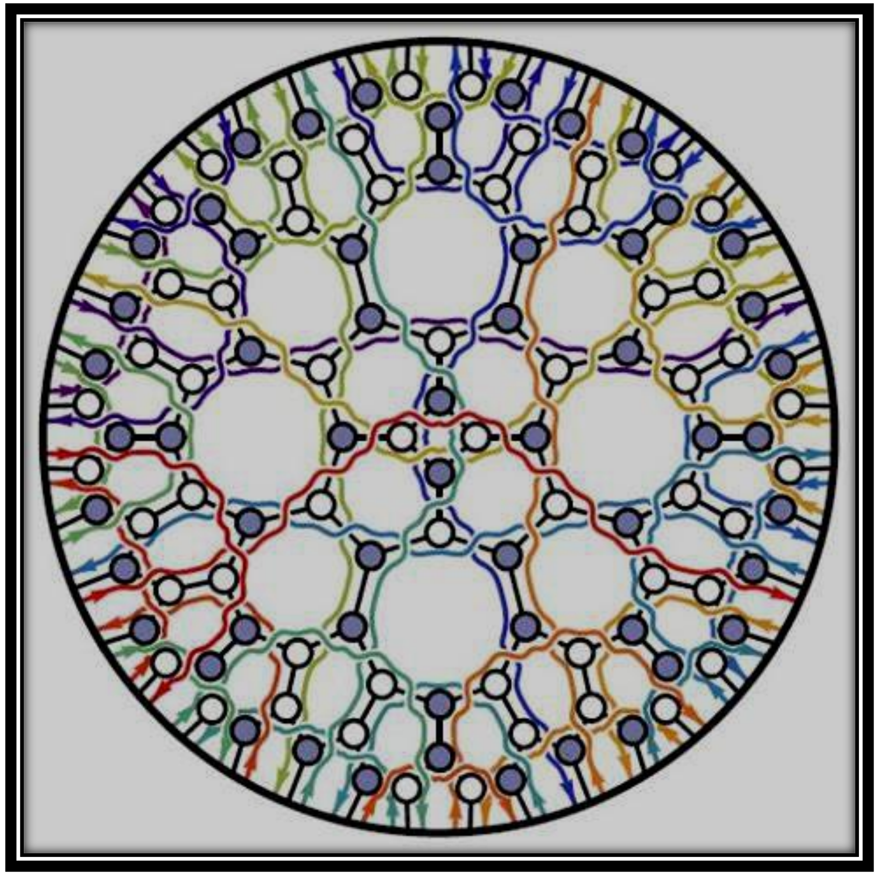


Too many diagrams…

Redundancy…

Complicated…



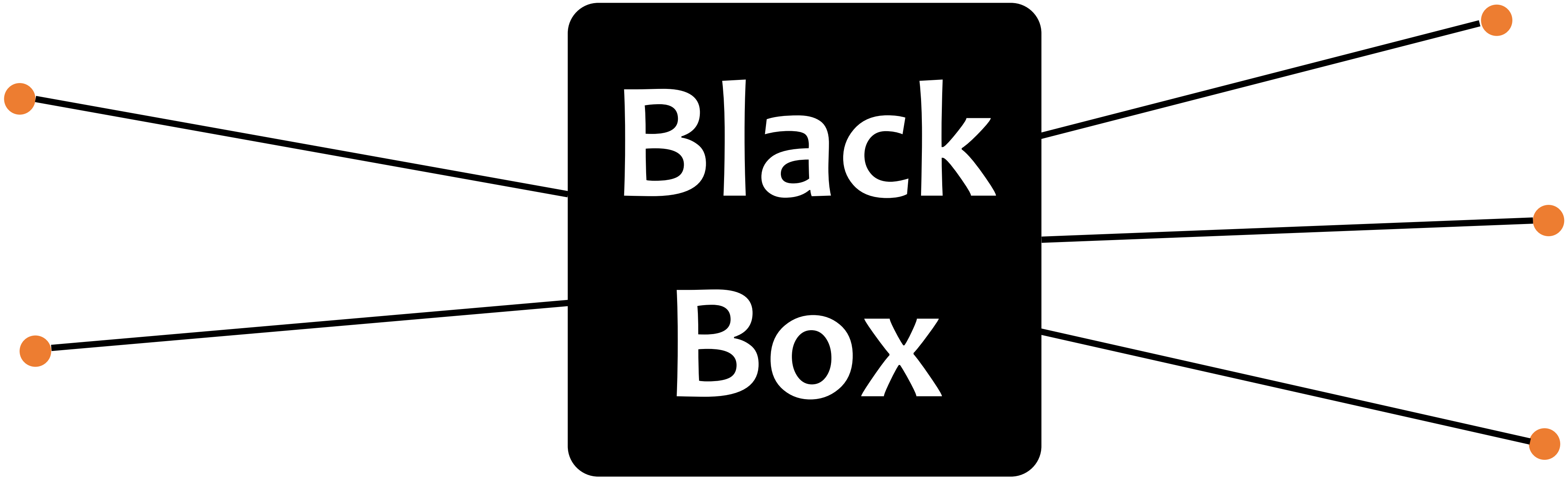


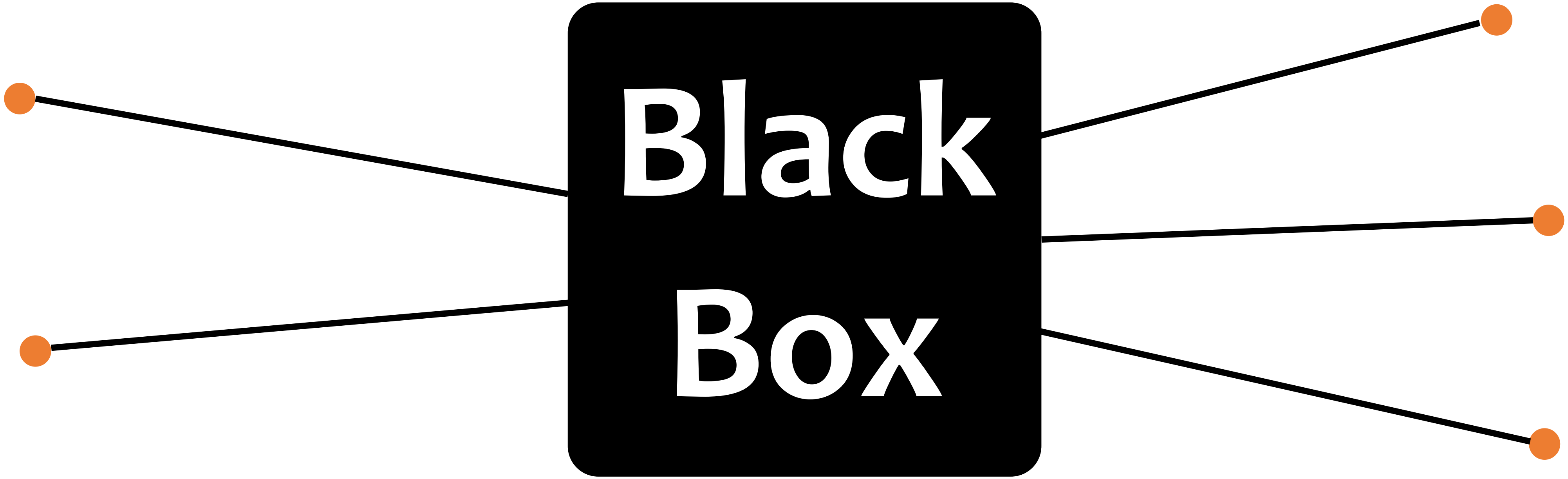
BCFW Symbol Unitarity-cut

BCJ Color-kinematic duality

Amplituhedron …

Black
Box


Black
Box



momentum
polarization
mass

Lagrangian
Path Integral
BCFW&3-pt amp
CHY

amplitude
cross section

Feynman Diagrams and on-shell methods

Lagrangian

↓

Feynman Rules

↓

Feynman Diagram

NOT THE REALITY

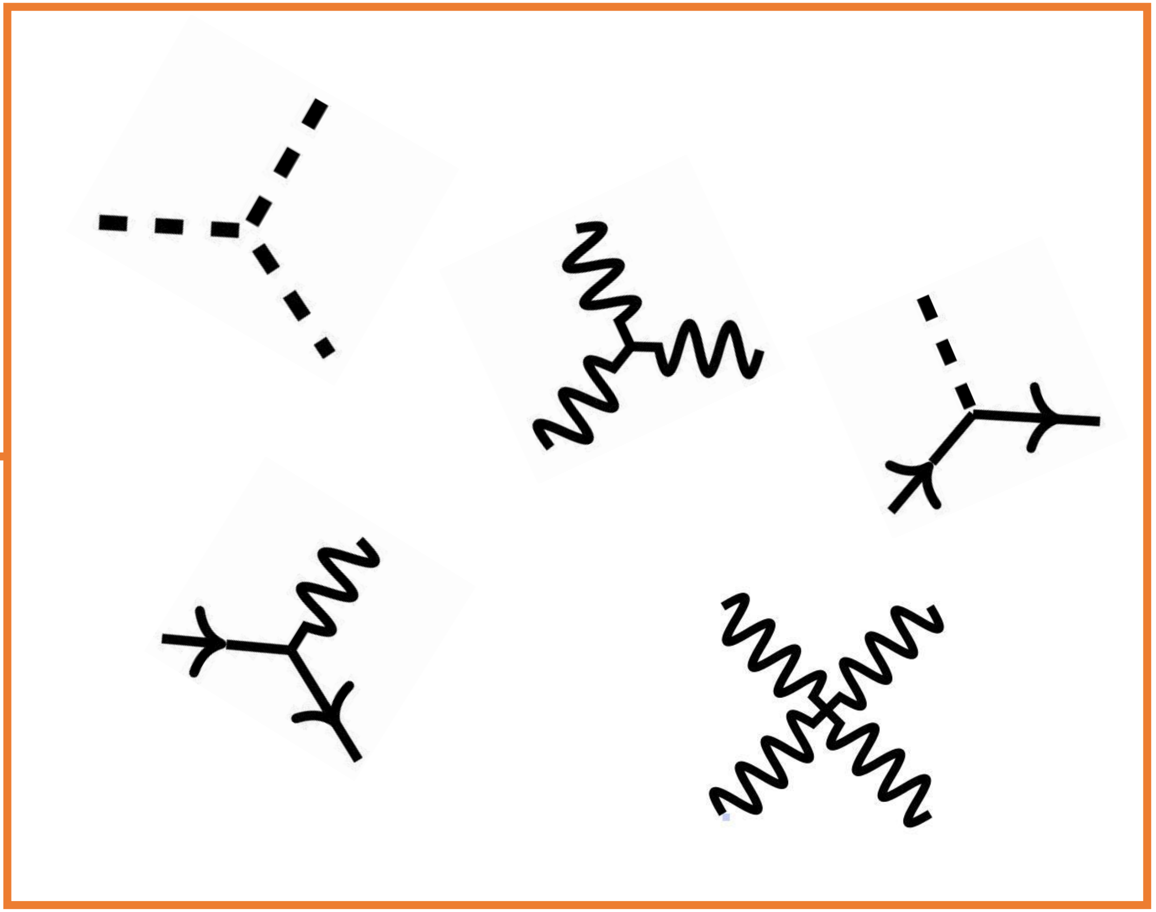
Lagrangian

↓

Feynman Rules

↓

Feynman Diagram



NOT THE REALITY

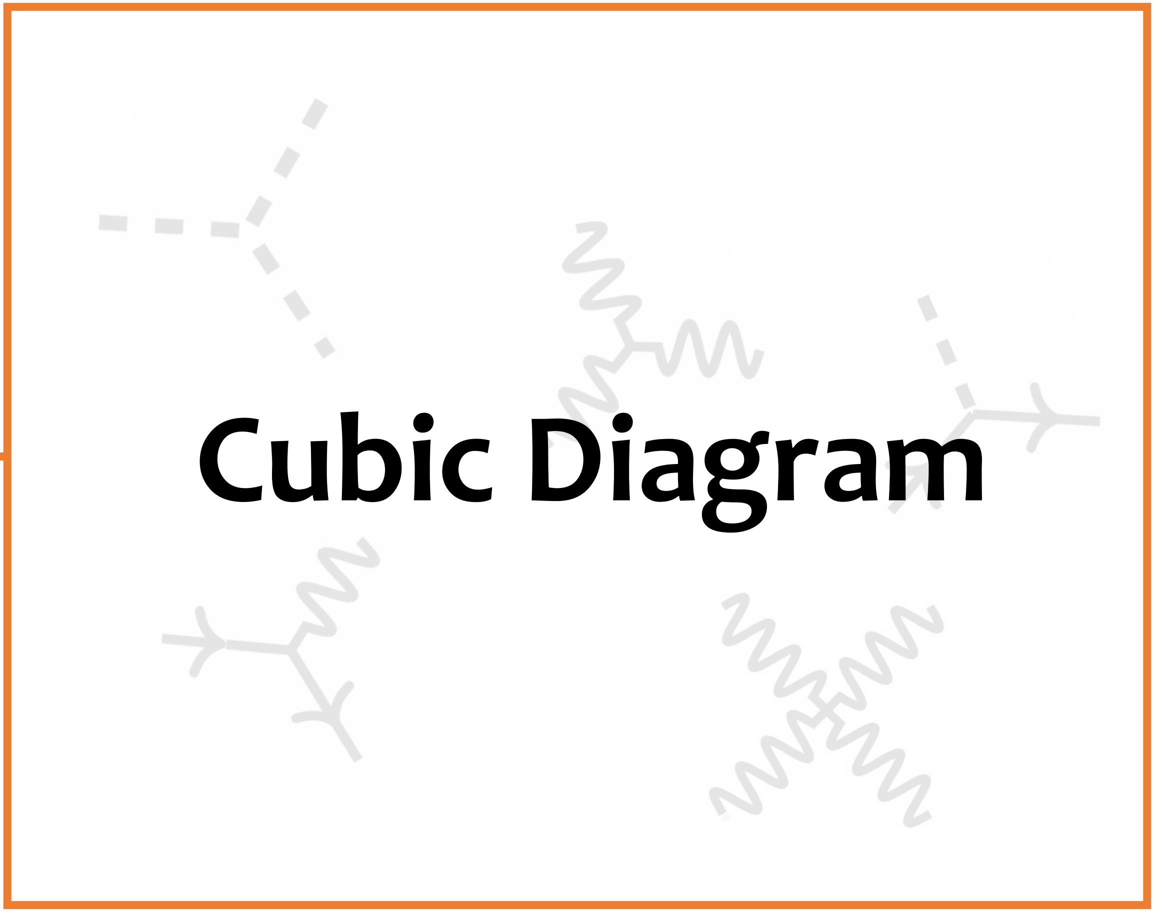
Lagrangian

↓

Feynman Rules

↓

Feynman Diagram



CHY-Formulation

CHY-Formulation

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude $=$ **Measure** $\times$ **CHY-Integrand**

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude = Measure × CHY-Integrand

$$\int \frac{dz_1 dz_2 \cdots dz_n}{\text{vol}\, SL(2,C)} \prod_{a=1}^{n}{}' \delta\left(\sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}}\right)$$

$$z_{ij} z_{jk} z_{ki} \prod_{a=1, a\neq i,j,k}^{n} \delta\left(\sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}}\right)$$

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

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$$z_{ij} z_{jk} z_{ki} \prod_{a=1,a\neq i,j,k}^{n} \delta\left(\sum_{b=1,b\neq a}^{n} \frac{s_{ab}}{z_{ab}}\right)$$

Scattering Equation

$$\mathcal{E}_a = \sum_{b=1,b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\ldots,n$$

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude = Measure × CHY-Integrand

$$\int \frac{dz_1 dz_2 \cdots dz_n}{\text{vol}\, SL(2,C)} \prod_{a=1}^{n}{}' \delta\left(\sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} \right)$$

$$z_{ij} z_{jk} z_{ki} \prod_{a=1, a\neq i,j,k}^{n} \delta\left(\sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} \right)$$

Integrated

$$\sum_{\{z_1,\ldots,z_n\}\in \text{solutions}} \frac{(z_{ij}z_{jk}z_{ki})(z_{pq}z_{qr}z_{rp})}{\text{Jacobian}}$$

$$\text{Det}[\Phi^{ijk}_{pqr}]$$

$$\Phi_{n\times n} = \begin{pmatrix} \frac{\partial \mathcal{E}_1}{\partial z_1} & \cdots & \frac{\partial \mathcal{E}_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{E}_n}{\partial z_1} & \cdots & \frac{\partial \mathcal{E}_n}{\partial z_n} \end{pmatrix}$$

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude $=$ **Measure** $\times$ **CHY-Integrand**

Bi-Scalar

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \times \frac{1}{z_{12}z_{23}\cdots z_{n1}}$$

Yang-Mills

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \times \mathrm{Pf}'\Psi$$

Gravity

$$\mathrm{Pf}'\Psi \times \mathrm{Pf}'\Psi$$

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude = Measure × CHY-Integrand

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Yang-Mills

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \times \mathrm{Pf}'\Psi$$

Gravity

$$\mathrm{Pf}'\Psi \times \mathrm{Pf}'\Psi$$

Reduced Pfaffian

$$\mathrm{Pf}'\,\Psi = 2\frac{(-)^{i+j}}{z_{ij}}\mathrm{Pf}(\Psi_{ij}^{ij})$$

$$\Psi_{2n\times 2n} = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

Pfaffian of anti-symmetric matrix

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude = Measure × CHY-Integrand

Bi-Scalar

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \times \frac{}{z_{12}z_{2}\cdots}$$

Yang-Mills

$$\cdots \frac{}{z_{n1}} \times \mathrm{P}\cdots$$

$$C_{ab} = \begin{cases} \dfrac{2\epsilon_a\cdot k_b}{z_{ab}}\,, a\neq b \\ -\displaystyle\sum_{j=1,j\neq a}^{n}\dfrac{2\epsilon_a\cdot k_j}{z_{aj}}\,, a=b \end{cases}$$

$$A_{ab} = \begin{cases} \dfrac{s_{ab}}{z_{ab}}\,, a\neq b \\ 0\,, a=b \end{cases}$$

Reduced Pfaffian

$$\mathrm{Pf}'\,\Psi = 2\frac{(-)^{i+j}}{z_{ij}}\mathrm{Pf}(\Psi^{ij}_{ij})$$

$$\Psi_{2n\times 2n} = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

$$B_{ab} = \begin{cases} \dfrac{2\epsilon_a\cdot \epsilon_b}{z_{ab}}\,, a\neq b \\ 0\,, a=b \end{cases}$$

Pfaffian of anti-symmetric matrix

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude = Measure × CHY-Integrand

Bi-Scalar	Yang-Mills	Gravity
$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \times \frac{1}{z_{12}z_{23}\cdots z_{n1}}$	$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \times \mathrm{Pf}'\Psi$	$\mathrm{Pf}'\Psi \times \mathrm{Pf}'\Psi$

A typical term of CHY-integrand

$$\frac{(\epsilon_4 \cdot k_3)(\epsilon_2 \cdot k_4)(\epsilon_1 \cdot \epsilon_3)}{z_{12}^2 z_{34}^2 z_{23} z_{41} z_{13} z_{24}}$$

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude = Measure × CHY-Integrand

Bi-Scalar

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \times \frac{1}{z_{12}z_{23}\cdots z_{n1}}$$

Yang-Mills

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}} \times \mathrm{Pf}'\Psi$$

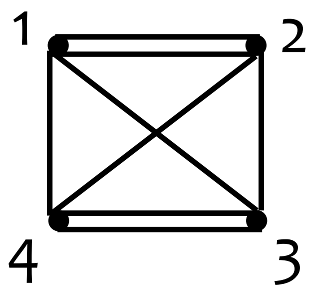
Gravity

$$\mathrm{Pf}'\Psi \times \mathrm{Pf}'\Psi$$

A typical term of CHY-integrand

$$\frac{(\epsilon_4\cdot k_3)(\epsilon_2\cdot k_4)(\epsilon_1\cdot\epsilon_3)}{z_{12}^2 z_{34}^2 z_{23}z_{41}z_{13}z_{24}}$$

4-regular graph



CHY-Formulation

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

CHY-Integrand

Gravity

Einstein-Maxwell(photon with flavor)

Einstein-Yang-Mills

Pure Yang-Mills

Non-linear sigma model

Extended Dirac-Born-Infeld

Einstein-Maxwell

Born-Infeld

Yang-Mills-scalar

Dirac-Born-Infeld

Bi-adjoint cubic scalar

Yang-Mills-scalar (special)

Phi-4 scalar

Special Galileon

CHY-Formulation

[F. Cachazo, S. He, E. Y. Yuan, 1306.6575; 1307.2199; 1309.0885; 1409.8256; 1412.3479]

Amplitude $=$ **Measure** $\times$ **CHY-Integrand**

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Amplitude = Measure × CHY-Integrand

$$A^{\text{CHY}} = \sum_{\{z_1,\ldots,z_n\}\in \text{solutions}} \frac{(z_{ij}z_{jk}z_{ki})(z_{pq}z_{qr}z_{rp})}{\text{Jacobian}} \times I^{\text{CHY}}(\epsilon_i, k_i, z_i)$$

$(n-3)$ equations of $(n-3)$ complex variables after Mobius invariance

$$\mathcal{E}_a = \sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\ldots,n$$

Solutions

Scattering Equations

$$\mathcal{E}_a = \sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\ldots,n$$

Scattering Equations

$$\mathcal{E}_a = \sum_{b=1,b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\dots,n$$

$n = 4$

$$\mathcal{E}_1 = \frac{s_{12}}{z_1 - z_2} + \frac{s_{13}}{z_1 - z_3} + \frac{s_{14}}{z_1 - z_4} = 0$$
$$\mathcal{E}_2 = \frac{s_{21}}{z_2 - z_1} + \frac{s_{23}}{z_2 - z_3} + \frac{s_{24}}{z_2 - z_4} = 0$$
$$\mathcal{E}_3 = \frac{s_{31}}{z_3 - z_1} + \frac{s_{32}}{z_3 - z_2} + \frac{s_{34}}{z_3 - z_4} = 0$$
$$\mathcal{E}_4 = \frac{s_{41}}{z_4 - z_1} + \frac{s_{42}}{z_4 - z_2} + \frac{s_{43}}{z_4 - z_3} = 0$$

Scattering Equations

$$\mathcal{E}_a = \sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\ldots,n$$

$n = 4$

$$\mathcal{E}_3 = \frac{s_{31}}{z_3 - z_1} + \frac{s_{32}}{z_3 - z_2} + \frac{s_{34}}{z_3 - z_4} = 0$$

Scattering Equations

$$\mathcal{E}_a = \sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\ldots,n$$

$n = 4$

$$\mathcal{E}_3 = \frac{s_{31}}{z_3 - z_1} + \frac{s_{32}}{z_3 - z_2} + \frac{s_{34}}{z_3 - z_4} = 0$$

Let $z_1 = \infty, z_2 = 1, z_4 = 0$

$$\mathcal{E}_3 = \frac{s_{31}}{z_3 - \infty} + \frac{s_{32}}{z_3 - 1} + \frac{s_{34}}{z_3 - 0} = 0$$

$$z_3 = \frac{s_{34}}{s_{23} + s_{34}}$$

Scattering Equations

[L. Dolan, P. Goddard, 1402.7374]

$$\mathcal{E}_a = \sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\ldots,n$$

Polynomial form of scattering equations

Scattering Equations

[L. Dolan, P. Goddard, 1402.7374]

$$\mathcal{E}_a = \sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\dots,n$$

Polynomial form of scattering equations

$n = 4$: $\quad h_1 = s_{12} + s_{13} z_3 = 0$

$n = 5$: $\quad h_1 = s_{12} + s_{13} z_3 + s_{14} z_4 = 0$

$\quad h_2 = s_{123} z_3 + s_{124} z_4 + s_{134} z_3 z_4 = 0$

$n = 6$: $\quad h_1 = s_{12} + s_{13} z_3 + s_{14} z_4 + s_{15} z_5 = 0$

$\quad h_2 = s_{123} z_3 + s_{124} z_4 + s_{125} z_5 + s_{134} z_3 z_4 + s_{135} z_3 z_5 + s_{145} z_4 z_5 = 0$

$\quad h_3 = s_{1234} z_3 z_4 + s_{1235} z_3 z_5 + s_{1345} z_3 z_4 z_5 = 0$

Equivalent to univariate polynomial equation of $(n-3)!$ order

Solutions

Scattering Equations

[L. Dolan, P. Goddard, 1402.7374]

$$\mathcal{E}_a = \sum_{b=1, b\neq a}^{n} \frac{s_{ab}}{z_{ab}} = 0 \quad \text{for} \quad a = 1,2,\ldots,n$$

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$\quad h_2 = s_{123} z_3 + s_{124} z_4 + s_{134} z_3 z_4 = 0$

$$z_3^{\pm} = \frac{s_{14}s_{123} - s_{13}s_{124} - s_{12}s_{134} \pm \sqrt{(s_{14}s_{123} - s_{13}s_{124} - s_{12}s_{134})^2 - 4 s_{12}s_{124}s_{13}s_{134}}}{2 s_{13}s_{134}}$$

$$z_4^{\pm} = \frac{s_{13}s_{124} - s_{12}s_{134} - s_{14}s_{123} \mp \sqrt{(s_{14}s_{123} - s_{13}s_{124} - s_{12}s_{134})^2 - 4 s_{12}s_{124}s_{13}s_{134}}}{2 s_{14}s_{134}}$$

Equivalent to univariate polynomial equation of $(n-3)!$ order

Scattering Equations

[L. Dolan, P. Goddard, 1402.7374]

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Polynomial form of scattering equations

$n = 4$: $\quad h_1 = s_{12} + s_{13} z_3 = 0$

$n = 5$: $\quad h_1 = s_{12} + s_{13} z_3 + s_{14} z_4 = 0$

$\quad h_2 = s_{123} z_3 + s_{124} z_4 + s_{134} z_3 z_4 = 0$

- **Analytic solution is impossible**
- **Exact numerical solution is impossible**
- **Approximate numerical solution is not enough**

Equivalent to univariate polynomial equation of $(n-3)!$ order

Problem: Let $I = \langle h_i \rangle$ be a zero-dimensional ideal in $R = \mathbb{C}[z_1, \ldots, z_n]$ generated by $h_i(z_1, \ldots, z_n) \in R, i = 1, \ldots, k$ and let $r(z_1, \ldots, z_n)$ be an arbitrary rational function in the fraction field of R. Because $\dim_{\mathbb{C}} I = 0, I = \sqcup_{j=1}^{N} \{z_j\}$ is a discrete set of N points. We wish to compute

$$\sum_{j=1}^{N} r(z_1, \ldots, z_n)$$

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companion matrix

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$$\sum_{j=1}^{N} r(z_1, \dots, z_n)$$

companion matrix

Stickelberger's Theorem: The complex roots $\{z_i\}$ of I are the vectors of simultaneous eigenvalues of the companion matrices $\{T_i\}$, i.e., the corresponding zero dimensional variety consists of the points

$$\mathcal{V}(I) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \ \forall\, i\colon\ T_i v = \lambda_i v\}$$

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$$\mathcal{V}(I) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \ \forall\, i\colon\ T_i v = \lambda_i v\}$$

Corollary:

$$\sum_{j=1}^{N} r(z_1, \dots, z_n) = \mathrm{Tr}[r(T_1, \dots, T_n)]$$

companion matrix

The ideal $I := \langle xy - z, yz - x, zx - y \rangle \subset R = \mathbb{C}[x, y, z]$

↓

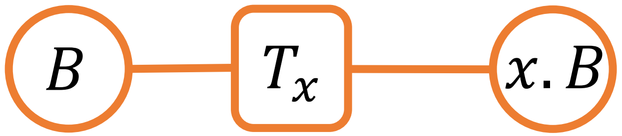
The Grobner basis $\mathrm{GB}(I) = \langle z^3 - z, yz^2 - y, y^2 - z^2, x - yz \rangle$

The Monomial basis $\mathrm{B} = \{1, y, yz, z, z^2\}$

↓

In the quotient ring R/I

$x.B = \{yz, z, z^2, y, yz\}$ $\quad$ $y.B = \{y, z^2, z, yz, y\}$ $\quad$ $z.B = \{z, yz, y, z^2, z\}$



$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ y \\ yz \\ z \\ z^2 \end{pmatrix} = \begin{pmatrix} yz \\ z \\ z^2 \\ u \\ yz \end{pmatrix}$$

Solutions

$$A^{\text{CHY}} = \sum_{\{z_1,\ldots,z_n\}\in \text{solutions}} \frac{(z_{ij}z_{jk}z_{ki})(z_{pq}z_{qr}z_{rp})}{\text{Jacobian}} \times I^{\text{CHY}}(\epsilon_i, k_i, z_i)$$

$r(z_1, \ldots, z_n)$

Solutions: the zeros of scattering equations (in polynomial form)

$I = \langle h_i \rangle$

[RH, J. Rao, B. Feng, YH. He, 1509,04483]

- Generate the **Grobner basis** and **Monomial basis** from I
- Compute the **Companion matrix** of variable z_i
- Replace $z_i \to T_i$ in $r(z_1, \ldots, z_n)$ to get matrix $r(T_1, \ldots, T_n)$
- Compute the **trace** of matrix $r(T_1, \ldots, T_n)$

A 5-point example

Ideal $I = \langle h_1 = s_{12} + s_{13}z_3 + s_{14}z_4 \quad , \quad h_2 = s_{123}z_3 + s_{124}z_4 + s_{134}z_3z_4 \rangle$

Grobner basis $\mathrm{GB}(I) = \langle s_{12}s_{45} + (s_{12}s_{25} - s_{13}s_{35} + s_{14}s_{45})z_4 + s_{14}s_{25}z_4^2,$

$s_{12} + s_{13}z_3 + s_{14}z_4 \ , \ s_{45}z_3 + s_{35}z_4 + s_{25}z_3z_4 \rangle$

Monomial basis $\mathrm{B} = \{1, z_4\}$

Companion matrix $T_{z_3}B = z_3B \ , \ T_{z_4}B = z_4B$

$$T_{z_3} = \begin{pmatrix} -\frac{s_{12}}{s_{13}} & -\frac{s_{14}}{s_{13}} \\ \frac{s_{12}s_{45}}{s_{13}s_{25}} & \frac{s_{14}s_{45} - s_{13}s_{35}}{s_{13}s_{35}} \end{pmatrix} \qquad T_{z_4} = \begin{pmatrix} 0 & 1 \\ -\frac{s_{12}s_{45}}{s_{14}s_{25}} & \frac{s_{13}s_{35} - s_{14}s_{45} - s_{12}s_{25}}{s_{14}s_{25}} \end{pmatrix}$$

A 5-point example

$$A = \sum_{\text{solutions}} \frac{z_{12}^2 z_{25}^2 z_{51}^2}{\text{Det}[\Phi_{125}^{125}]} \frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{51}^2} = \sum_{\text{solutions}} \frac{P(z_3, z_4)}{Q(z_3, z_4)}$$

$$T_{z_3} = \begin{pmatrix} -\frac{s_{12}}{s_{13}} & -\frac{s_{14}}{s_{13}} \\ \frac{s_{12}s_{45}}{s_{13}s_{25}} & \frac{s_{14}s_{45} - s_{13}s_{35}}{s_{13}s_{35}} \end{pmatrix} \qquad T_{z_4} = \begin{pmatrix} 0 & 1 \\ -\frac{s_{12}s_{45}}{s_{14}s_{25}} & \frac{s_{13}s_{35} - s_{14}s_{45} - s_{12}s_{25}}{s_{14}s_{25}} \end{pmatrix}$$

A 5-point example

$$A = \sum_{\text{solutions}} \frac{z_{12}^2 z_{25}^2 z_{51}^2}{\text{Det}[\Phi_{125}^{125}]} \frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{51}^2} = \sum_{\text{solutions}} \frac{P(z_3, z_4)}{Q(z_3, z_4)}$$

$$P = z_3^2 (z_4 - 1)^2$$

$$\begin{aligned} Q = {} & (s_{35}(z_3 - 1)^2 + s_{23} z_3^2)(z_3 - z_4)^2 (s_{45}(z_4 - 1)^2 + s_{24} z_4^2) \\ & + s_{34} \Big[s_{45}(z_3 - 1)^2 z_3^2 (z_4 - 1)^2 \\ & + z_4^2 (z_3^2 (s_{24}(z_3 - 1)^2 + s_{23}(z_4 - 1)^2) + s_{35} (z_3 - 1)^2 (z_4 - 1)^2) \Big] \end{aligned}$$

$$T_{z_3} = \begin{pmatrix} -\dfrac{s_{12}}{s_{13}} & -\dfrac{s_{14}}{s_{13}} \\ \dfrac{s_{12} s_{45}}{s_{13} s_{25}} & \dfrac{s_{14} s_{45} - s_{13} s_{35}}{s_{13} s_{35}} \end{pmatrix} \qquad T_{z_4} = \begin{pmatrix} 0 & 1 \\ -\dfrac{s_{12} s_{45}}{s_{14} s_{25}} & \dfrac{s_{13} s_{35} - s_{14} s_{45} - s_{12} s_{25}}{s_{14} s_{25}} \end{pmatrix}$$

A 5-point example

$$A = \sum_{\text{solutions}} \frac{z_{12}^2 z_{25}^2 z_{51}^2}{\text{Det}[\Phi_{125}^{125}]} \frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{51}^2} = \sum_{\text{solutions}} \frac{P(z_3, z_4)}{Q(z_3, z_4)}$$

$$= \text{Tr}[P(T_{z_3}, T_{z_4}) Q^{-1}(T_{z_3}, T_{z_4})]$$

$$\frac{1}{s_{15}s_{23}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{15}s_{34}} + \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}}$$

$$T_{z_3} = \begin{pmatrix} -\frac{s_{12}}{s_{13}} & -\frac{s_{14}}{s_{13}} \\ \frac{s_{12}s_{45}}{s_{13}s_{25}} & \frac{s_{14}s_{45} - s_{13}s_{35}}{s_{13}s_{35}} \end{pmatrix} \qquad T_{z_4} = \begin{pmatrix} 0 & 1 \\ -\frac{s_{12}s_{45}}{s_{14}s_{25}} & \frac{s_{13}s_{35} - s_{14}s_{45} - s_{12}s_{25}}{s_{14}s_{25}} \end{pmatrix}$$

A 5-point example

$$A=\sum_{\text{solutions}}\frac{z_{12}^2 z_{25}^2 z_{51}^2}{\mathrm{Det}[\Phi^{125}_{125}]}\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{51}^2}=\sum_{\text{solutions}}\frac{P(z_3,z_4)}{Q(z_3,z_4)}$$

$$=\mathrm{Tr}[P(T_{z_3},T_{z_4})Q^{-1}(T_{z_3},T_{z_4})]$$

$$\frac{1}{s_{15}s_{23}}+\frac{1}{s_{12}s_{34}}+\frac{1}{s_{15}s_{34}}+\frac{1}{s_{12}s_{45}}+\frac{1}{s_{23}s_{45}}$$

$$T_{z_3}=\begin{pmatrix}-\frac{s_{12}}{s_{13}} & -\frac{s_{14}}{s_{13}}\\ \frac{s_{12}s_{45}}{s_{13}s_{25}} & \frac{s_{14}s_{45}-s_{13}s_{35}}{s_{13}s_{35}}\end{pmatrix}\qquad T_{z_4}=\begin{pmatrix}0 & 1\\ -\frac{s_{12}s_{45}}{s_{14}s_{25}} & \frac{s_{13}s_{35}-s_{14}s_{45}-s_{12}s_{25}}{s_{14}s_{25}}\end{pmatrix}$$

$(n-3)!\times(n-3)!$ matrix

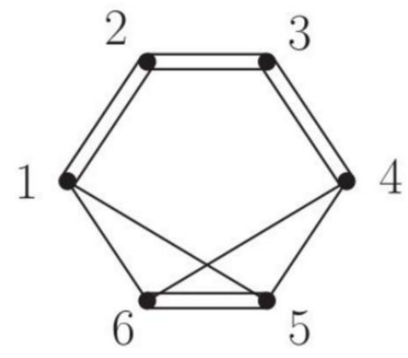
Integration Rules

[C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, 1506.06137; 1507.00997]

$I^{\text{CHY}}(z_1, \dots, z_n) \leftrightarrow$ 4-regular graph

A 6-point example

$$\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{56}^2 z_{45} z_{61} z_{46} z_{15}} \quad \longleftrightarrow$$



[C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, 1506.06137; 1507.00997]

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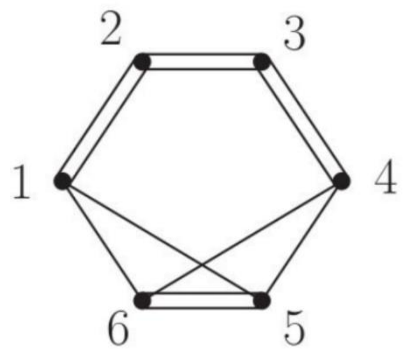

The pole index

$$\chi[A] = \mathbb{L}[A] - 2(|A| - 1)$$

- $\chi[A] < 0$ no such pole
- $\chi[A] = 0$ single pole $1/s_A$
- $\chi[A] > 0$ higher-order pole $1/s_A^{\chi+1}$

[C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, 1506.06137; 1507.00997]

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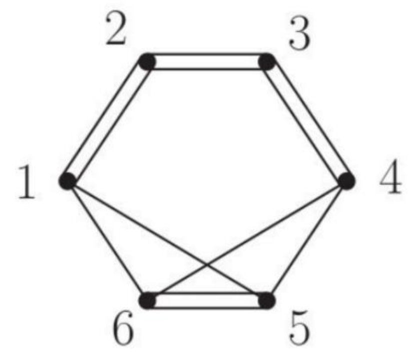

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- $\chi[A] > 0$ higher-order pole $1/s_A^{\chi+1}$

{1,2}=2-2(2-1)=0

{1,3}=0-2(2-1)=-2

{1,5,6}=3-2(3-1)=-1

{1,2,3}=4-2(3-1)=0

…….

[C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, 1506.06137; 1507.00997]

$I^{\mathrm{CHY}}(z_1, \dots, z_n) \leftrightarrow$ 4-regular graph

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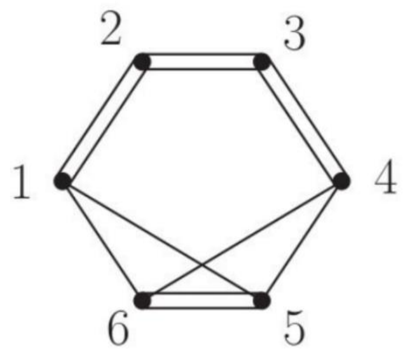


Integration Rules, only valid for CHY-integrand with single poles

- Find all possible single poles: {1,2} {2,3} {3,4} {5,6} {1,2,3} {2,3,4}
- Construct all compatible combinations:

{{1,2}, {3,4}, {5,6}} {{1,2,3}, {1,2}, {5,6}} {{1,2,3}, {2,3}, {5,6}}
{{2,3,4}, {2,3}, {5,6}} {{2,3,4}, {3,4}, {5,6}}

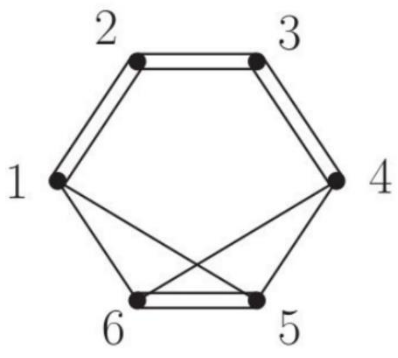
- Sum over all compatible combinations.

[C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, 1506.06137; 1507.00997]

$I^{\text{CHY}}(z_1, \ldots, z_n) \leftrightarrow$ 4-regular graph

A 6-point example

$$\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{56}^2 z_{45} z_{61} z_{46} z_{15}} \longleftrightarrow$$



Integration Rules, ... poles

For n-point amplitude, select $(n-3)$ subsets which are compatible

- Find all possible s... ... {2,3,4}
- Construct all compatible combinations:

{{1,2}, {3,4}, {5,6}} {{1,2,3}, {1,2}, {5,6}} {{1,2,3}, {2,3}, {5,6}}

{{2,3,4}, {2,3}, {5,6}} {{2,3,4}, {3,4}, {5,6}}

- Sum over all compatible combinations.

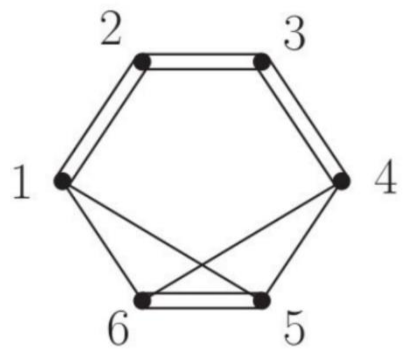
Integration Rules

[C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, 1506.06137; 1507.00997]

$I^{\text{CHY}}(z_1, \ldots, z_n) \leftrightarrow$ 4-regular graph

A 6-point example

$$\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{56}^2 z_{45} z_{61} z_{46} z_{15}} \longleftrightarrow$$



Integration Rules

$$\frac{1}{s_{12}s_{34}s_{56}} + \frac{1}{s_{123}s_{12}s_{56}} + \frac{1}{s_{123}s_{23}s_{56}} + \frac{1}{s_{234}s_{23}s_{56}} + \frac{1}{s_{234}s_{34}s_{56}}$$

- Find all possible
- Construct all con

{{1,2}, {3,4}, {5,6}} {{1,2,3}, {1,2}, {5,6}} {{1,2,3}, {2,3}, {5,6}}

{{2,3,4}, {2,3}, {5,6}} {{2,3,4}, {3,4}, {5,6}}

- Sum over all compatible combinations.

[C. Baadsgaard, N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard, 1506.06137; 1507.00997]

$I^{\text{CHY}}(z_1, \ldots, z_n) \leftrightarrow$ 4-regular graph

A 6-point example

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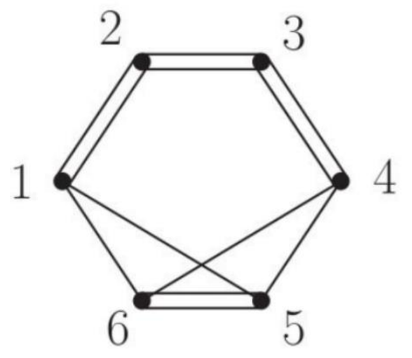
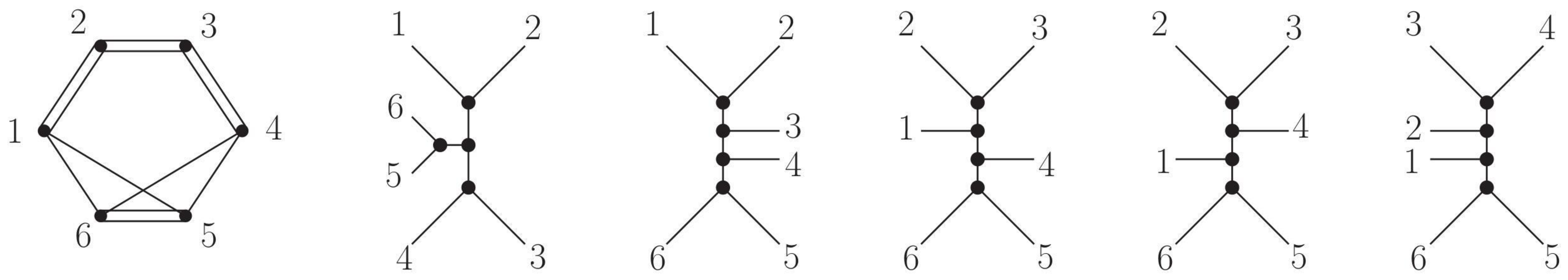
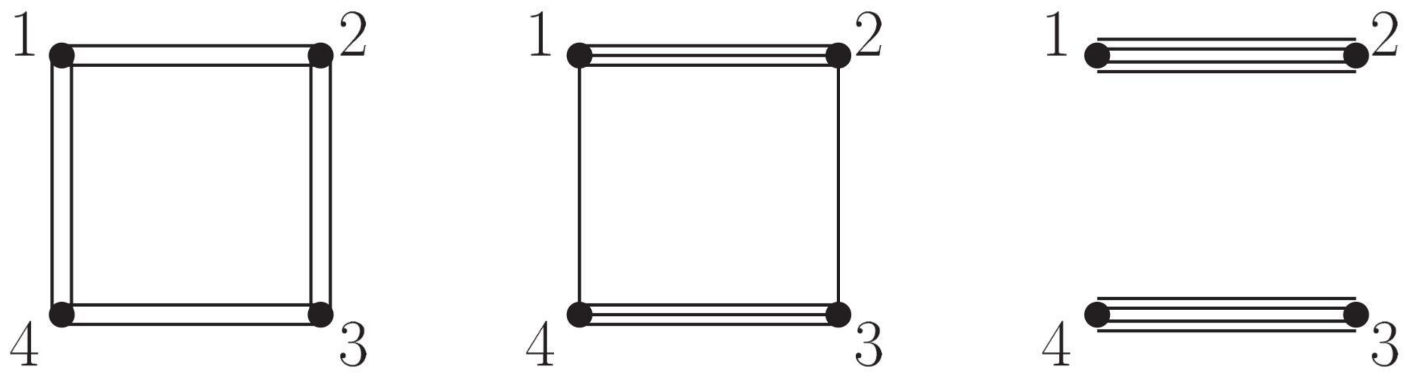




Integration rules for higher-order poles?

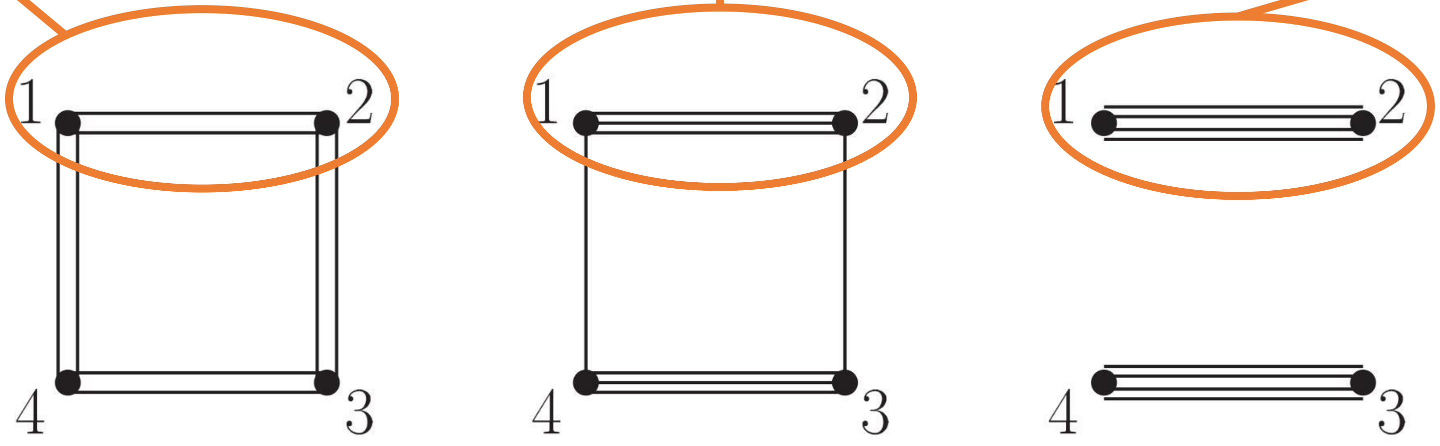


Integration rules for higher-order poles?

$\chi = 2 - 2(2-1) = 0$

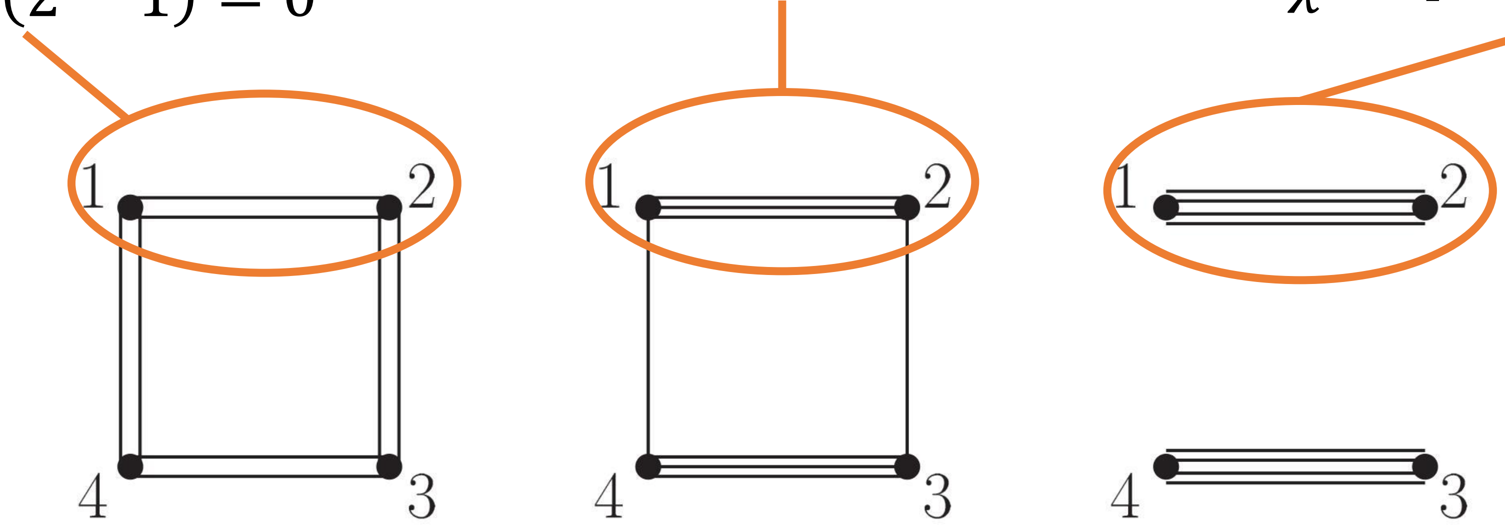
$\chi = 3 - 2(2-1) = 1$

$\chi = 4 - 2(2-1) = 2$



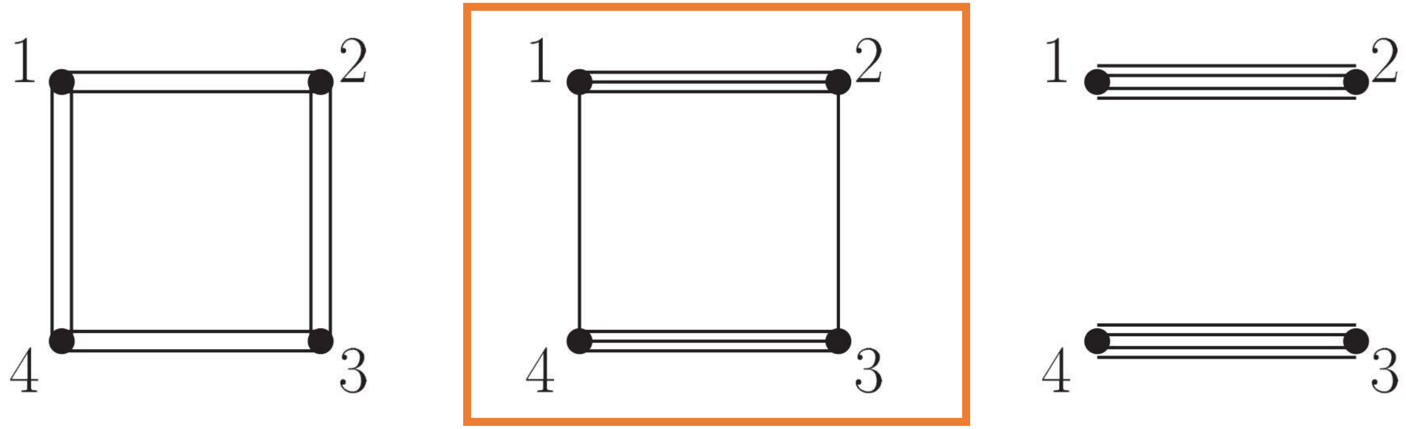
Integration rules for higher-order poles?

$\chi = 2 - 2(2-1) = 0$ $\chi = 3 - 2(2-1) = 1$ $\chi = 4 - 2(2-1) = 2$

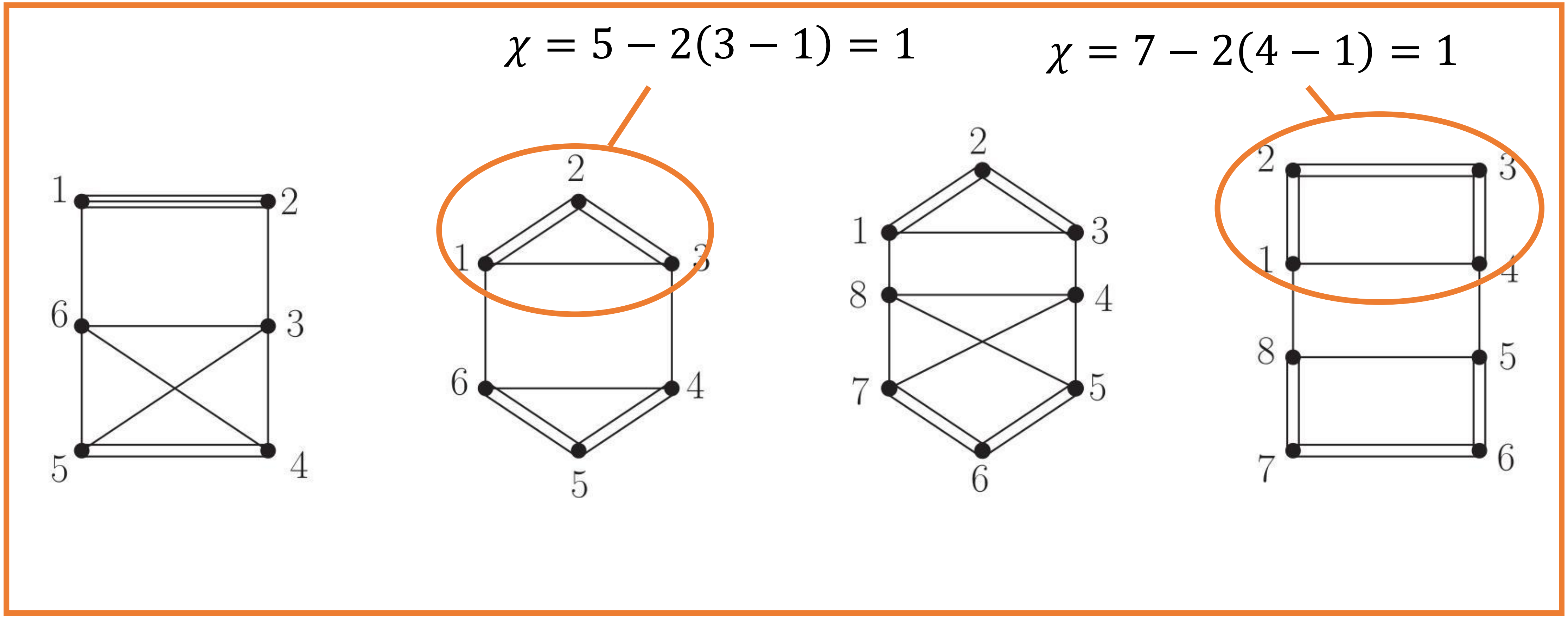


$$\frac{1}{s_{12}} + \frac{1}{s_{14}} \qquad \frac{s_{13}}{s_{12}^2} \qquad \frac{s_{13}s_{14}}{s_{12}^3}$$

Integration rules for higher-order poles?

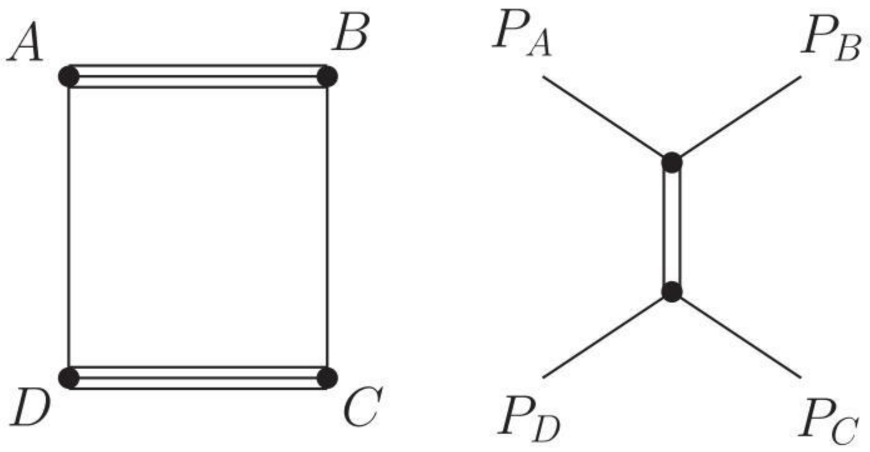


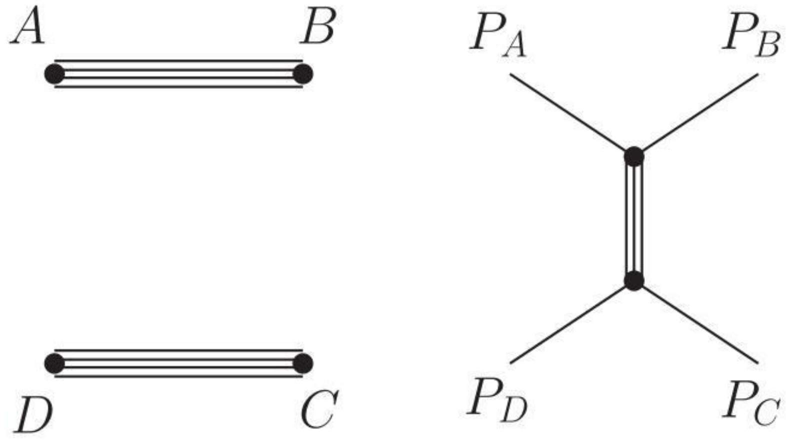
Integration rules for higher-order poles?



Integration Rules

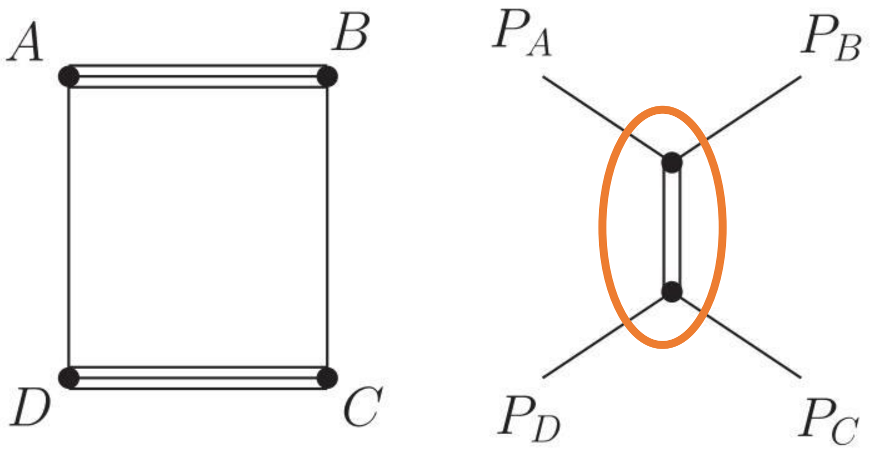
Integration rules for higher-order poles?

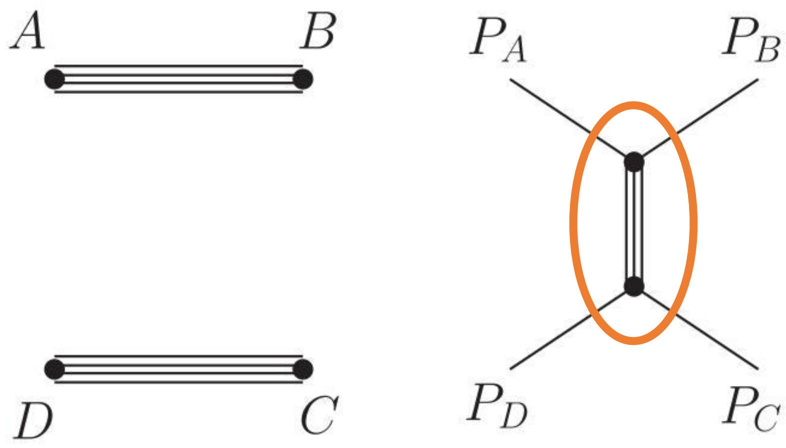




Integration rules for higher-order poles?

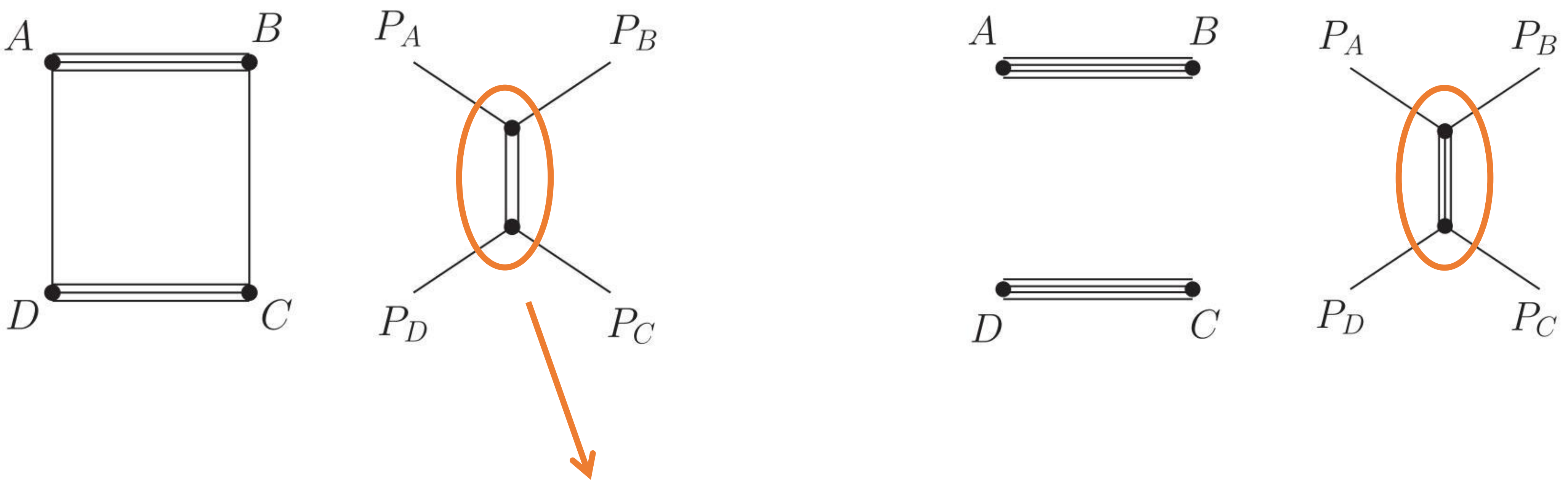
[RH, B. Feng, M.X. Luo, C.J. Zhu, 1604.07314]





Integration rules for higher-order poles?

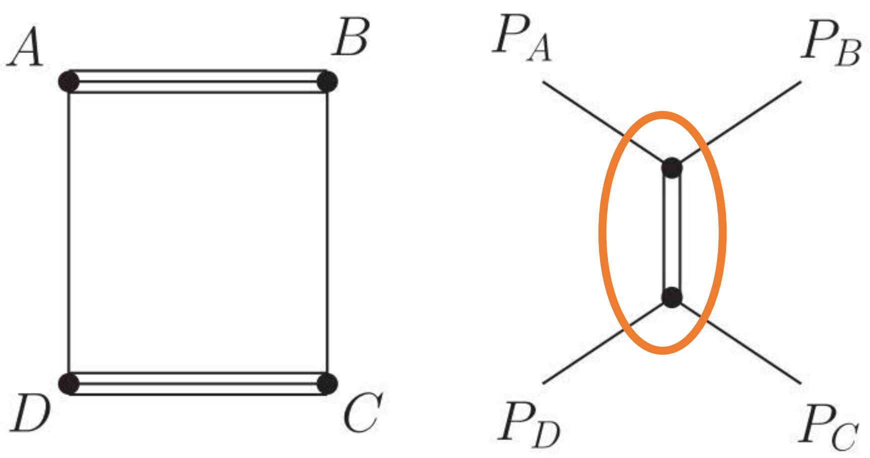
[RH, B. Feng, M.X. Luo, C.J. Zhu, 1604.07314]

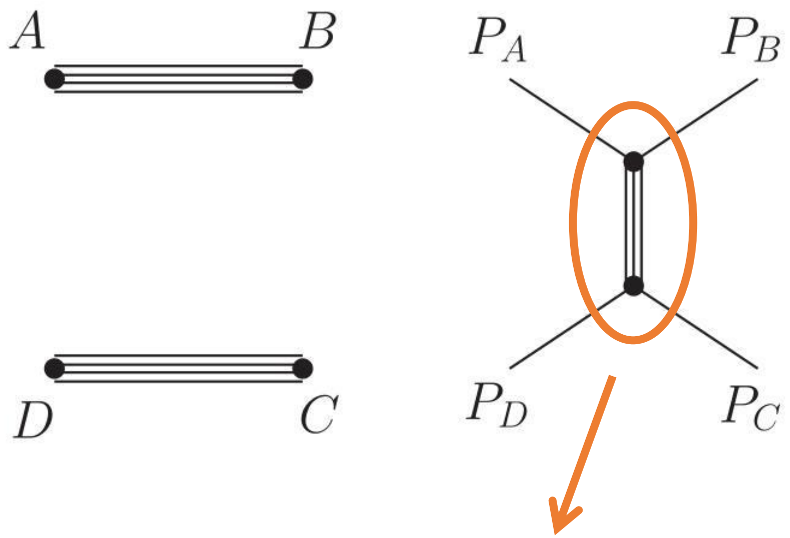


$$\mathcal{R}^1_{\text{ule}}[P_A, P_B, P_C, P_D] = \frac{2P_A \cdot P_C + 2P_B \cdot P_D}{s_{AB}^2}$$

Integration rules for higher-order poles?

[RH, B. Feng, M.X. Luo, C.J. Zhu, 1604.07314]

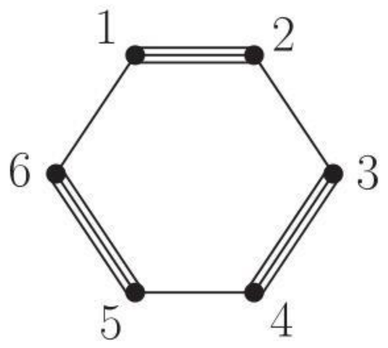


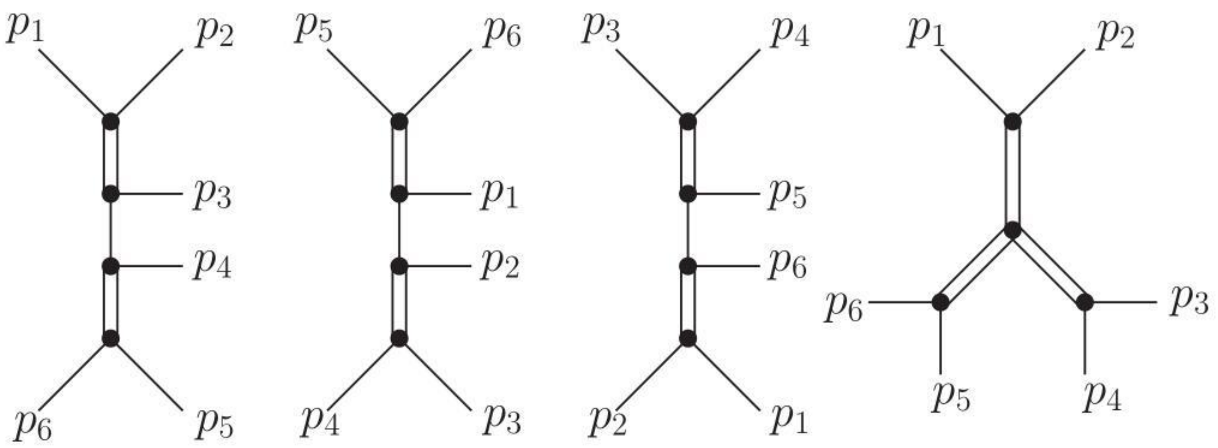


$$\mathcal{R}^2_{\text{ule}}[P_A, P_B, P_C, P_D]$$
$$= \frac{(2P_A \cdot P_C)(2P_A \cdot P_D) + (2P_B \cdot P_C)(2P_B \cdot P_D) + (2P_C \cdot P_A)(2P_C \cdot P_B) + (2P_D \cdot P_A)(2P_D \cdot P_B)}{4 s_{AB}^3}$$
$$- \frac{(P_A^2 - P_B^2)^2 + (P_C^2 - P_D^2)^2}{4 s_{AB}^3} + \frac{2}{9} \frac{(P_A^2 + P_B^2)(P_C^2 + P_D^2)}{4 s_{AB}^3}$$

Integration rules for higher-order poles?

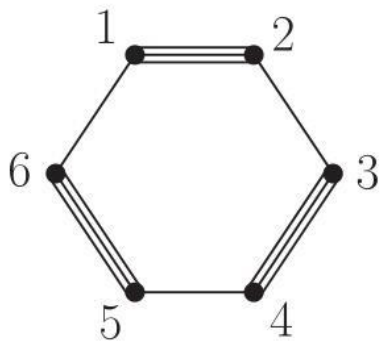
[RH, B. Feng, M.X. Luo, C.J. Zhu, 1604.07314]

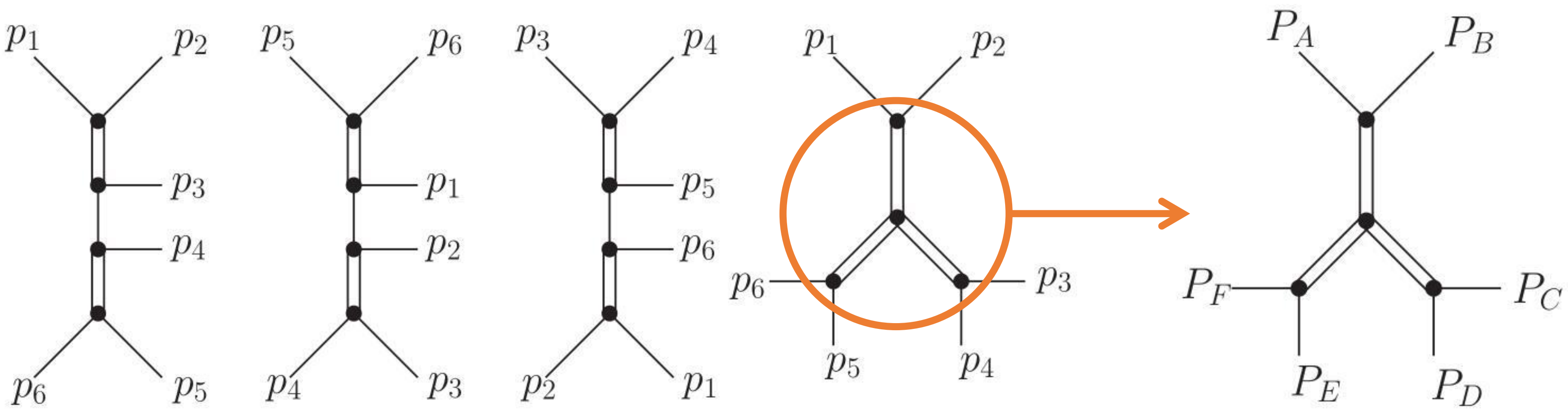




Integration rules for higher-order poles?

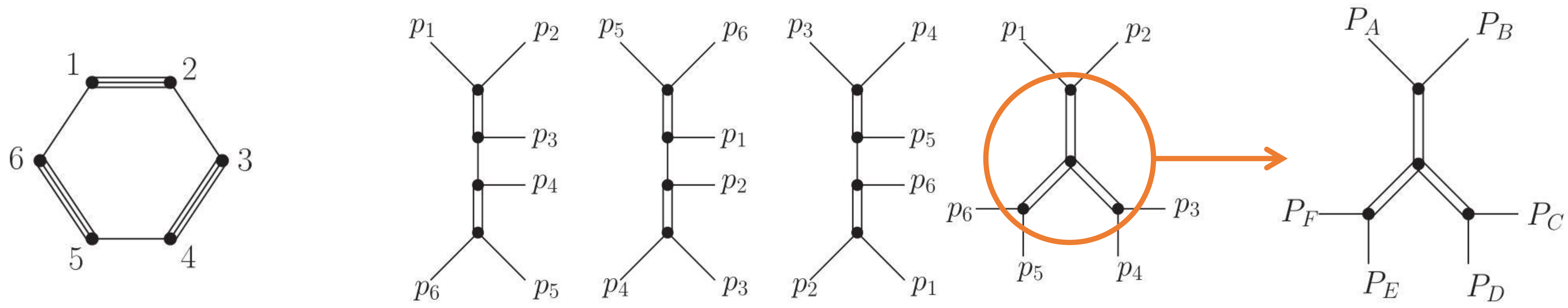
[RH, B. Feng, M.X. Luo, C.J. Zhu, 1604.07314]





Integration rules for higher-order poles?

[RH, B. Feng, M.X. Luo, C.J. Zhu, 1604.07314]

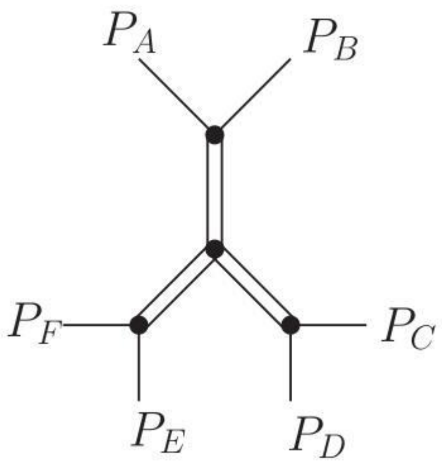


$$\mathcal{R}^{\rm IX}_{\rm ule}[P_A,P_B,P_C,P_D,P_E,P_F]$$
$$= \left(\frac{\mathcal{R}_{11}}{8s_{AB}^2 s_{CD}^2}+\frac{\mathcal{R}_{12}}{8s_{CD}^2 s_{EF}^2}+\frac{\mathcal{R}_{13}}{8s_{EF}^2 s_{AB}^2}\right)+\left(\frac{\mathcal{R}_{21}}{2s_{AB}^2 s_{CD} s_{EF}}+\frac{\mathcal{R}_{22}}{2s_{CD}^2 s_{EF} s_{AB}}+\frac{\mathcal{R}_{23}}{2s_{EF}^2 s_{AB} s_{CD}}\right)$$
$$+\left(\frac{\mathcal{R}_{31}}{2s_{AB}^2 s_{CD}^2 s_{EF}}+\frac{\mathcal{R}_{32}}{2s_{CD}^2 s_{EF}^2 s_{AB}}+\frac{\mathcal{R}_{33}}{2s_{EF}^2 s_{AB}^2 s_{CD}}\right)+\frac{\mathcal{R}_{4}}{2s_{AB}^2 s_{CD}^2 s_{EF}^2}+\frac{1}{s_{AB} s_{CD} s_{EF}}$$
$$-(P_A^2+P_B^2+P_C^2+P_D^2+P_E^2+P_F^2)\left(\frac{1}{4s_{AB}^2 s_{CD}^2}+\frac{1}{4s_{AB}^2 s_{EF}^2}+\frac{1}{4s_{CD}^2 s_{EF}^2}\right),$$

Integration rules for higher-order poles?

[RH, B. Feng, M.X. Luo, C.J. Zhu, 1604.07314]

$$\mathcal{R}^{\text{IX}}_{\text{ule}}[P_A,P_B,P_C,P_D,P_E,P_F]$$
$$= \left(\frac{\mathcal{R}_{11}}{8s_{AB}^2 s_{CD}^2}+\frac{\mathcal{R}_{12}}{8s_{CD}^2 s_{EF}^2}+\frac{\mathcal{R}_{13}}{8s_{EF}^2 s_{AB}^2}\right)+\left(\frac{\mathcal{R}_{21}}{2s_{AB}^2 s_{CD}s_{EF}}+\frac{\mathcal{R}_{22}}{2s_{CD}^2 s_{EF}s_{AB}}+\frac{\mathcal{R}_{23}}{2s_{EF}^2 s_{AB}s_{CD}}\right)$$
$$+\left(\frac{\mathcal{R}_{31}}{2s_{AB}^2 s_{CD}^2 s_{EF}}+\frac{\mathcal{R}_{32}}{2s_{CD}^2 s_{EF}^2 s_{AB}}+\frac{\mathcal{R}_{33}}{2s_{EF}^2 s_{AB}^2 s_{CD}}\right)+\frac{\mathcal{R}_4}{2s_{AB}^2 s_{CD}^2 s_{EF}^2}+\frac{1}{s_{AB}s_{CD}s_{EF}}$$
$$-(P_A^2+P_B^2+P_C^2+P_D^2+P_E^2+P_F^2)\left(\frac{1}{4s_{AB}^2 s_{CD}^2}+\frac{1}{4s_{AB}^2 s_{EF}^2}+\frac{1}{4s_{CD}^2 s_{EF}^2}\right),$$

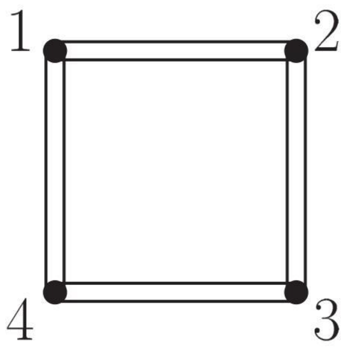


$$\mathcal{R}_{11} = 2(\widetilde{s}_{EC}+\widetilde{s}_{FB}-\widetilde{s}_{EB}-\widetilde{s}_{FC})-(\widetilde{s}_{ABC}+\widetilde{s}_{BCD}+\widetilde{s}_{CDE}+\widetilde{s}_{DEF}+\widetilde{s}_{EFA}+\widetilde{s}_{FAB}),$$
$$\mathcal{R}_{12} = 2(\widetilde{s}_{AE}+\widetilde{s}_{BD}-\widetilde{s}_{AD}-\widetilde{s}_{BE})-(\widetilde{s}_{ABC}+\widetilde{s}_{BCD}+\widetilde{s}_{CDE}+\widetilde{s}_{DEF}+\widetilde{s}_{EFA}+\widetilde{s}_{FAB}),$$
$$\mathcal{R}_{13} = 2(\widetilde{s}_{CA}+\widetilde{s}_{DF}-\widetilde{s}_{CF}-\widetilde{s}_{DA})-(\widetilde{s}_{ABC}+\widetilde{s}_{BCD}+\widetilde{s}_{CDE}+\widetilde{s}_{DEF}+\widetilde{s}_{EFA}+\widetilde{s}_{FAB}),$$
$$\mathcal{R}_{21} = \widetilde{s}_{AF}+\widetilde{s}_{BC}+\widetilde{s}_{AC}+\widetilde{s}_{BF}-(\widetilde{s}_{ACE}+\widetilde{s}_{BDF}),$$
$$\mathcal{R}_{22} = \widetilde{s}_{CB}+\widetilde{s}_{DE}+\widetilde{s}_{CE}+\widetilde{s}_{DB}-(\widetilde{s}_{ACE}+\widetilde{s}_{BDF}),$$
$$\mathcal{R}_{23} = \widetilde{s}_{ED}+\widetilde{s}_{FA}+\widetilde{s}_{EA}+\widetilde{s}_{FD}-(\widetilde{s}_{ACE}+\widetilde{s}_{BDF}),$$

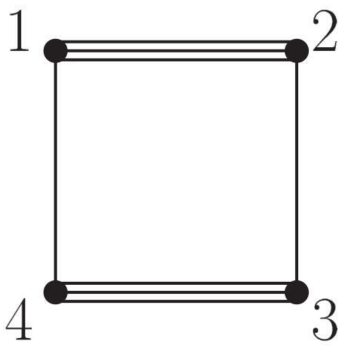
$$\mathcal{R}_{31} = \widetilde{s}_{BC}(\widetilde{s}_{ED}+\widetilde{s}_{FA}-\widetilde{s}_{EC}-\widetilde{s}_{FB}) + (\widetilde{s}_{CA}\widetilde{s}_{DE}+\widetilde{s}_{BD}\widetilde{s}_{AF}-\widetilde{s}_{BF}\widetilde{s}_{CA}-\widetilde{s}_{CE}\widetilde{s}_{BD})+(\widetilde{s}_{CA}-\widetilde{s}_{BD})(\widetilde{s}_{CE}-\widetilde{s}_{BF}),$$
$$\mathcal{R}_{32} = \widetilde{s}_{DE}(\widetilde{s}_{AF}+\widetilde{s}_{BC}-\widetilde{s}_{AE}-\widetilde{s}_{BD}) + (\widetilde{s}_{EC}\widetilde{s}_{FA}+\widetilde{s}_{DF}\widetilde{s}_{CB}-\widetilde{s}_{DB}\widetilde{s}_{EC}-\widetilde{s}_{EA}\widetilde{s}_{DF})+(\widetilde{s}_{EC}-\widetilde{s}_{DF})(\widetilde{s}_{EA}-\widetilde{s}_{DB}),$$
$$\mathcal{R}_{33} = \widetilde{s}_{FA}(\widetilde{s}_{CB}+\widetilde{s}_{DE}-\widetilde{s}_{CA}-\widetilde{s}_{DF}) + (\widetilde{s}_{AE}\widetilde{s}_{BC}+\widetilde{s}_{FB}\widetilde{s}_{ED}-\widetilde{s}_{FD}\widetilde{s}_{AE}-\widetilde{s}_{AC}\widetilde{s}_{FB})+(\widetilde{s}_{AE}-\widetilde{s}_{FB})(\widetilde{s}_{AC}-\widetilde{s}_{FD}),$$
$$\mathcal{R}_4 = \widetilde{s}_{BC}(\widetilde{s}_{CE}\widetilde{s}_{EA}+\widetilde{s}_{BF}\widetilde{s}_{FD})+\widetilde{s}_{DE}(\widetilde{s}_{EA}\widetilde{s}_{AC}+\widetilde{s}_{DB}\widetilde{s}_{BF})+\widetilde{s}_{FA}(\widetilde{s}_{AC}\widetilde{s}_{CE}+\widetilde{s}_{FD}\widetilde{s}_{DB}) + \widetilde{s}_{BC}\widetilde{s}_{DE}(\widetilde{s}_{EA}+\widetilde{s}_{BF})+\widetilde{s}_{DE}\widetilde{s}_{FA}(\widetilde{s}_{AC}+\widetilde{s}_{DB})+\widetilde{s}_{FA}\widetilde{s}_{BC}(\widetilde{s}_{CE}+\widetilde{s}_{FD})+2\widetilde{s}_{BC}\widetilde{s}_{DE}\widetilde{s}_{FA}$$

Integration Rules

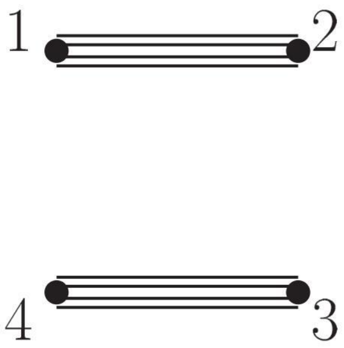
Cross-ratio Identities



$$\frac{1}{s_{12}}+\frac{1}{s_{14}}$$

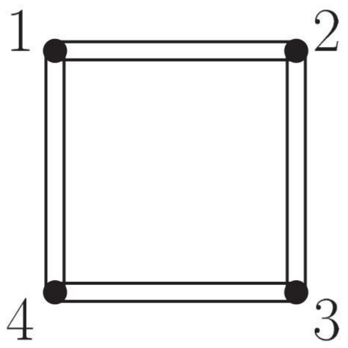


$$\frac{s_{13}}{s_{12}^2}$$

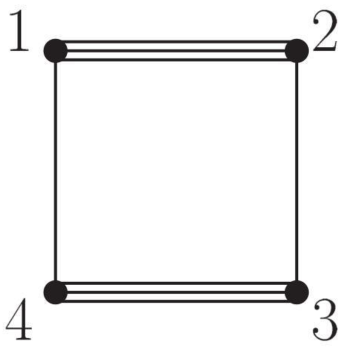


$$\frac{s_{13}s_{14}}{s_{12}^3}$$

Cross-ratio Identities



$$\frac{1}{s_{12}} + \frac{1}{s_{14}}$$



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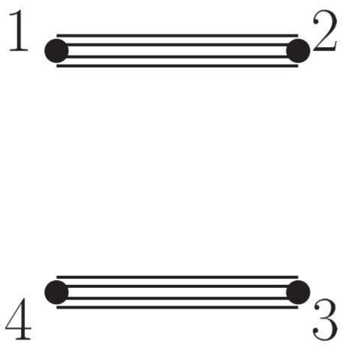
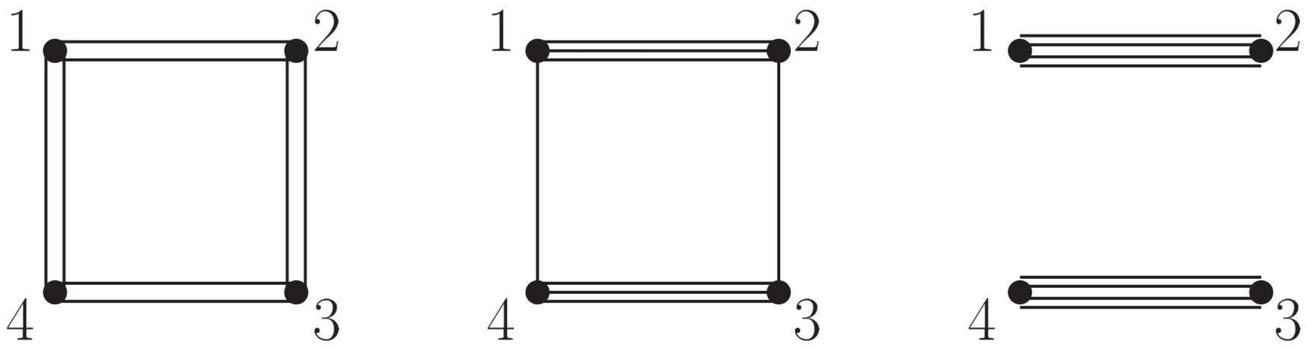


$$\frac{s_{13}s_{14}}{s_{12}^3}$$

$$\mathcal{E}_1 = \frac{s_{12}}{z_{12}} + \frac{s_{13}}{z_{13}} + \frac{s_{14}}{z_{14}} = 0$$

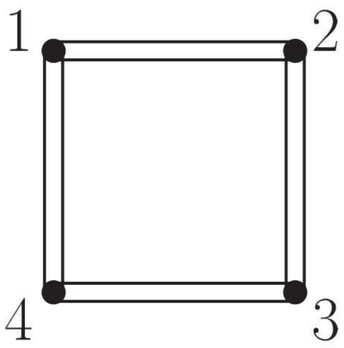
Cross-ratio Identities



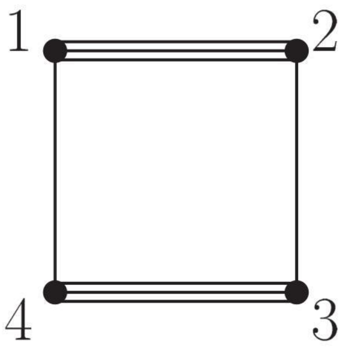
$$\frac{1}{s_{12}} + \frac{1}{s_{14}} \qquad \frac{s_{13}}{s_{12}^2} \qquad \frac{s_{13}s_{14}}{s_{12}^3}$$

$$\frac{z_{12}}{s_{12}} \left(\frac{s_{12}}{z_{12}} + \frac{s_{13}}{z_{13}} + \frac{s_{14}}{z_{14}} \right) = 0$$

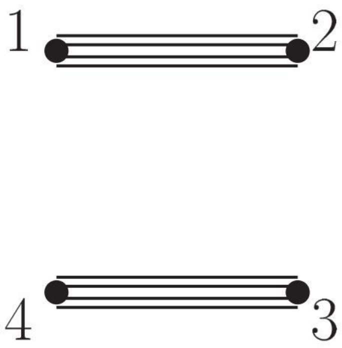
Cross-ratio Identities



$$\frac{1}{s_{12}} + \frac{1}{s_{14}}$$



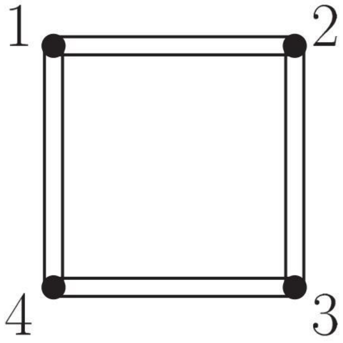
$$\frac{s_{13}}{s_{12}^2}$$



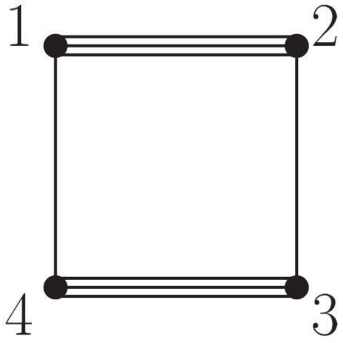
$$\frac{s_{13}s_{14}}{s_{12}^3}$$

$$1 = -\frac{s_{14}}{s_{12}}\frac{z_{12}z_{43}}{z_{14}z_{23}}$$

Cross-ratio Identities



$$\frac{1}{s_{12}} + \frac{1}{s_{14}}$$



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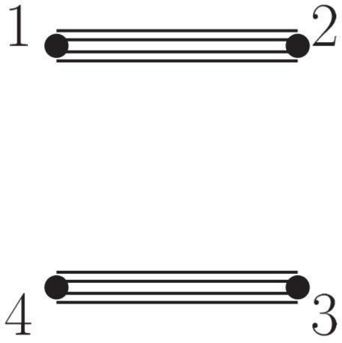

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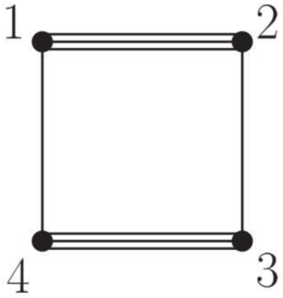
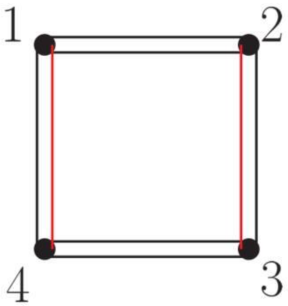
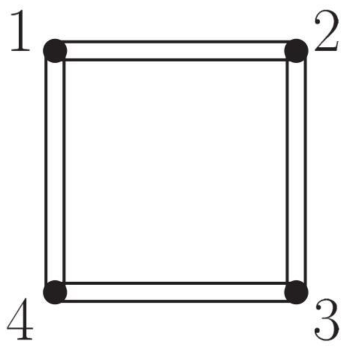


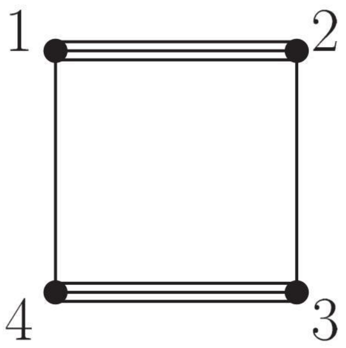
$$\longrightarrow -\frac{s_{14}}{s_{12}}$$



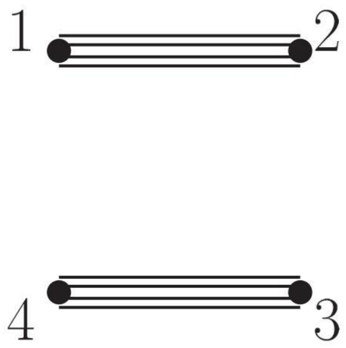
Cross-ratio Identities



$$\frac{1}{s_{12}} + \frac{1}{s_{14}}$$

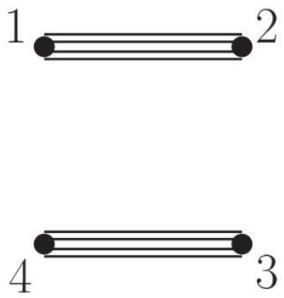


$$\frac{s_{13}}{s_{12}^2}$$

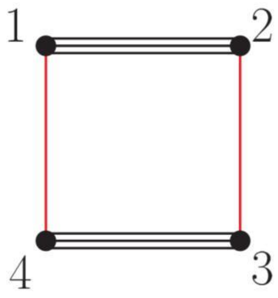


$$\frac{s_{13}s_{14}}{s_{12}^3}$$

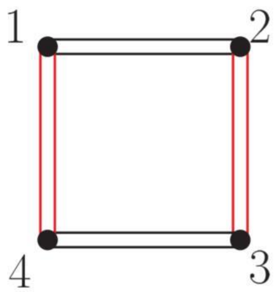
$$1 = -\frac{s_{14}}{s_{12}}\frac{z_{12}z_{43}}{z_{14}z_{23}}$$



$$\longrightarrow -\frac{s_{14}}{s_{12}}$$



$$\longrightarrow \frac{s_{14}^2}{s_{12}^s}$$



Cross-ratio Identities

[C. Cardona, B. Feng, H. Gomez, RH, 1606.00670]

The n-point amplitude, two-particle pole

$$1 = -\sum_{b=1,b\neq a,q,p}^{n} \frac{s_{ab}}{s_{aq}} \frac{z_{aq} z_{bp}}{z_{ab} z_{qp}}$$

Cross-ratio Identities

[C. Cardona, B. Feng, H. Gomez, RH, 1606.00670]

The n-point amplitude, two-particle pole

Cross-ratio, Mobius invariant

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Cross-ratio Identities

[C. Cardona, B. Feng, H. Gomez, RH, 1606.00670]

The n-point amplitude, two-particle pole

Cross-ratio, Mobius invariant

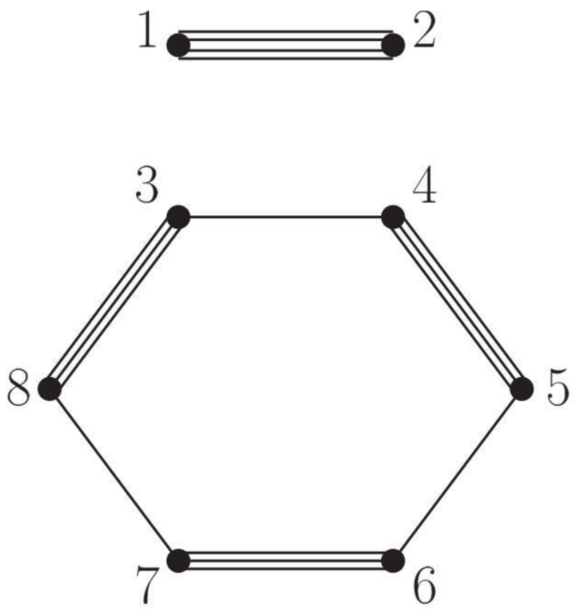
$$1 = -\sum_{b=1, b\neq a,q,p}^{n} \frac{s_{ab}}{s_{aq}} \frac{z_{aq} z_{bp}}{z_{ab} z_{qp}}$$

The n-point amplitude, generic pole

$$1 = -\sum_{i\in A\setminus\{a\}} \sum_{j\in \bar{A}\setminus\{b\}} \frac{s_{ij}}{s_A} \frac{z_{ia} z_{jb}}{z_{ij} z_{ab}}$$

Cross-ratio Identities

[C. Cardona, B. Feng, H. Gomez, RH, 1606.00670]



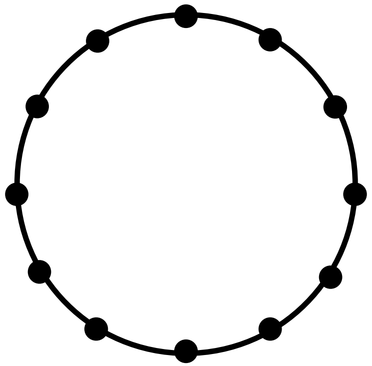
$\chi = 2$ {1,2}

$\chi = 1$ {3,8} , {4,5} , {6,7} , {1,2,3,8} , {1,2,4,5} , {1,2,6,7}

	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	Round 8
#[ALL]	5	25	121	613	2779	7543	9914	9922
#[H]	5	25	121	464	615	301	2	0

to the PT factor

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$



[RH, Y.J. Du, B. Feng, 1702.05840]

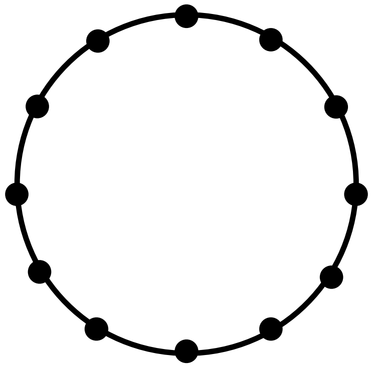
to the PT factor

The cross-ratio identity

The open-up relation

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$

$$\frac{[z_1, z_n]}{\langle z_1, \alpha, z_n, \beta \rangle} = (-)^{n_\beta+1}\sum_{\blacksquare} \frac{1}{[z_1, \alpha \blacksquare \beta^T, z_n]}$$



[RH, Y.J. Du, B. Feng, 1702.05840]

to the PT factor

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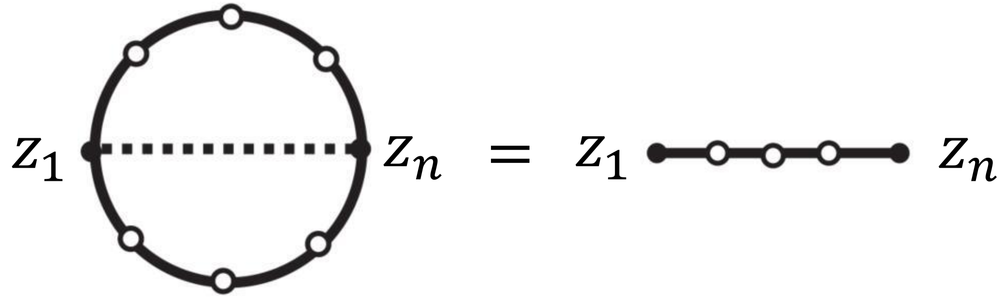

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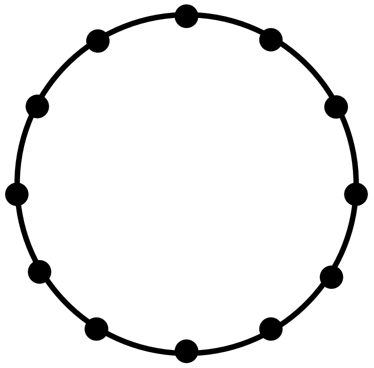

[RH, Y.J. Du, B. Feng, 1702.05840]

to the PT factor

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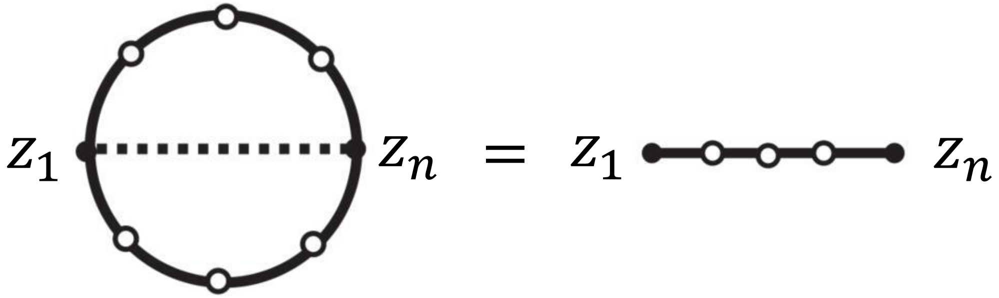
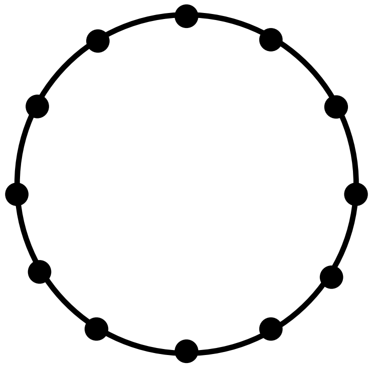


$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$



$$\frac{[z_1, z_n]}{\langle z_1, \alpha, z_n, \beta \rangle} = (-)^{n_\beta+1} \sum_{\blacksquare} \frac{1}{[z_1, \alpha \blacksquare \beta^T, z_n]}$$

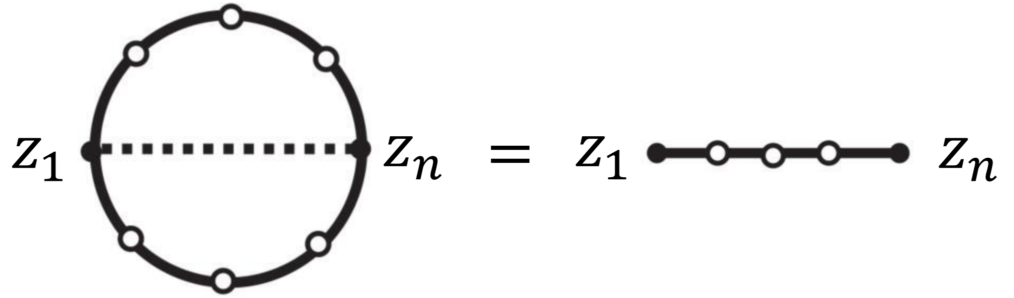
$$\frac{1}{\langle z_1, z_2, \cdots, z_n \rangle} \frac{[z_i, z_j][z_k, z_l]}{[z_i, z_k][z_j, z_l]}$$

[RH, Y.J. Du, B. Feng, 1702.05840]

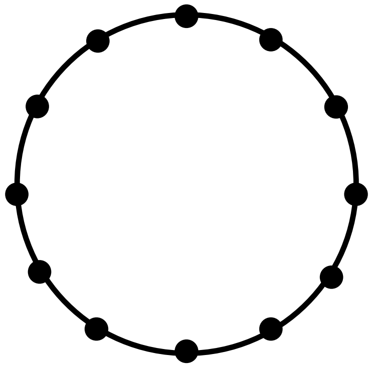
to the PT factor

The cross-ratio identity

The open-up relation



$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$



$$\frac{[z_1,z_n]}{\langle z_1,\alpha,z_n,\beta\rangle} = (-)^{n_\beta+1}\sum_{\blacksquare}\frac{1}{[z_1,\alpha\blacksquare\beta^T,z_n]}$$

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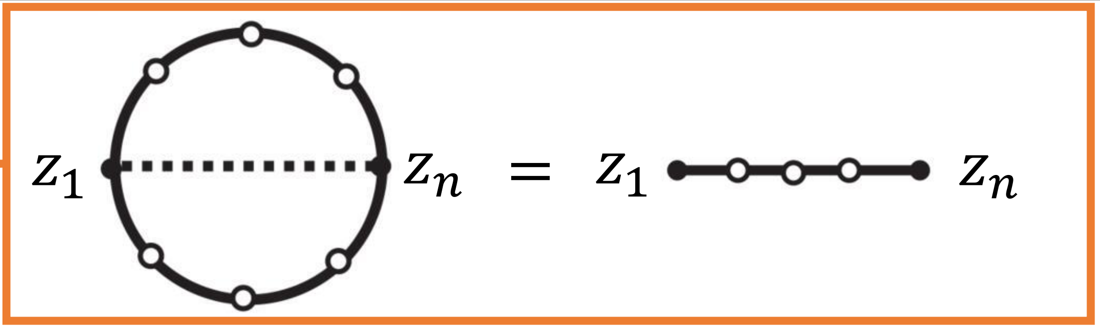
[RH, Y.J. Du, B. Feng, 1702.05840]

Integration Rules

to the PT factor

The cross-ratio identity

The open-up relation



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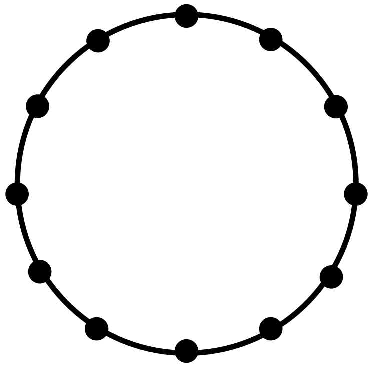
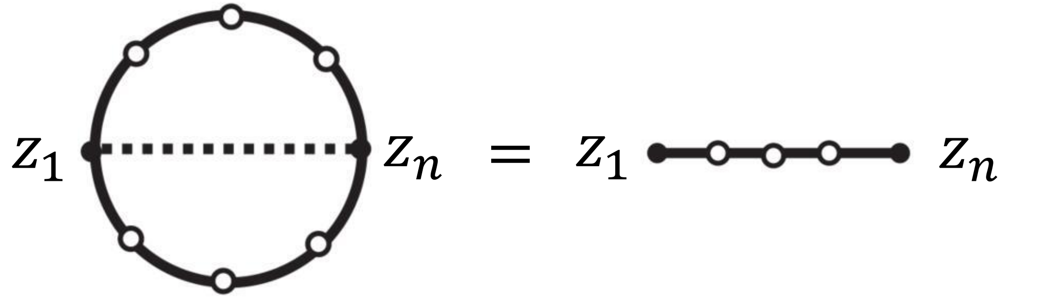


[RH, Y.J. Du, B. Feng, 1702.05840]

to the PT factor

The cross-ratio identity

The open-up relation



$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$

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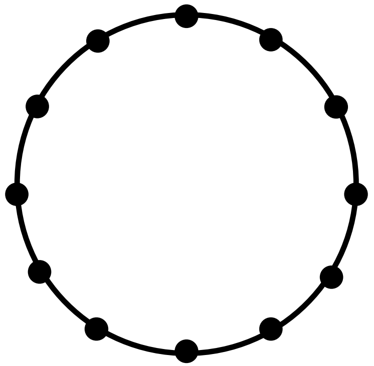

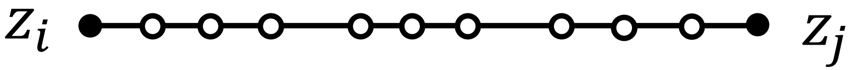

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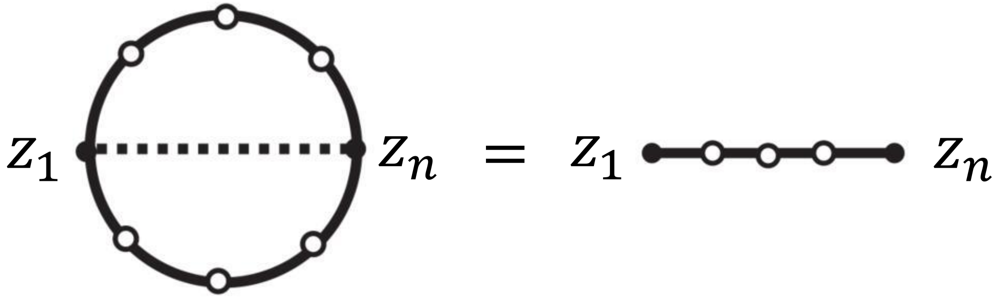
The cross-ratio identity

The open-up relation



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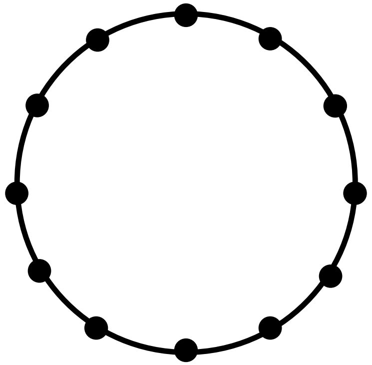

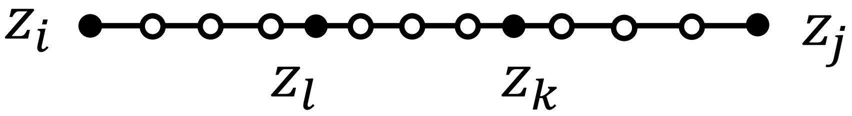

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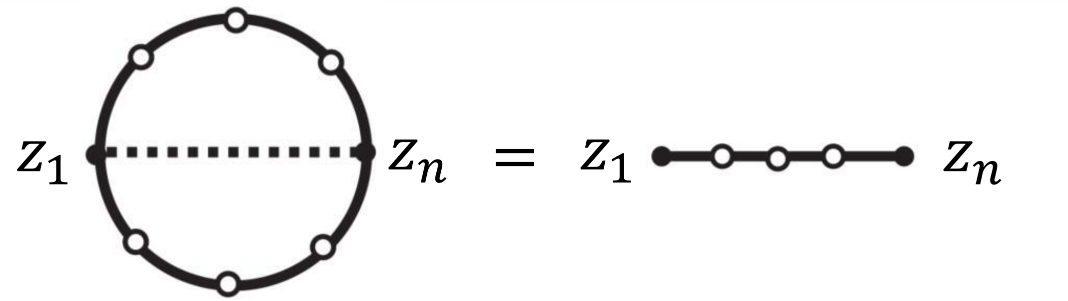

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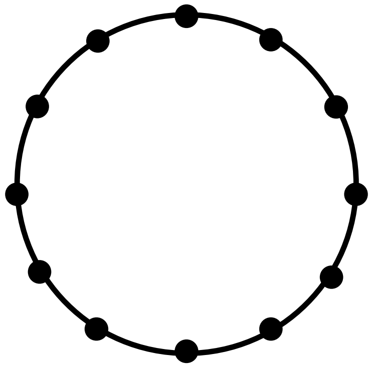

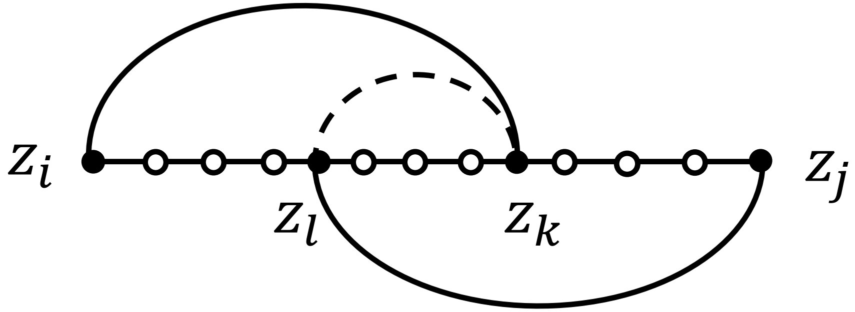

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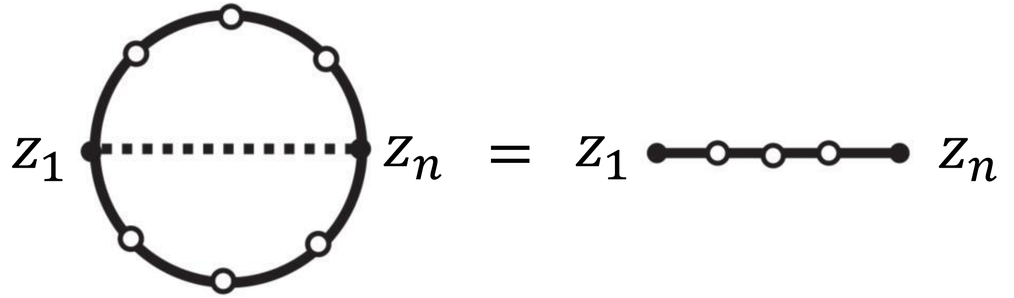

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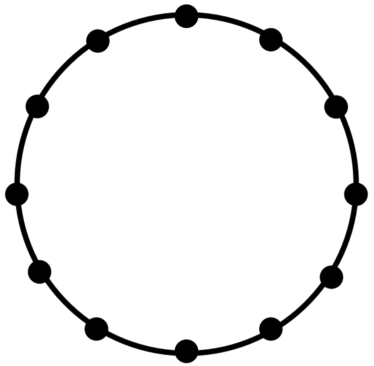
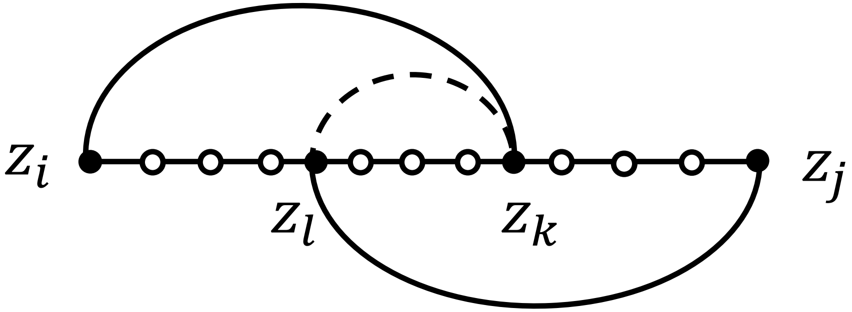
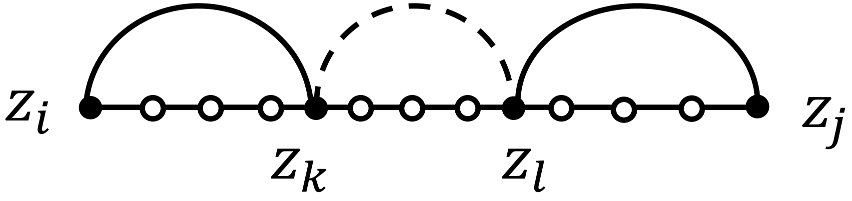
$$\frac{[z_1,z_n]}{\langle z_1,\alpha,z_n,\beta\,\rangle} = (-)^{n_\beta+1}\sum_{\blacksquare}\frac{1}{[z_1,\alpha\blacksquare\beta^T,z_n]}$$



$$\frac{1}{\langle z_1,z_2,\cdots,z_n\rangle}\frac{[z_i,z_j][z_k,z_l]}{[z_i,z_k][z_j,z_l]} = \left(\frac{[z_i,z_j]}{\langle z_1,z_2,\cdots,z_n\rangle}\right)\frac{[z_k,z_l]}{[z_i,z_k][z_j,z_l]}$$



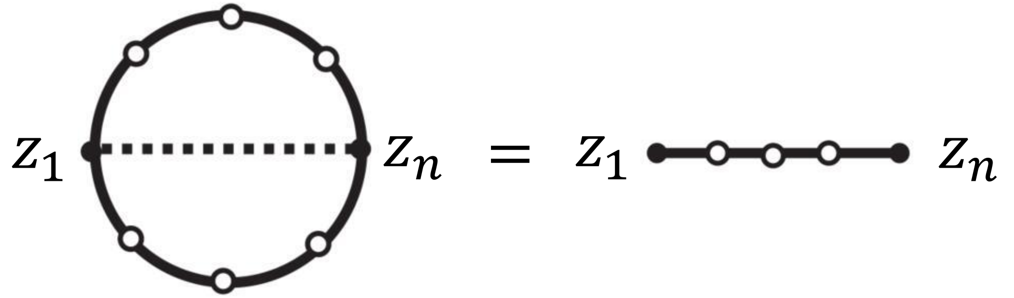


[RH, Y.J. Du, B. Feng, 1702.05840]

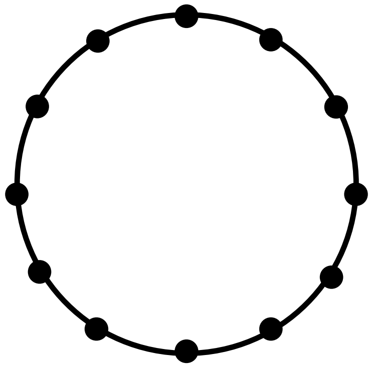
to the PT factor

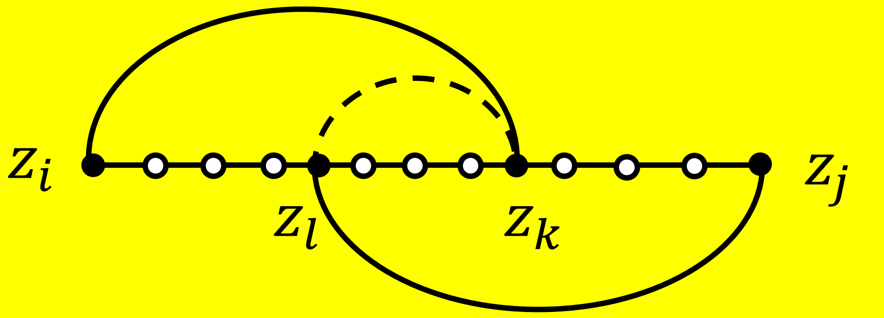
The cross-ratio identity

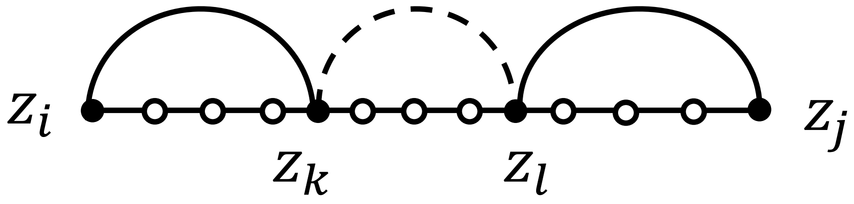
The open-up relation



$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$







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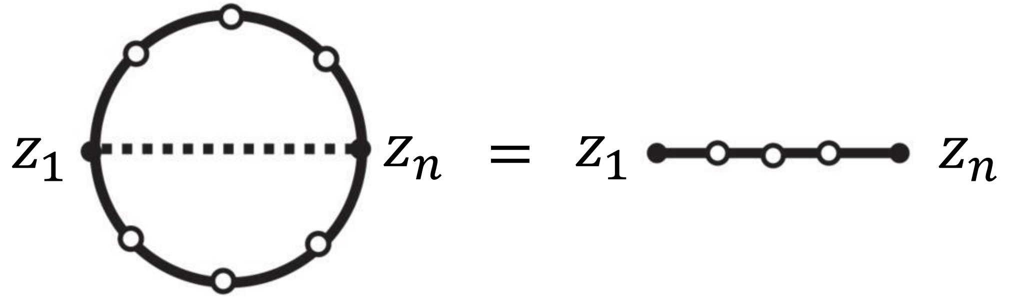
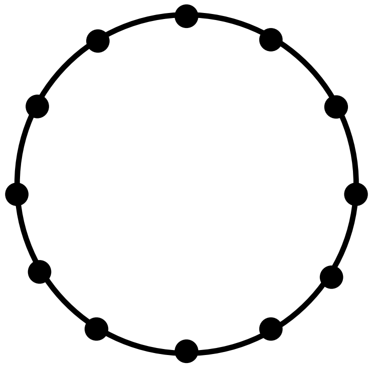
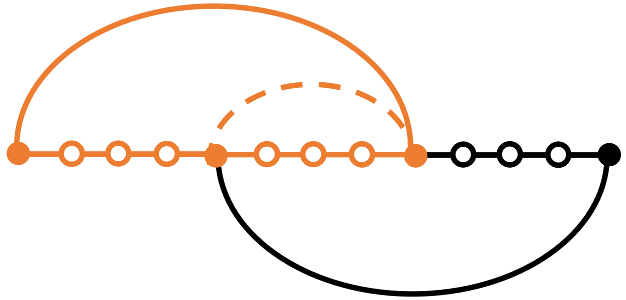
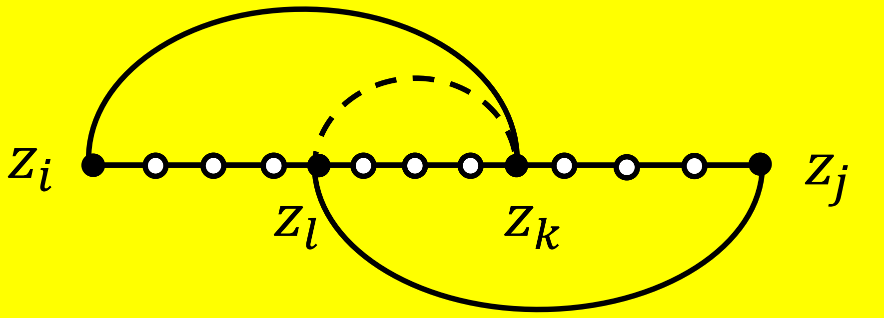
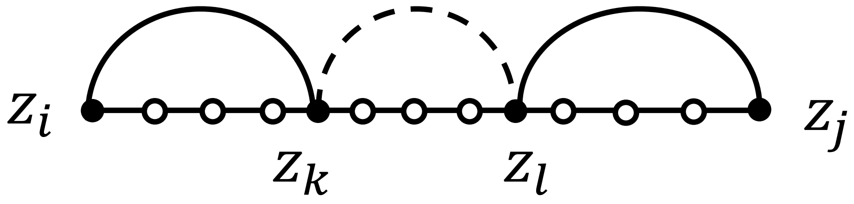


$$\frac{1}{z_{12} z_{23} \cdots z_{n1}}$$









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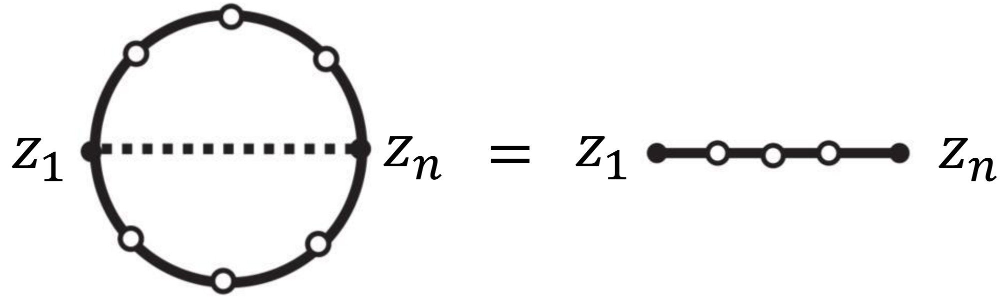
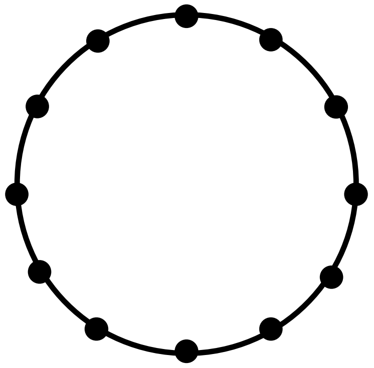
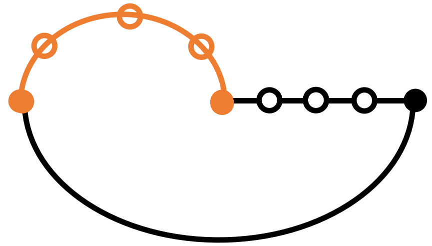
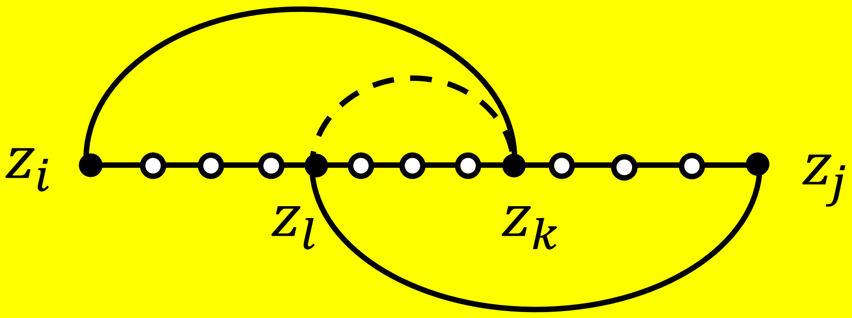
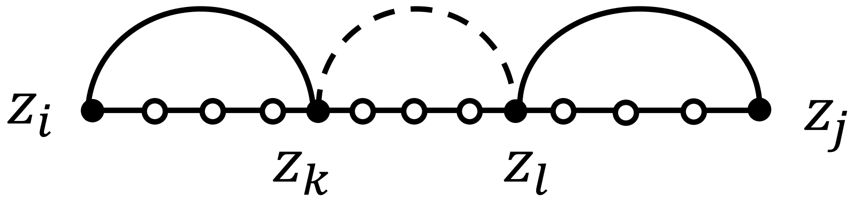


$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$









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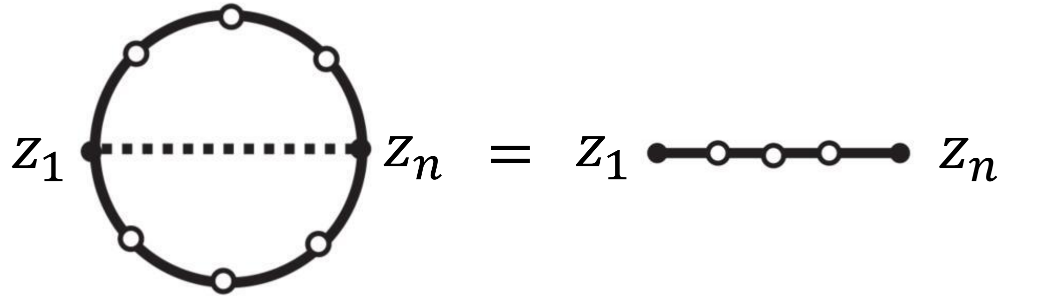
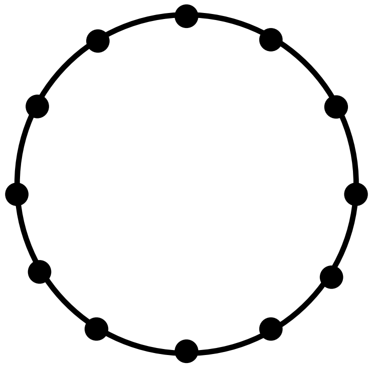
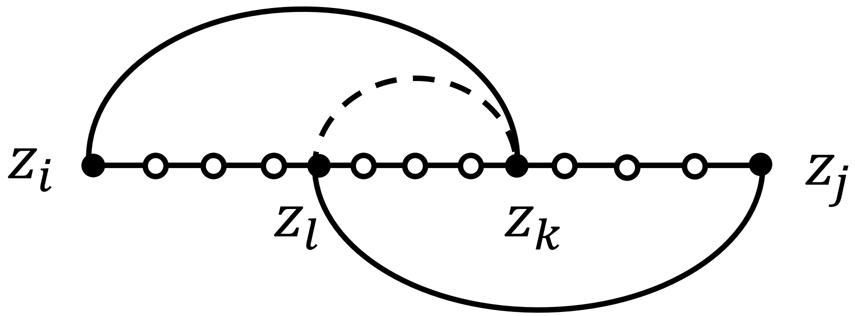
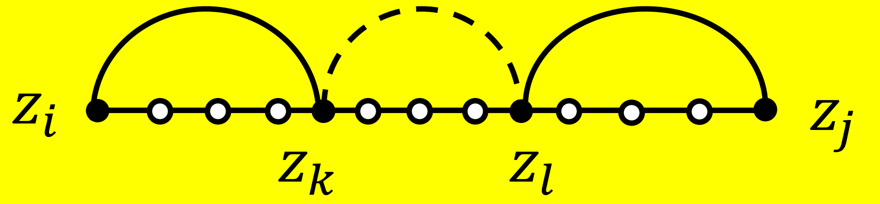


$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$







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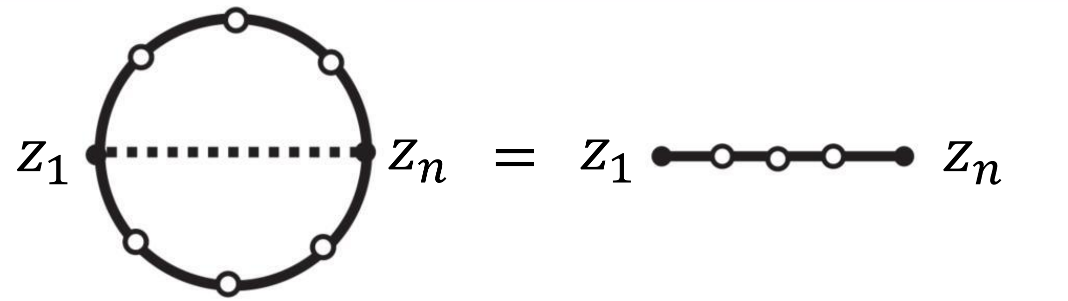
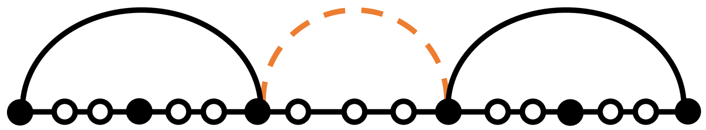
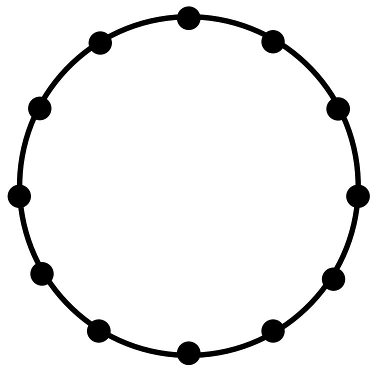
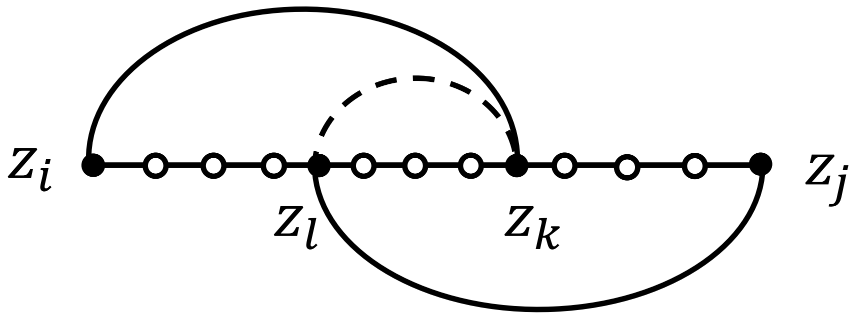


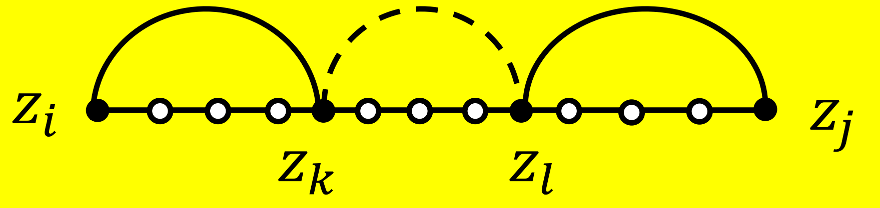
$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$

Situation 1









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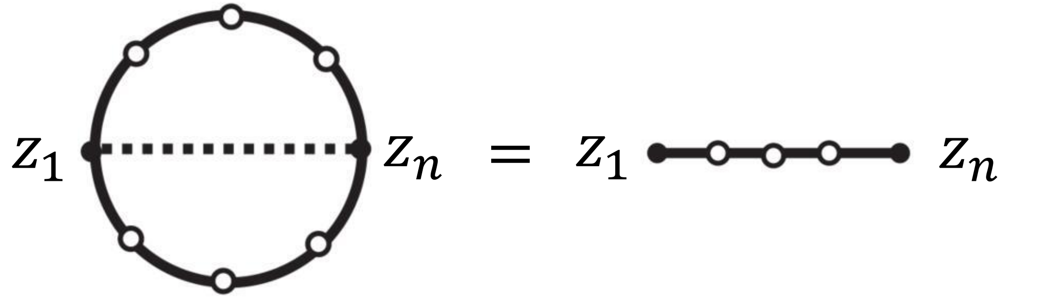

$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$

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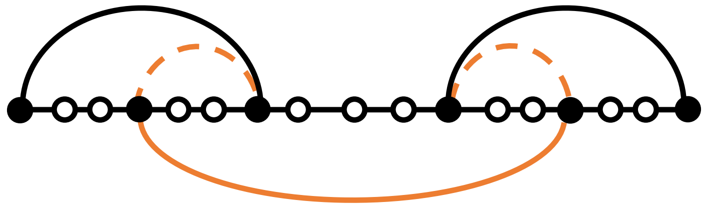

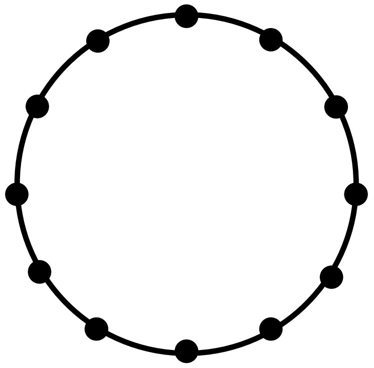

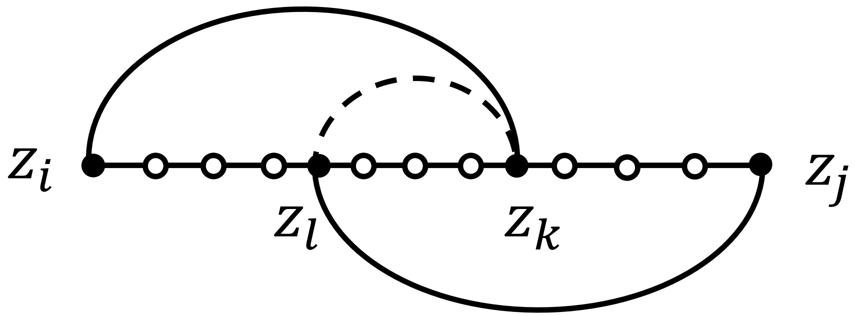

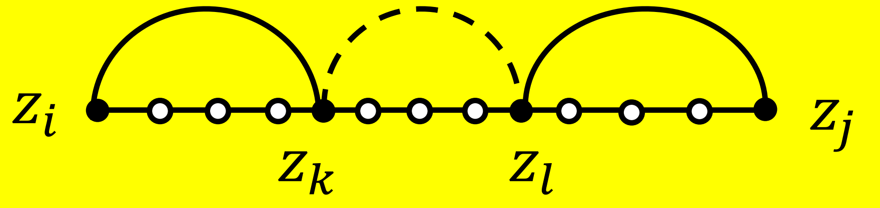

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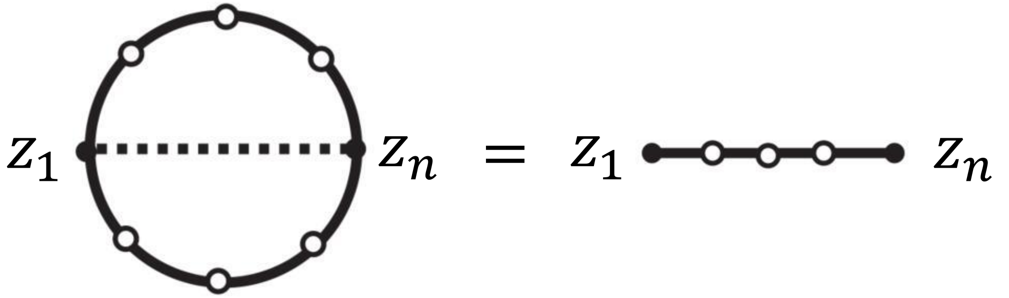
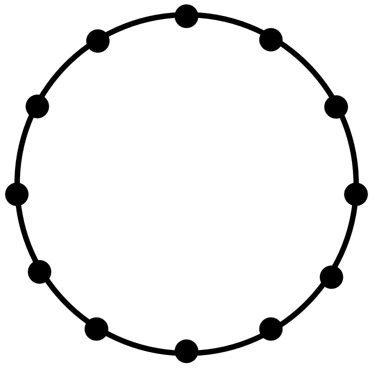


$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$



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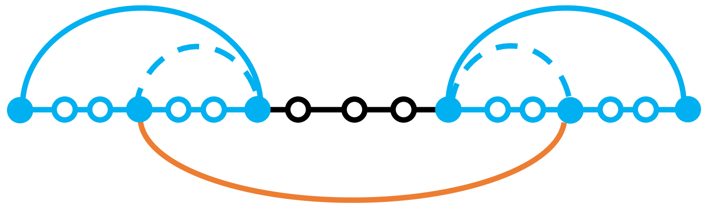

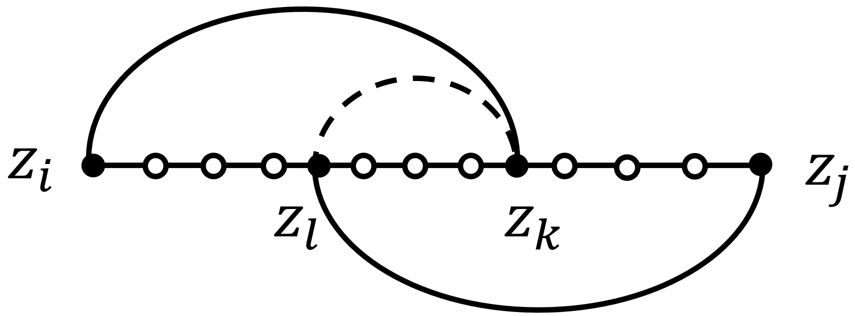

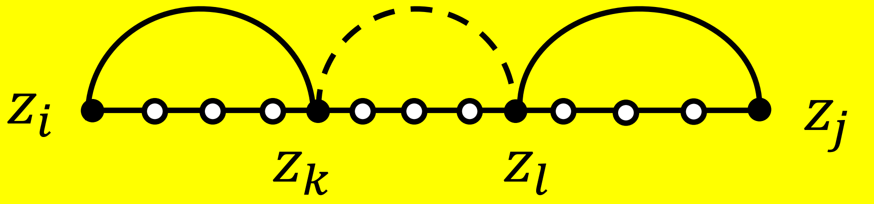

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Integration Rules

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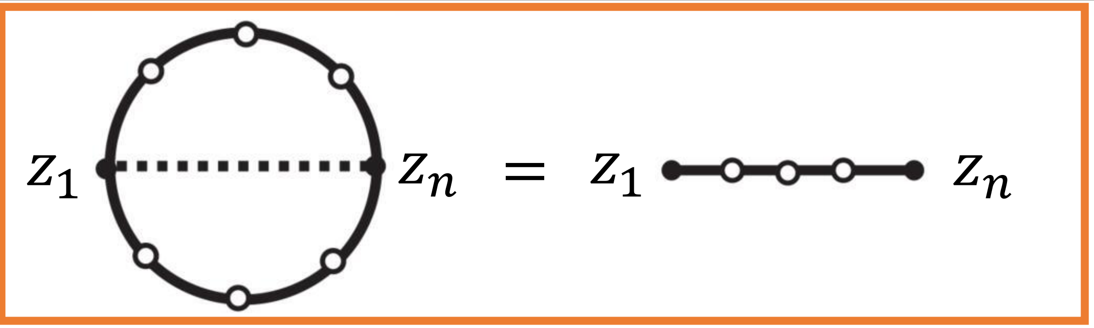
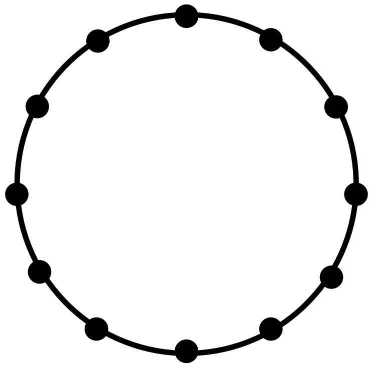


$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$



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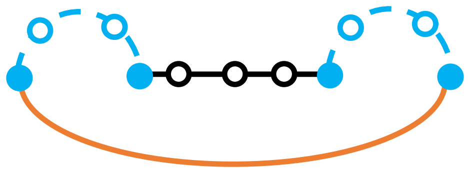

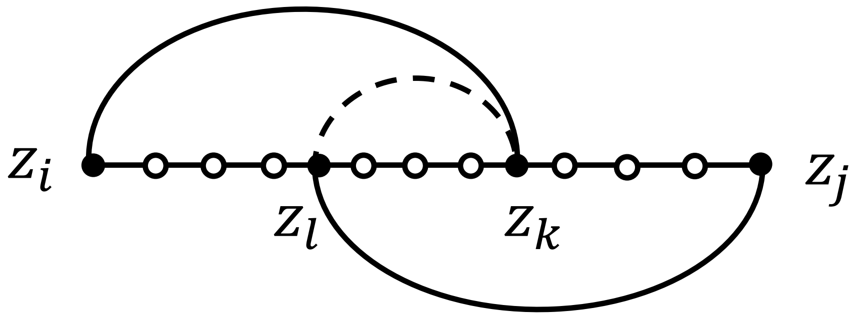

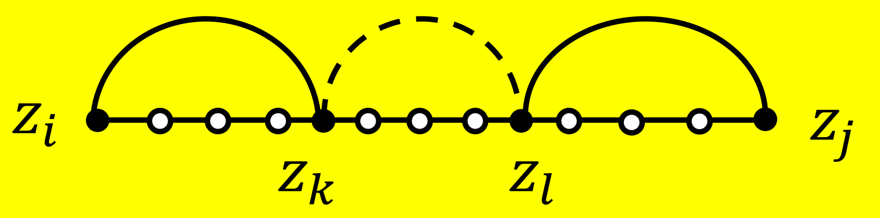

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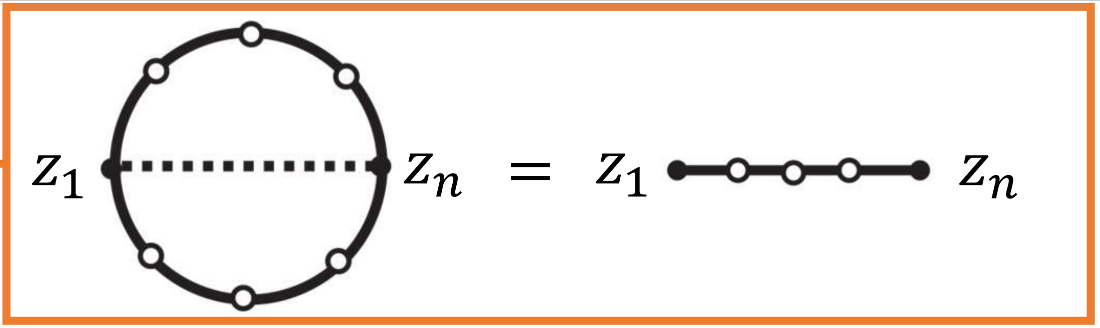
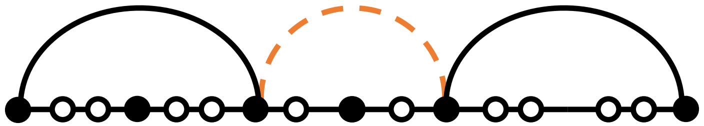
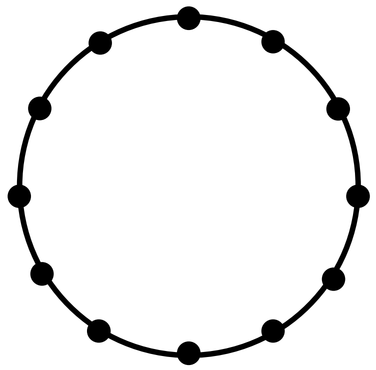
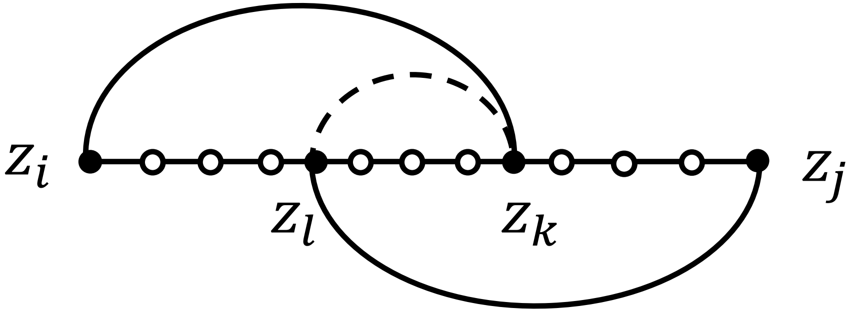


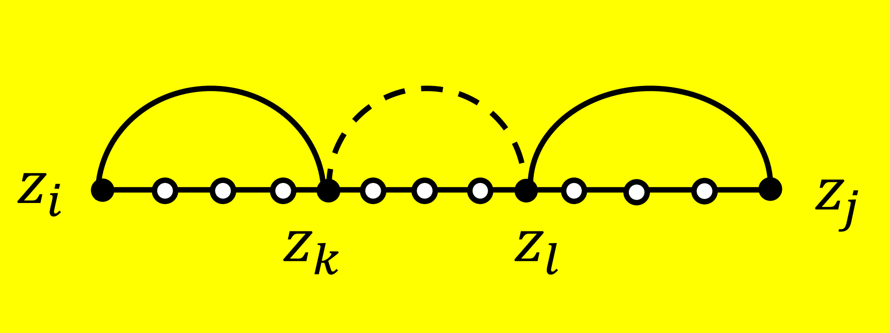
$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$

Situation 2









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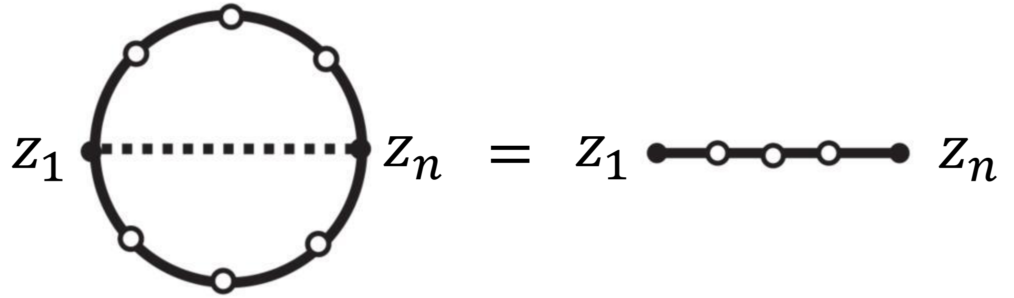

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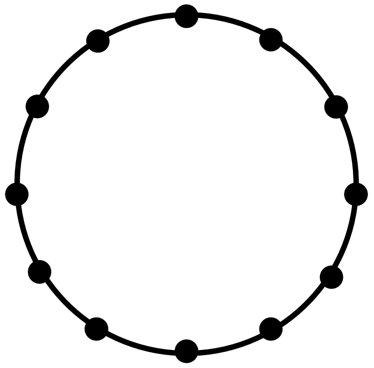

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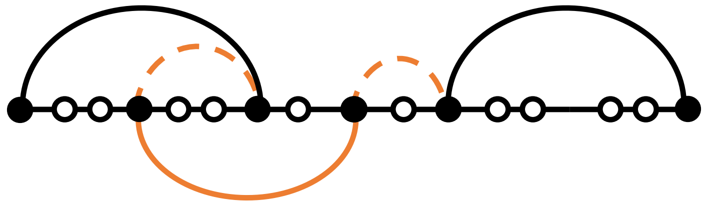

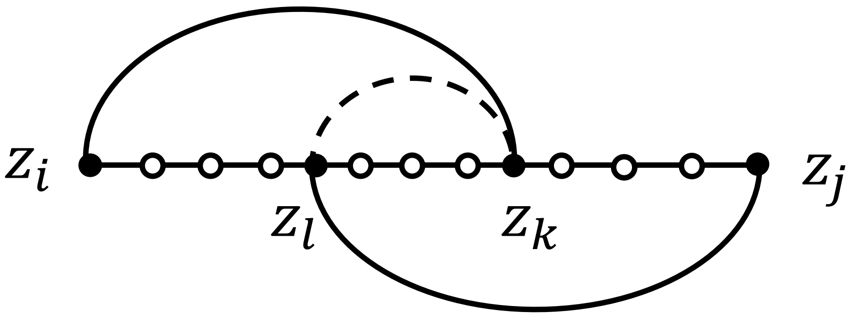

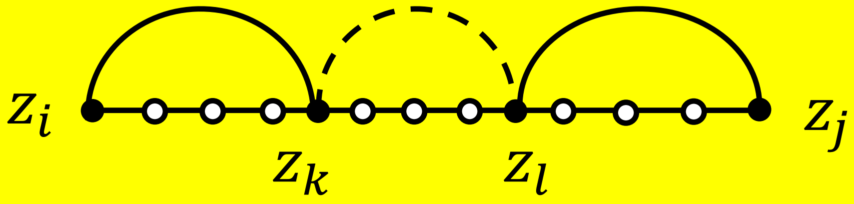

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Integration Rules

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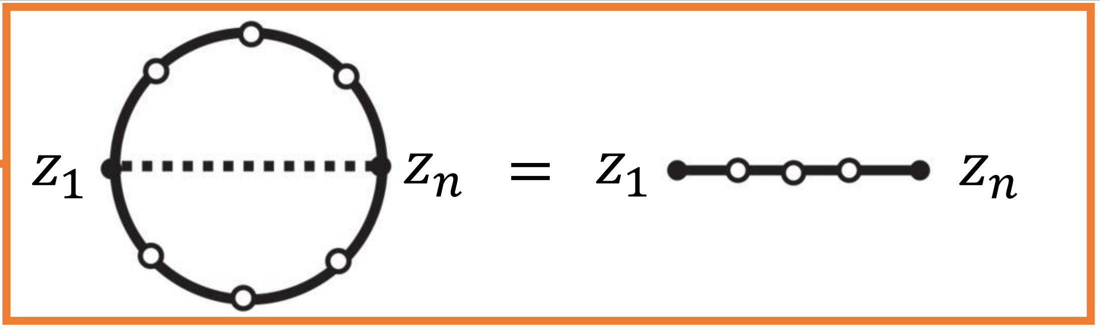

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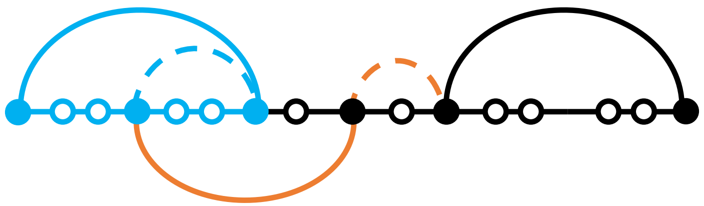

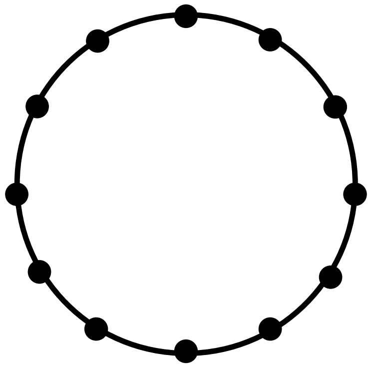

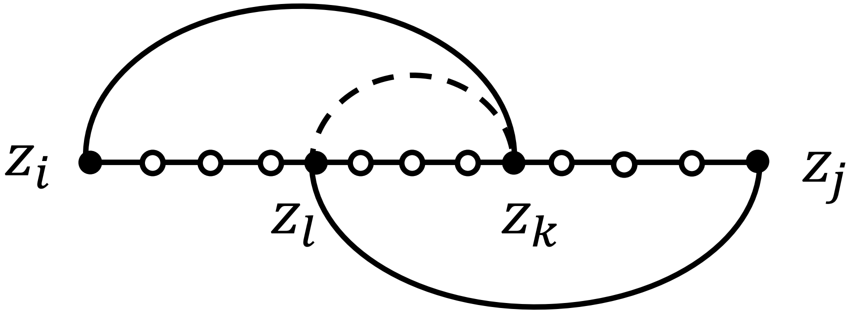

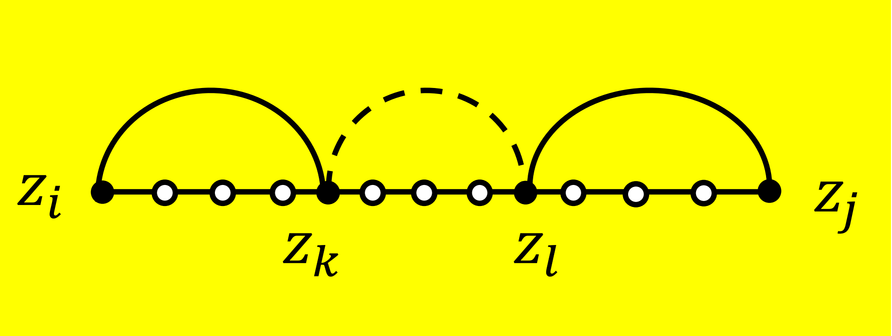

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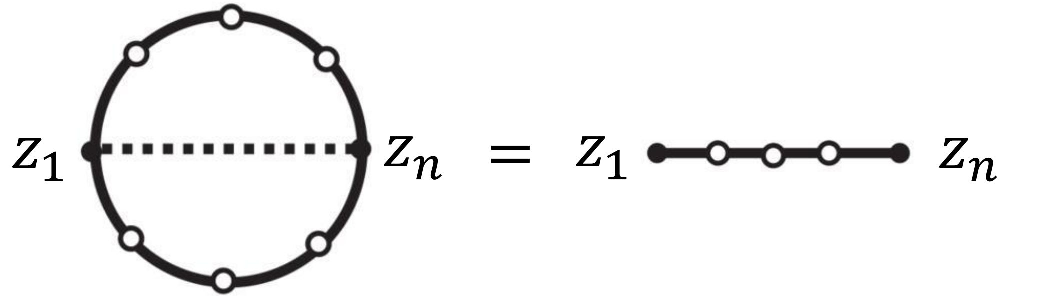

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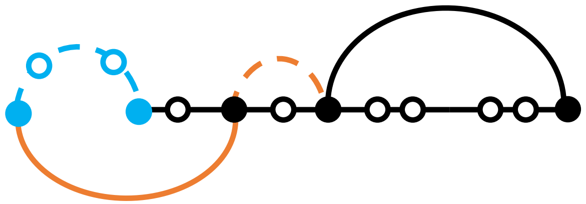

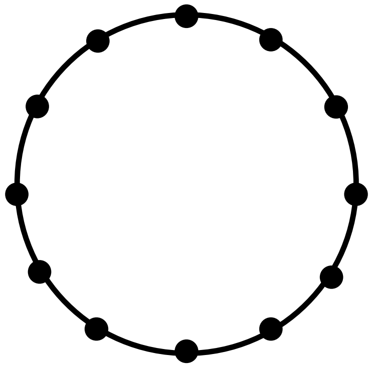

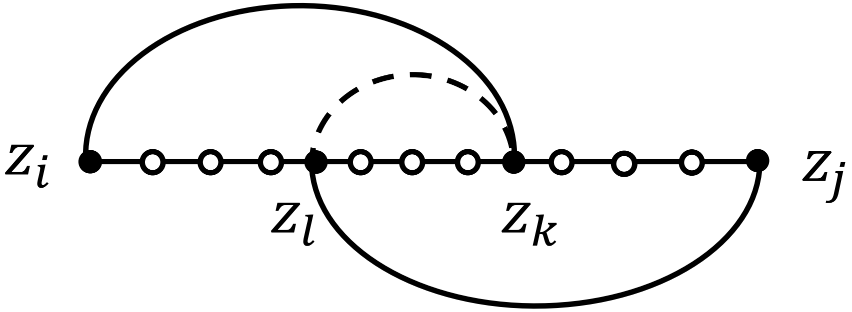

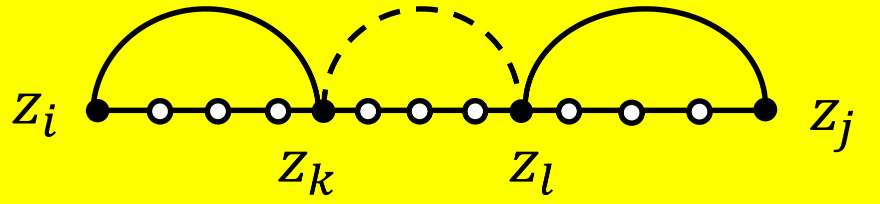

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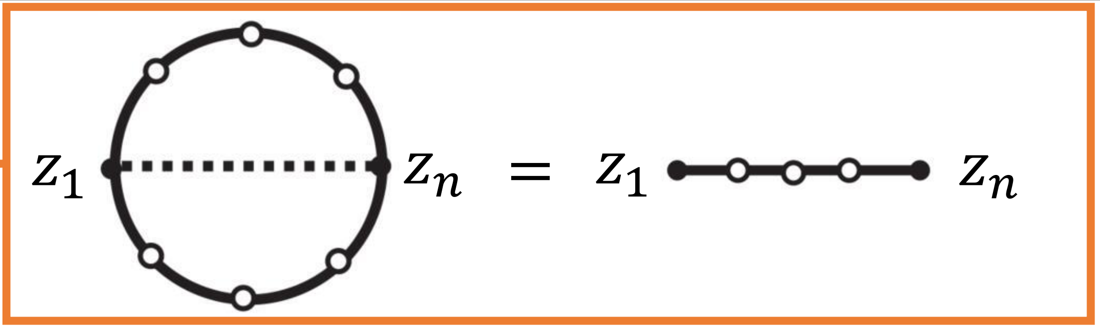

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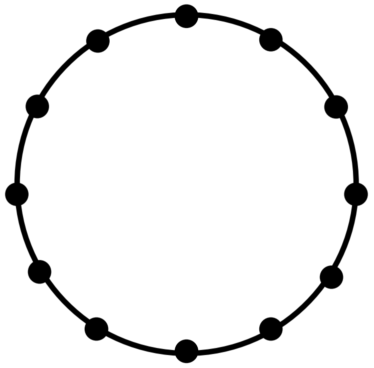

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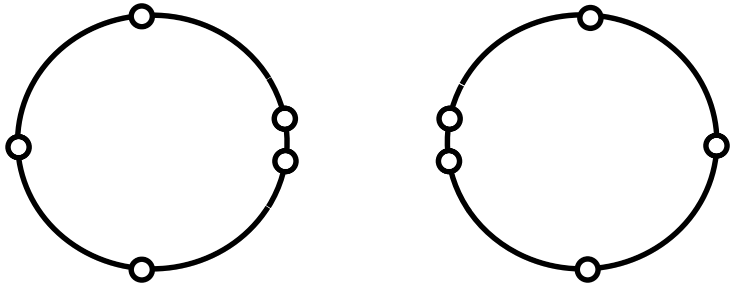

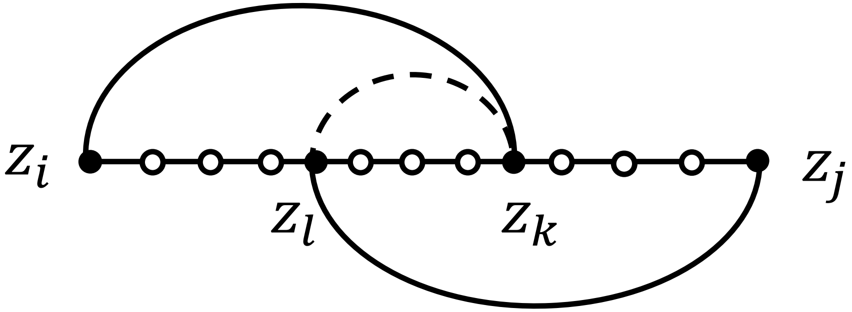

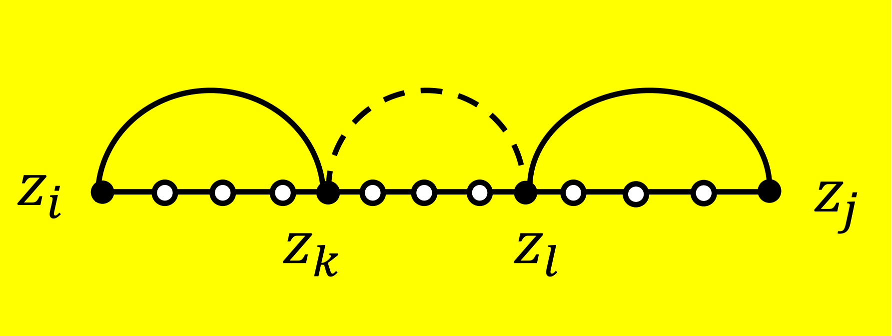

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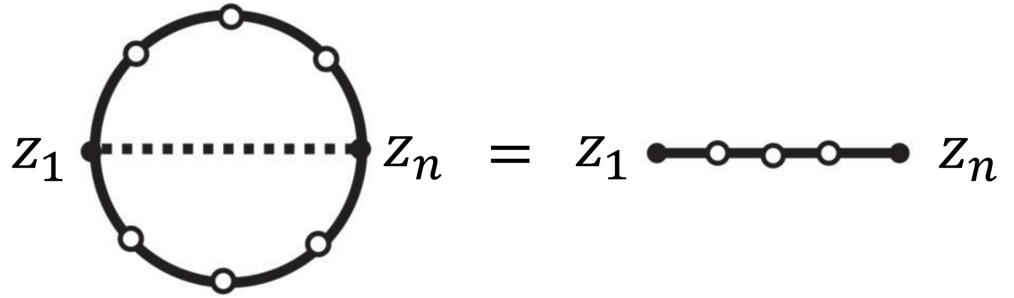

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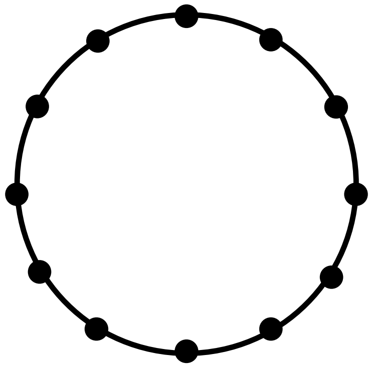
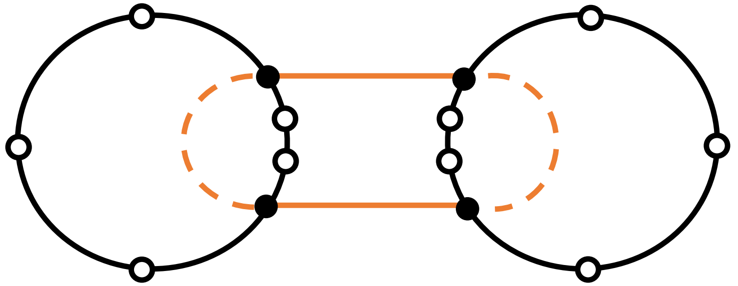
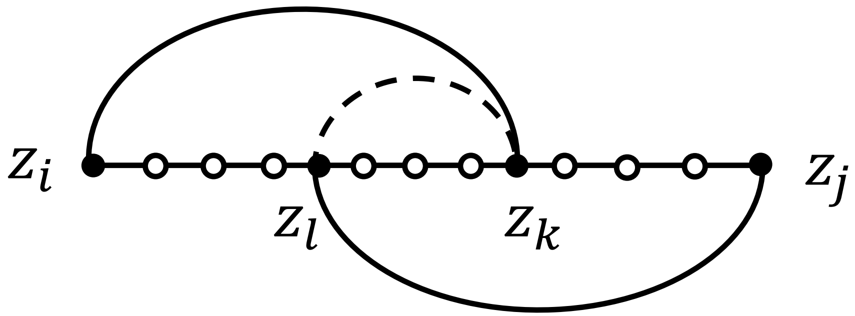
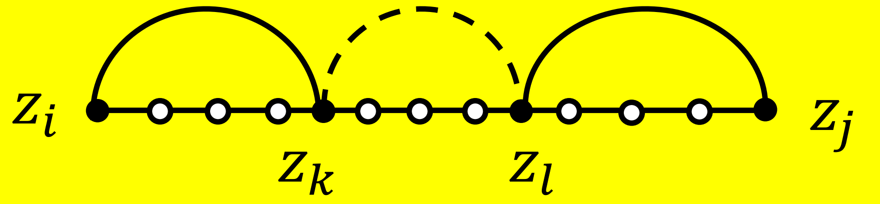


Situation 2





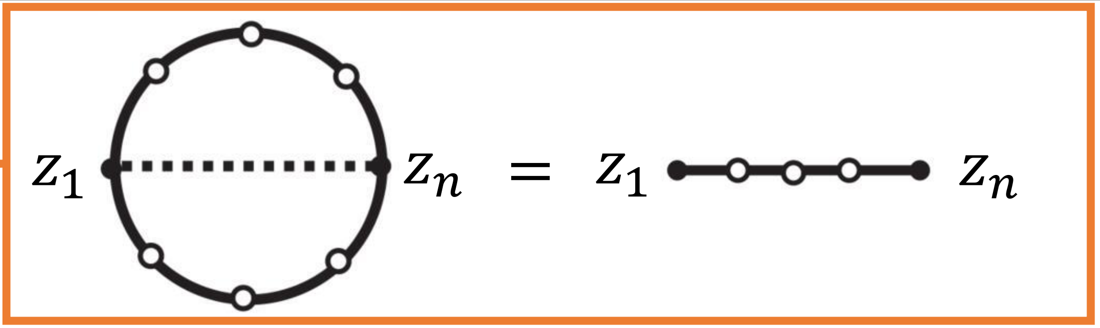


[RH, Y.J. Du, B. Feng, 1702.05840]

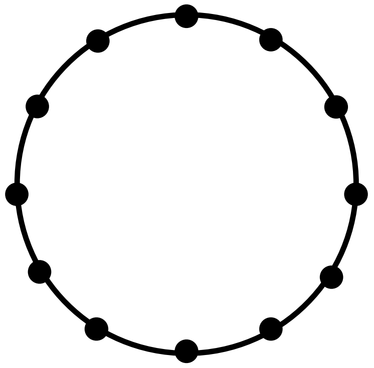
to the PT factor

The cross-ratio identity

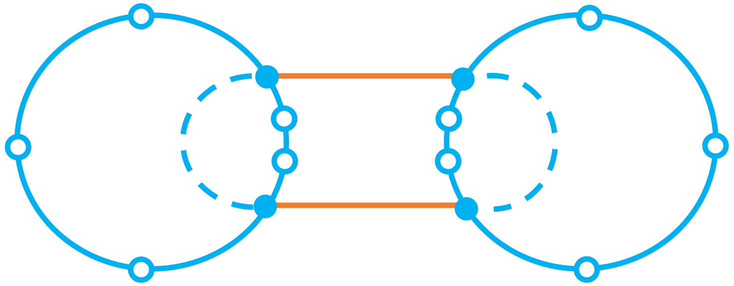
The open-up relation

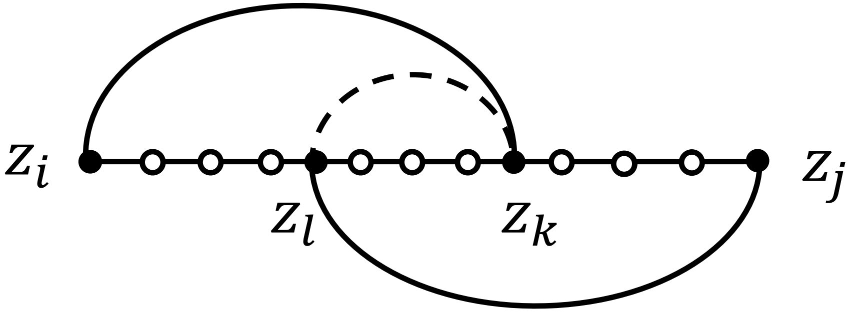


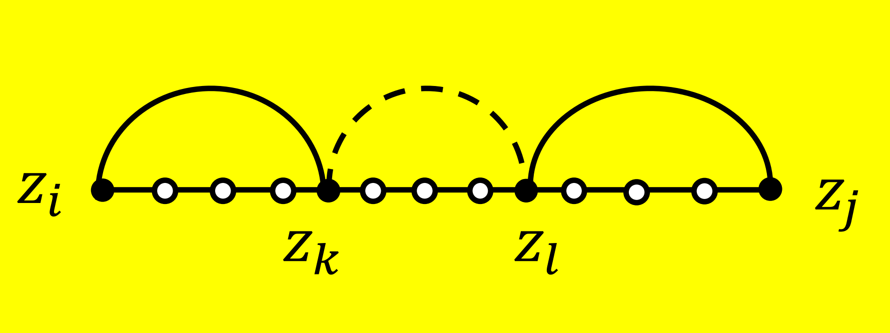
$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$



Situation 2







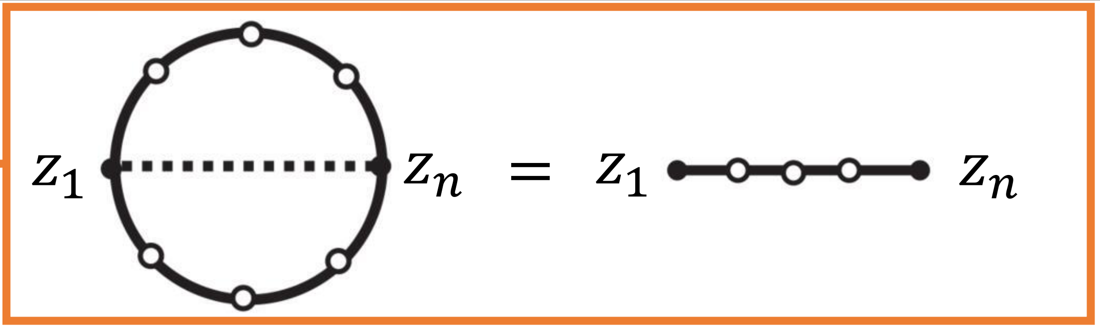
[RH, Y.J. Du, B. Feng, 1702.05840]

Integration Rules

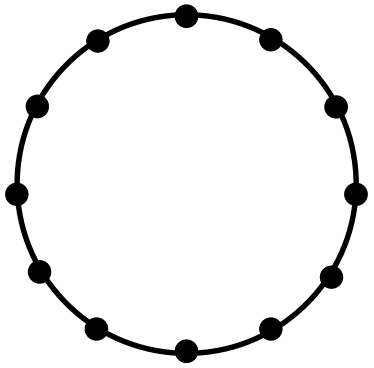
to the PT factor

The cross-ratio identity

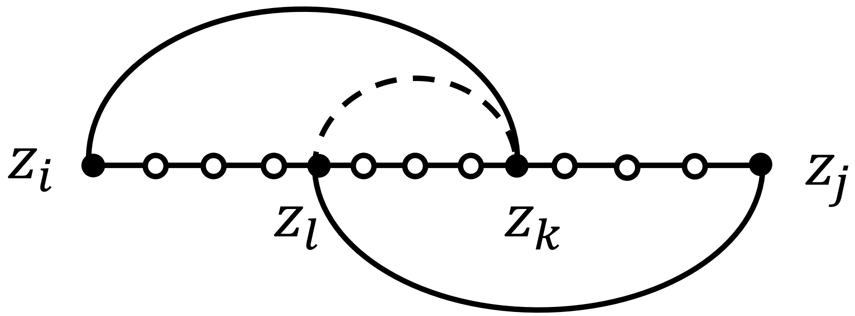
The open-up relation

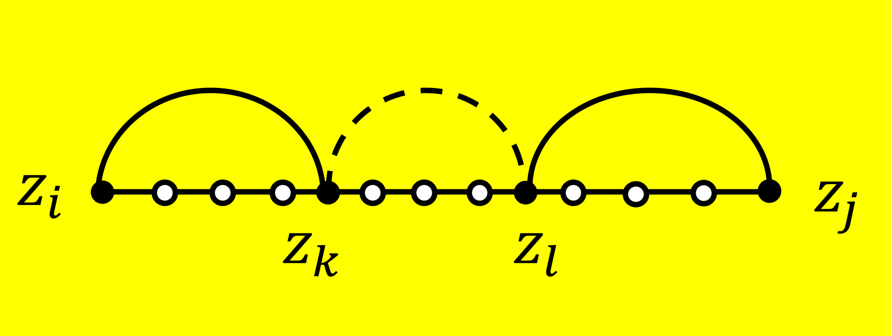


$$\frac{1}{z_{12}z_{23}\cdots z_{n1}}$$



Situation 2





[RH, Y.J. Du, B. Feng, 1702.05840]

$$\text{CHY-integrand} = \sum \text{Coefficient} \times \frac{1}{(\text{PT Factor})} \times \frac{1}{(\text{PT Factor})}$$

Integration Rules

$$\text{CHY-integrand} = \sum \text{Coefficient} \times \frac{1}{(\text{PT Factor})} \times \frac{1}{(\text{PT Factor})}$$

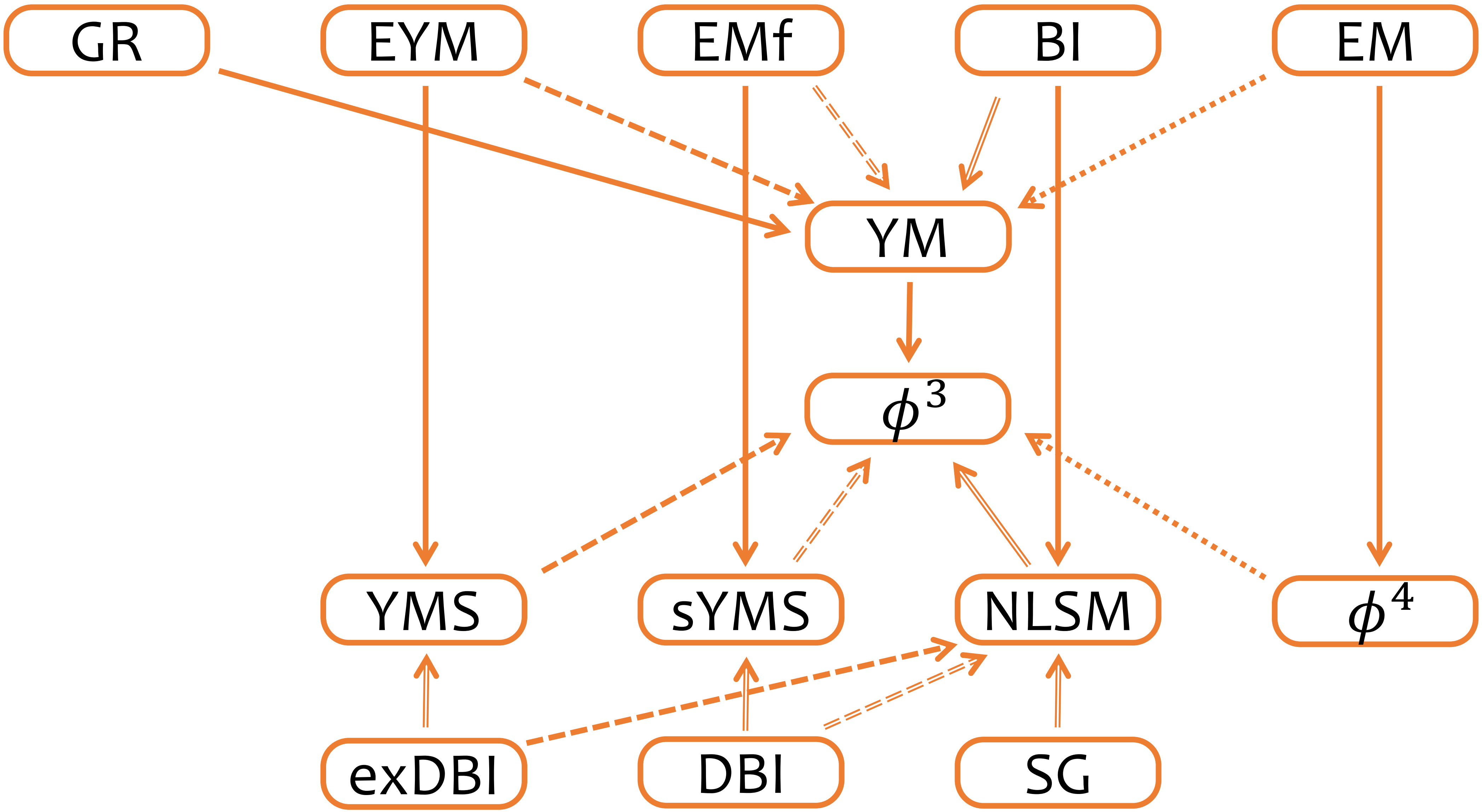
↓ Amplitude ↓

$$\text{Field Theory} = \sum \text{Coefficient} \times (\text{Cubic Scalar Theory})$$

Network of Field Theories

Network of Field Theories

[K. Zhou, B. Feng, 1808.06835; K. Zhou, 1908.10272]



EFT & Amplitude Workshop @ USTC

Scattering Amplitude in CHY-Formulation

Rijun Huang
Nanjing Normal University