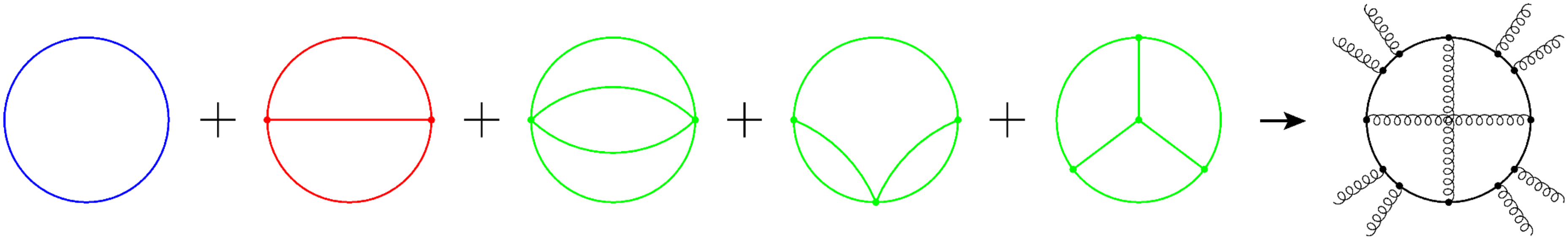
Towards a general solution for multi-loop calculations



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Based on works done with Xiao Liu(刘霄) and Chen-Yu Wang(王辰宇)

1711.09572, 1801.10523 and works in preparation

有效场论与散射振幅专题研讨会

2019/09/07-08, USTC, Hefei

Outline

I. Introduction

II. A New Representation

III. Reduction

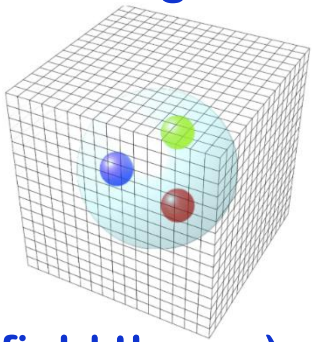
IV. Outlook

Quantum field theory

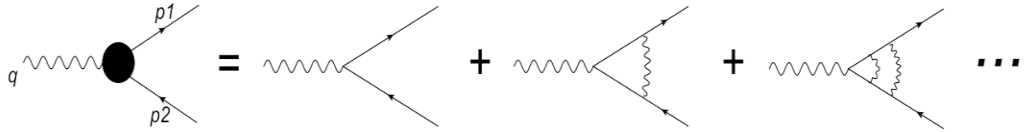
➢ **QFT: the underlying theory of modern physics**

- Solving QFT is important for testing the SM and discovering NP

➢ **How to solve QFT:**



- **Nonperturbatively** (e.g. lattice field theory):
  discretize spacetime, numerical simulation
  complicated, application limited



- **Perturbatively** (small coupling constant):
  generate and calculate Feynman amplitudes,
  relatively simpler, the primary method

Super computer

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules

见张扬、黄日俊、靳庆军报告

- New developments: unitarity, recurrence relation

2. Calculate Feynman loop integrals

3. Calculate phase-space integrals

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals

$$\int \frac{d^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \lim_{\eta\to 0^+}\int \frac{d^D p}{(2\pi)^D}\left(\frac{i}{p^2+i\eta}+\frac{-i}{p^2-i\eta}\right)$$

见刘晓辉报告

Feynman loop integrals

➢ The key to apply pQFT

$$\lim_{\eta\to 0^+}\int \prod_{i=1}^{L}\frac{\mathrm{d}^D\ell_i}{\mathrm{i}\pi^{D/2}}\prod_{\alpha=1}^{N}\frac{1}{(q_\alpha^2-m_\alpha^2+\mathrm{i}\eta)^{\nu_\alpha}}$$

- q_α: linear combination of loop momenta and external momenta
- Taking $\eta \to 0^+$ before taking $D \to 4$

➢ Theorem:

Smirnov, Petukhov, 1004.4199

For a given set of propagators, Feynman integrals form a finite-dimensional linear space

Multi-loop: a challenge for intelligence

- One-loop calculation: (up to 4 legs) satisfactory approaches existed as early as 1970s

’t Hooft, Veltman, NPB (1979); Passarino, Veltman, NPB (1979); Oldenborgh, Vermaseren (1990)

Developments of unitarity-based method in the past decade made the calculation efficient for multi-leg problems

Britto, Cachazo, Feng, 0412103; Ossola, Papadopoulos, Pittau, 0609007; Giele, Kunszt, Melnikov, 0801.2237

- About 40 years later, a satisfactory method for multi-loop calculation is still missing

Main strategy

1) Reduce loop integrals to basis (Master Integrals)

- Integration-by-parts (IBP) reduction: Chetyrkin, Tkachov, NPB (1981) Laporta, 0102033

  the only way (before our method), main bottleneck

  extremely time consuming for multi-scale problems

  unitarity-based reduction is efficient but cannot give complete reduction

2) Calculate MIs/original integrals

- Differential equations (depends on reduction and BCs) Kotikov, PLB (1991)
- Difference equations (depends on reduction and BCs) Laporta, 0102033
- Sector decomposition (extremely time-consuming) Binoth, Heinrich, 0004013
- Mellin-Barnes representation (nonplanar, time) Usyukina (1975) Smirnov, 9905323

IBP redution

➢ A result of dimensional regularization

Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(v_k^\mu \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}} \right) = 0 \,, \quad \forall j, k$$

$$\Downarrow$$

- **Linear equations:** $\sum_{i=1} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$
- M_i **scalar integrals,** Q_i **polynomials in** $D, \vec{s}, \eta$

➢ For each problem, the number of MIs is FINITE

Smirnov, Petukhov, 1004.4199

- **Feynman integrals form a finite dimensional linear space**
- **Reduce thousands of loop integrals to much less MIs**

Difficulty of IBP reduction

➢ Solve IBP equations

Laporta's algorithm (2000)

$$\sum_{i=1} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta)=0$$

- Very large scale of linear equations (can be billions of)
- Coupled, it is hard to solve
- Hard to do analytic Gaussian elimination for many variables $D, \vec{s}, \eta$
- Too slow if solving it numerically for each phase space point

➢ Cutting-edge problems

- Hundreds GB RAM
- Months of runtime using super computer

Unitarity Cuts

➢ Integrand-level reduction

$$\text{Integrand} = \sum c_i \times I_i$$

Physical singularities $\Rightarrow$ **Coefficients**

$$\mathcal{M}^{(1)}(2\to 2) = \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \left(\frac{\Delta_4(\ell)}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_4} + \sum_{i_1 i_2 i_3} \frac{\Delta_{3,i_1 i_2 i_3}(\ell)}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} + \sum_{i_1 i_2} \frac{\Delta_{2,i_1 i_2}(\ell)}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}} + \text{tadpoles} \right)$$

$$D_1 = D_2 = D_3 = D_4 = 0 \quad \Rightarrow \quad \Delta_4$$

$$D_{i_1} = D_{i_2} = D_{i_3} = 0 \quad \Rightarrow \quad \Delta_{3,i_1 i_2 i_3} \qquad \ldots$$

Needs further IBP reduction at multi-loop level!

Unitarity Cuts

➢ History

- **1994 Z. Bern, L. Dixon, D. Dunbar, D. Kosower**
  "One-loop n-point gauge theory amplitudes, unitarity and collinear limits"
- **2005 R. Britto, F. Cachazo, B. Feng**
  "Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills"
- **2007 G. Ossola, C. Papadopoulos, R. Pittan**
  "Reducing full one-loop amplitudes to scalar integrals at the integrand level"
- **2008 G. Ossola, C. Papadopoulos, R. Pittan ->** CutTools
  "CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes"
- **2011 P. Mastrolia, G. Ossola**
  "On the integrand-reduction method for two-loop scattering amplitudes"
- **2012 Y. Zhang**
  "Integrand-level reduction of loop amplitudes by computational algebraic geometry methods"
- **2017 J. Bosma, M. Sogaard, Y. Zhang**
  "Maximal cuts in arbitrary dimension"

Sector Decomposition

➢ Feynman parametric representation

$$I(D;\{\nu_\alpha\}) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2)^{\nu_\alpha}}$$

where $q_\alpha = c_\alpha^i \ell_i + d_\alpha^i p_i$

$$I(D;\{\nu_\alpha\}) = (-1)^\nu \frac{\Gamma\left(\nu - LD/2\right)}{\prod_k \Gamma(\nu_k)} \int \prod_\alpha (x_\alpha^{\nu_\alpha - 1} \mathrm{d}x_\alpha) \times \delta\left(x - 1\right) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{\mathcal{F}^{\nu - LD/2}}$$

$$\mathcal{U}(\vec{x}) = \sum_{T\in T_1} \prod_{i\notin T_1} x_i$$

$\mathcal{U} \sim x^L$ — **Spanning 1-tree, sub UV divergences**

$$\mathcal{F}_0(\vec{x}) = -\sum_{T\in T_2} s_T \prod_{i\notin T_2} x_i$$

$\mathcal{F} \sim x^{L+1}$ — **Spanning 2-tree, IR divergences**

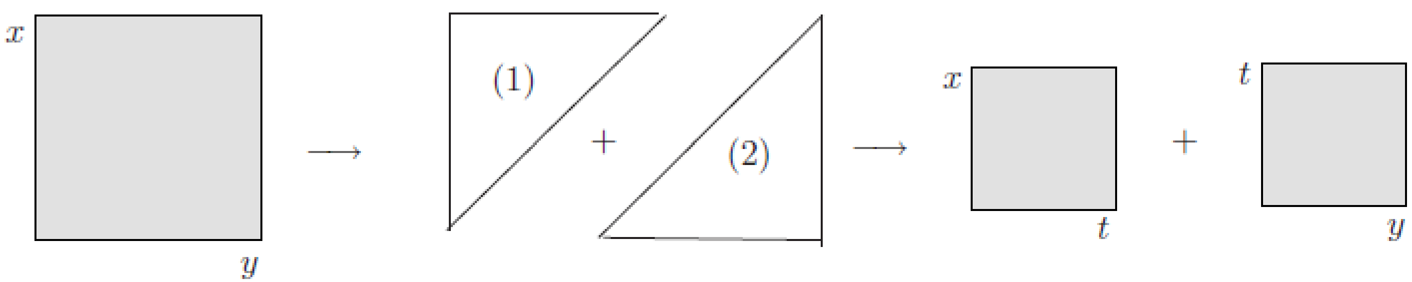
$$\mathcal{F}(\vec{x}) = \mathcal{F}_0(\vec{x}) + \mathcal{U}(\vec{x}) \sum_{\alpha=1}^{N} x_\alpha m_\alpha^2$$

See e.g. Heinrich (2008)

Sector Decomposition

➢ Sector decomposition: basic example

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$



$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\, t\Big)^{-1}$$

$$+ \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\, t\Big)^{-1}$$

Sector Decomposition

➢ Apply to Calculation of Feynman Integrals

- Generate primary sectors

Binoth, Heinrich (2000), …

- Generate subsectors iteratively
- Take epsilon expansion

$$I = (-1)^\nu \Gamma(\nu - LD/2) \sum_{i=1}^{N} \sum_{j=1}^{\Lambda(i)} I_{ij}, \quad I_{ij} = \sum_{k=-2L}^{r} C_{ij,k}\epsilon^k + \mathcal{O}(\epsilon^{r+1})$$

- Evaluate the finite integrals numerically

$$C_{ij,k} \quad \xrightarrow{\text{M-C}} \quad \text{number}$$

Sector Decomposition

➢ History

- **1966 K. Hepp (BPHZ)**

  "Proof of the Bogoliubov-Parasiuk Theorem on Renormalization"

- **2000 T. Binoth, G. Heinrich**

  "An automatized algorithm to compute infrared divergent multi-loop integrals"

- **2008 A. Smirnov, M.N. Tentyukov, et.al ->** FIESTA

  "Feynman Integral Evaluation by a Sector decomposiTion Approach (FIESTA)"

- **2010 J. Carter, G. Heinrich, et.al ->** SecDec

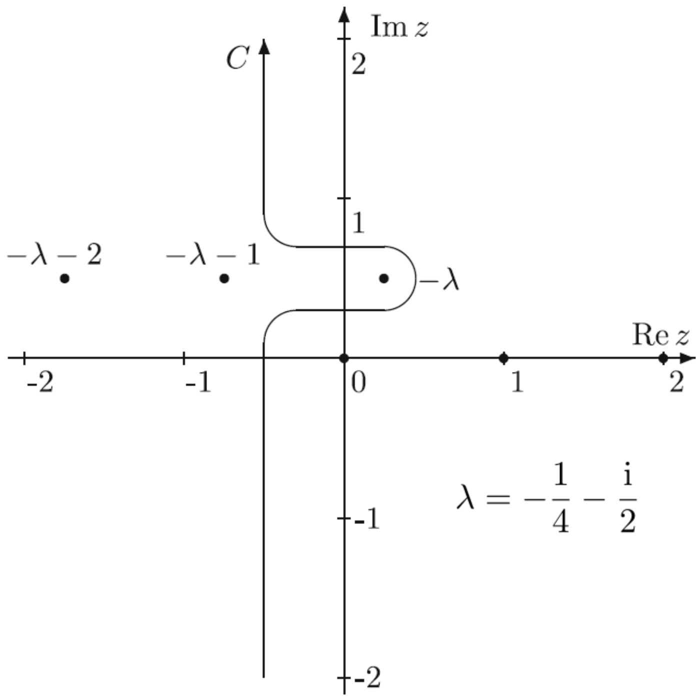
  "SecDec: A general program for sector decomposition"

- **2017 S. Borowka, G. Heinrich, et.al ->** pySecDec

  "pySecDec: a toolbox for the numerical evaluation of multi-scale integrals"

➢ Basic Relation

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\Gamma(\lambda+z)\Gamma(-z)\frac{Y^z}{X^{\lambda+z}}$$

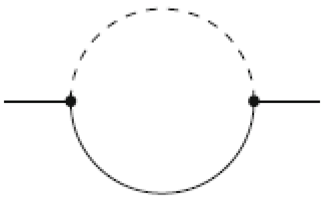


Rules:
Poles of $\Gamma(\cdots + z)$ are to the left of the contour.
Poles of $\Gamma(\cdots - z)$ are to the right of the contour.

Mellin-Barnes Representation

- Apply to massive propagator

$$\frac{1}{(\ell^2 - m^2)^\lambda} = \frac{1}{(\ell^2)^\lambda} \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \Gamma(\lambda + z) \Gamma(-z) \left(-\frac{m^2}{\ell^2} \right)^z$$



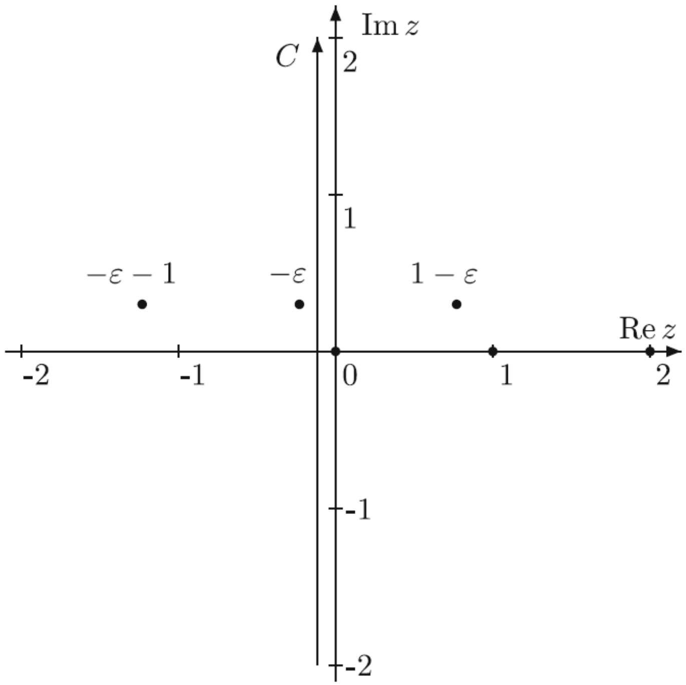
$$= \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - m^2)(\ell + p)^2}$$

$$\longrightarrow \int \mathrm{d}z \frac{\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z)}{\Gamma(2 - 2\epsilon - z)} \left(-\frac{m^2}{p^2} \right)^z$$

The contour is pinched.

There is a UV divergence. We need to resolve the singularity.

Mellin-Barnes Representation



Strategy A: MBresolve.m A. Smirnov, V. Smirnov (2009)

Deform the integration contours.

Strategy B: MB.m M. Czakon (2005)

Fix the integration contours and tends ϵ to 0.

➢ Practical procedure

- Obtain MB representation
- Resolve epsilon singularities
- Perform epsilon expansion
- Evaluate the finite integrals numerically

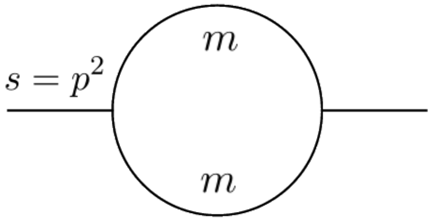
Mellin-Barnes Representation

➢ History

- **1975 N. Usyukina**
  "On a representation for the three-point function"
- **1999 V. Smirnov**
  "Analytical result for dimensionally regularized massless on-shell double box"
- **2005 M. Czakon->** MB.m
  "Automatized analytic continuation of Mellin-Barnes integrals"
- **2007 J. Gluza, K. Kajda, T. Riemann ->** AMBRE.m
  "AMBRE – a Mathematica package for the construction of Mellin-Barnes representations for Feynman integals"
- **2009 A. Smirnov, V. Smirnov, et.al ->** MBresolve.m
  "On the resolution of singularities of multiple Mellin-Barnes integrals"
- **2014 J. Blumlein, I. Dubovyk, et.al**
  "Non-planar Feynman integals, Mellin-Barnes representations, multiple sums"
- **2015 M. Ochman, T. Riemann ->** MBsums.m
  "Mbsums – a Mathematica package for the representation of Mellin-Barnes integrals by multiple sums"

Differential Equation Method

➢ Differential Equation + Boundary Condition



$$I(D;\{1,1\}) = \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - m^2)[(\ell+p)^2 - m^2]}$$

$$\begin{aligned}
\frac{\partial}{\partial m^2} I(D;\{1,1\}) &= I(D;\{2,1\}) + I(D;\{1,2\}) \\
&\overset{\text{IBP}}{=} \frac{2(D-3)}{4m^2 - s} I(D;\{1,1\}) - \frac{D-2}{m^2(4m^2 - s)} I(D;\{1,0\})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial m^2} I(D;\{1,0\}) &= I(D;\{2,0\}) \\
&\overset{\text{IBP}}{=} \frac{D-2}{2m^2} I(D;\{1,0\})
\end{aligned}$$

$$I(D;\{1,1\})|_{m^2=0} = \Gamma(2 - D/2)(-s)^{D/2-2} \frac{\Gamma(D/2-1)^2}{\Gamma(D-2)}, \quad I(D;\{1,0\})|_{m^2=0} = \cdots$$

Differential Equation Method

➢ Step1: Set up the differential equation

- Differentiate w.r.t. invariants, such as m^2, p^2

Kotikov, 1991

- IBP relations $$\frac{\partial}{\partial x}\vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)$$

➢ Step2: Calculate boundary condition

- Calculate integrals at special value of m^2, p^2
- General method?

➢ Step3: Solve the differential equation

Henn 2013

- Canonical form (special cases) $$\partial_x\vec{I}(x;\epsilon) = \epsilon A(x)\vec{I}(x;\epsilon)$$
- Numerical

Differential Equation Method

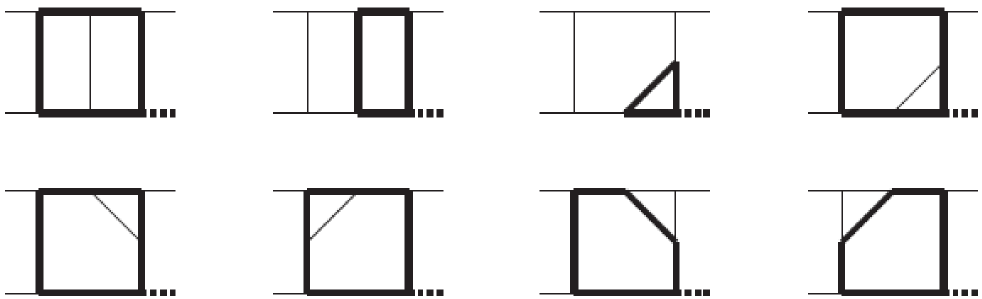
➢ History

- **1991 A. Kotikov**
  "Differential equations method: the calculation of N point Feynman diagrams"
- **1991 A. Kotikov**
  "Differential equations method: new technique for massive Feynman diagrams calculation"
- **1997 E. Remiddi**
  "Differential equations for Feynman graph ampltides"
- **2000 T. Gehrmann, E. Remiddi**
  "Differential equations for two-loop four-point functions"
- **2013 J. Henn ->** Canonical form
  "Multiloop integrals in dimensional regularization made simple"
- **2014 R. Lee**
  "Reducing differential equations for multiloop master integrals"
- **2017 L. Adams, E. Chaubey, S. Weinzierl**
  "Simplifying differential equations for multiscale Feynman integrals beyond multiple polylogarithms"

Difficulty of MIs calculation

- **Analytical: Higgs → 3 partons (Euclidean Region)**

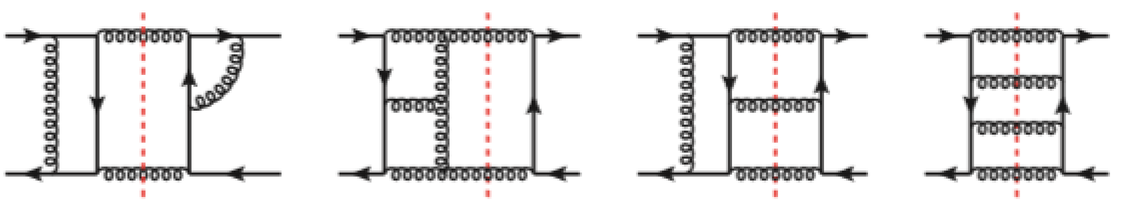
R. Bonciani, et.al 2016



200MB, 10 min

- **Numerical: Quarkonium decay at NNLO**

Feng, Jia, Sang, 1707.05758



NNLO (Virtual Squared) NNLO (Double Virtual) NNLO (Virtual – Real) NNLO (Double Real)

10^5 CPU core-hour

Recent developments

➢ Improvements for IBP reduction

- Finite field method Manteuffel, Schabinger, 1406.4513
- Direct solution Kosower, 1804.00131
- Syzygies method Böhm, Georgoudis, Larsen, Schönemann, Zhang, 1805.01873
- Obtain one coefficient at each step Chawdhry, Lim, Mitov, 1805.09182
- Expansion of small parameters Xu, Yang, 1810.12002; Mishima, 1812.04373
- Intersection Numbers Frellesvig, et. al., 1901.11510

➢ Improvements for evaluating scalar integrals

- Quasi-Monte Carlo method Li, Wang, Yan, Zhao, 1508.02512
- Finite basis Manteuffel, Panzer, Schabinger, 1510.06758
- Uniform-transcendental basis Boels, Huber, Yang, 1705.03444
- Loop-tree duality Capatti, Hirschi, Kermanschah, Ruijl, 1906.06138

State-of-the-art computation

- 2→2 process with massive particles at two-loop order: almost done $g+g\to t+\bar{t}$, $g+g\to H+H(g)$
- Very time-consuming
  - Two-loop $g+g\to H+H\ (g)$: complete IBP reduction cannot be achieved within tolerable time
    Borowka et. al., 1604.06447
    Jones, Kerner, Luisoni, 1802.00349
  - Two-loop decay $Q+\bar{Q}\to g+g$, MIs cost $O(10^5)$ CPU core-hour
    Feng, Jia, Sang, 1707.05758
- Current frontier: 2→3 processes at two loop

  5-gluon scattering may be feasible; hard for massive particles

New ideas are badly needed

MY philosophy

- Reducing/evaluating FIs analytically may not be possible for sufficiently complicated problems

- A general solution for FIs calculation, if exists, should be a numerical method

  Only numerical numbers are needed to compare with experimental data

Evaluation of FIs

- **Sufficient conditions for a good solution:**
    1. Systematic: can be applied to any problem
    2. Efficient: the amount of computation is linearly dependent on the number of FIs and the number of effective digits, and it is insensitive to the number of mass scales involved
    3. "Analytical": knows all singularities, and can calculate coefficients of asymptotic expansion at any given singular point

This talk: A method may satisfy these conditions

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Modified FIs

- **Modify Feynman loop integral by keeping finite η**

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}} \qquad \mathcal{D}_\alpha \equiv q_\alpha^2 - m_\alpha^2$$

- Take it as an analytical function of η
- Physical result is defined by

$$\mathcal{M}(D,\vec{s},0) \equiv \lim_{\eta\to 0^+} \mathcal{M}(D,\vec{s},\eta)$$

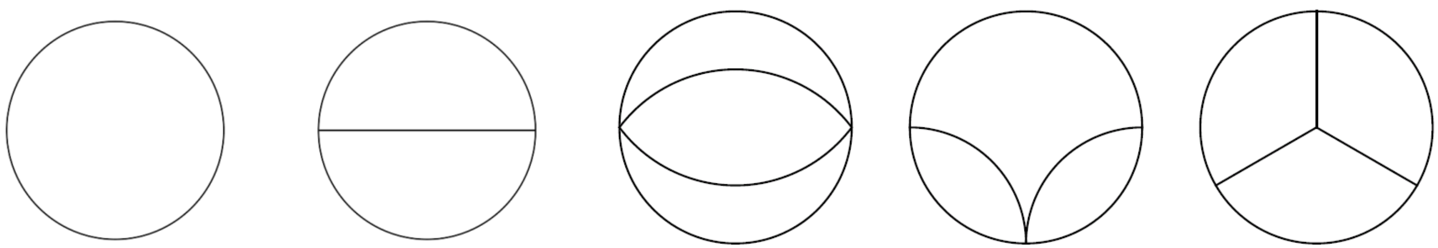
Expansion at infinity

➢ Expansion of propagators around $\eta = \infty$

$$\frac{1}{[(\ell+p)^2 - m^2 + \mathrm{i}\eta]^\nu} = \frac{1}{(\ell^2 + \mathrm{i}\eta)^\nu} \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \left(\frac{-2\ell \cdot p - p^2 + m^2}{\ell^2 + \mathrm{i}\eta} \right)^n$$

- Only one region in the method of region: $l^\mu \sim |\eta|^{1/2}$
- No external momenta in denominator, vacuum integrals
- Simple enough to deal with

➢ Vacuum MIs with equal internal masses



- Analytical results are known up to 3-loop
- Numerical results are known up to 5-loop

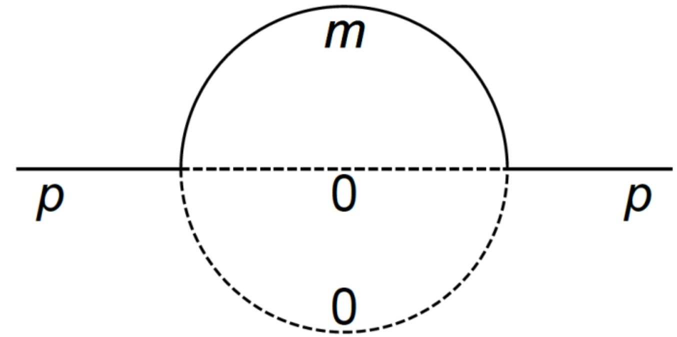
Davydychev,Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136

Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

Example

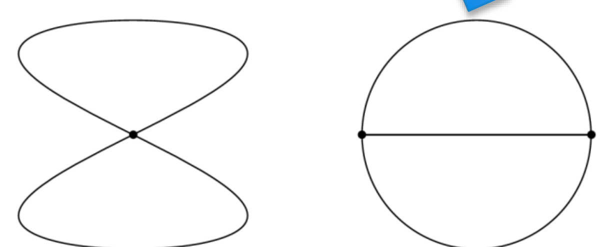
➢ Sunrise integral

$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$



$$\mathcal{D}_1 =(\ell_1+p)^2-m^2,\ \mathcal{D}_2=\ell_2^2,\ \mathcal{D}_3=(\ell_1+\ell_2)^2$$

$$I_{111} = \eta^{D-3}\left\{\left[1-\frac{D-3}{3}\frac{m^2}{\mathrm{i}\eta}+\frac{(D+4)(D-3)}{9D}\frac{p^2}{\mathrm{i}\eta}\right]I_{2,2}^{\mathrm{bub}} - \mathrm{i}\left[\frac{(D-2)^2}{3D}\frac{p^2}{\mathrm{i}\eta}\right]I_{2,1}^{\mathrm{bub}}+\mathcal{O}(\eta^{-2})\right\}$$



A new representation

➢ Asymptotic expansion

$$\mathcal{M}(D,\vec{s},\eta) = \eta^{LD/2-\sum_\alpha \nu_\alpha} \sum_{\mu_0=0}^{\infty} \eta^{-\mu_0} \mathcal{M}_{\mu_0}^{\text{bub}}(D,\vec{s})$$

$$\mathcal{M}_{\mu_0}^{\text{bub}}(D,\vec{s}) = \sum_{k=1}^{B_L} I_{L,k}^{\text{bub}}(D) \sum_{\vec{\mu}\in\Omega_{\mu_0}^r} C_k^{\mu_0\ldots\mu_r}(D) s_1^{\mu_1}\cdots s_r^{\mu_r}$$

- $I_{L,k}^{\text{bub}}(D)$: k-th master vacuum integral at L-loop order
- $C_k^{\mu_0\ldots\mu_r}(D)$: rational functions of D

➢ A new representation

- Uniqueness theorem of analytical functions: physical FI is uniquely determined by this asymptotic series via analytical continuation
- A new series representation of FIs
- All FIs (therefore scattering amplitudes) are determined by equal-mass vacuum integrals

Remarks: series representations

- **Quantities present in all Feynman integrals:**
  - Space-time dimension $D \to 4$ and Feynman prescription $\eta \to 0^+$
- **Baikov's series representation:**
  - Asymptotic expansion of FIs at $D \to \infty$ — Baikov 0507053
  - According our test: calculation of the series is expensive
  - Try hard to see if it is possible to improve the speed
- **Our series representation:**
  - Asymptotic expansion of FIs at $\eta \to \infty$
  - Calculation is cheaper, due to all coefficients are polynomials

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What is reduction

➢ Reduction

- Find relations between loop integrals
- Use them to express all loop integrals as linear combinations of MIs

➢ Relations among $G \equiv \{M_1, M_2, \ldots, M_n\}$

$$\sum_{i=1}^{n} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$

- $Q_i(D, \vec{s}, \eta)$: homogeneous polynomials of $\vec{s}, \eta$ of degree d_i

➢ Constraints from mass dimension

$$2d_1 + \mathrm{Dim}(\mathcal{M}_1) = \cdots = 2d_n + \mathrm{Dim}(\mathcal{M}_n)$$

- Only 1 degree of freedom in $\{d_i\}$, chosen as $d_{\max} \equiv \mathrm{Max}\,\{d_i\}$

Find relations

$$\sum_{i=1}^{n} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta) = 0$$

- **Decomposition of** $Q_i(D,\vec{s},\eta)$

$$Q_i(D,\vec{s},\eta) = \sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} Q_i^{\lambda_0\ldots\lambda_r}(D)\,\eta^{\lambda_0} s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

$$\Rightarrow \sum_{k,\rho_0,\vec{\rho}} f_k^{\rho_0\ldots\rho_r}\,\mathcal{I}_{L,k}^{\mathrm{bub}}(D)\,\eta^{\rho_0} s_1^{\rho_1}\cdots s_r^{\rho_r} = 0$$

- **Linear equations:** $f_k^{\rho_0\ldots\rho_r} = 0$
  - **With enough constraints** $\Rightarrow Q_i^{\lambda_0\ldots\lambda_r}(D)$
  - **With finite field technique, only integers in a finite field are involved, equations can be efficiently solved**

- **Relations among** $G \equiv \{M_1, M_2, \ldots, M_n\}$ **with fixed** $d_{\max}$ **are fully determined**

Reduction

- **With $G = G_1 \cup G_2$, satisfy**
  - G_1 is more complicated than G_2
  - G_1 can be reduced to G_2
- **Algorithm** ***Search for simplest relations***
  1. Set $d_{\max} = 0$
  2. Find out all reduction relations among G with fixed $d_{\max}$
  3. If obtained relations are enough to determine G_1 by G_2, stop;
     else, $d_{\max} = d_{\max} + 1$ and go to step 2
- **Conditions for G_1 and G_2**
  1. Relations among G_1 and G_2 are not too complicated: easy to find
  2. $\#G_1$ is not too large: numerically diagonalize relations easily

Reduction scheme with only dots

- **FIs:** $\vec{\nu} = (\nu_1, \dots, \nu_N), \nu_i \geq 0$
  - $\mathbf{0}^{\pm} \equiv \mathbf{Identity}$, $\mathbf{m}^{\pm} \equiv (\mathbf{m}-\mathbf{1})^{\pm}\mathbf{1}^{\pm}$
  - $\mathbf{1}^{+}(5,1,0,3) = \{(6,1,0,3), (5,2,0,3), (5,1,0,4)\}$
  - $\mathbf{1}^{-}(5,1,0,3) = \{(4,1,0,3), (5,0,0,3), (5,1,0,2)\}$
- **1-loop:** $G_1 = \mathbf{1}^{+}\vec{\nu}, G_2 = \mathbf{1}^{-}\mathbf{1}^{+}\vec{\nu}$ Duplancic and Nizic, 0303184
- **Multi-loop:**

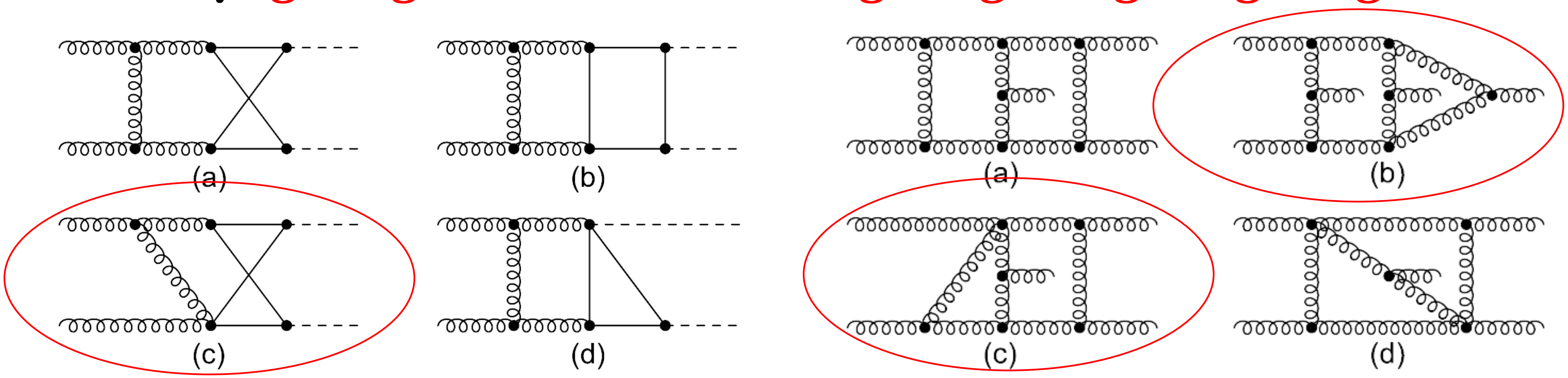
$$G_1 = \mathbf{m}^{+}\vec{\nu}, G_2 = \{\mathbf{1}^{-}\mathbf{m}^{+}, \mathbf{1}^{-}(\mathbf{m}-\mathbf{1})^{+}, \dots, \mathbf{1}^{-}\mathbf{1}^{+}\}\vec{\nu}$$

  - $m = 2,3$ in examples, $\#G_1$ is not too large, include dozens of integrals
  - Relations among G_1 and G_2 are not too complicated, see examples

A step-by-step reduction is realized!

Examples

➢ 2-loop $g+g\to H+H$ and $g+g\to g+g+g$



$g+g\to H+H$				$g+g\to g+g+g$			
Sector	Type	$d_{\max}$	$\mathbf{m}^+$	Sector	Type	$d_{\max}$	$\mathbf{m}^+$
1(a)	7-NP	1	$\mathbf{3}^+$	2(a)	8-NP	1	$\mathbf{3}^+$
1(b)	7-P	1	$\mathbf{3}^+$	2(b)	8-NP	3	$\mathbf{3}^+$
1(c)	6-NP	5	$\mathbf{3}^+$	2(c)	7-NP	4	$\mathbf{3}^+$
1(d)	6-P	4	$\mathbf{2}^+$	2(d)	6-NP	2	$\mathbf{3}^+$

Difficulty:

- More legs > less legs
- Nonplanar > Planar
- $\mathbf{m}^+\vec{e} > \mathbf{m}^+\vec{\nu}$

- Relations can be obtained by a single-core laptop in a few hours
- Diagonalizing at each phase space point (floating number): 0.01 second
- Results checked numerically by FIRE

Reduction of numerators

- **Method similar to the reduction of denominators, work in progress**

- **Use η expansion to directly reduce amplitudes**

  Wang, Li, Basat, 1901.09390

Analytical continuation

- **Set up and solve DEs of MIs**

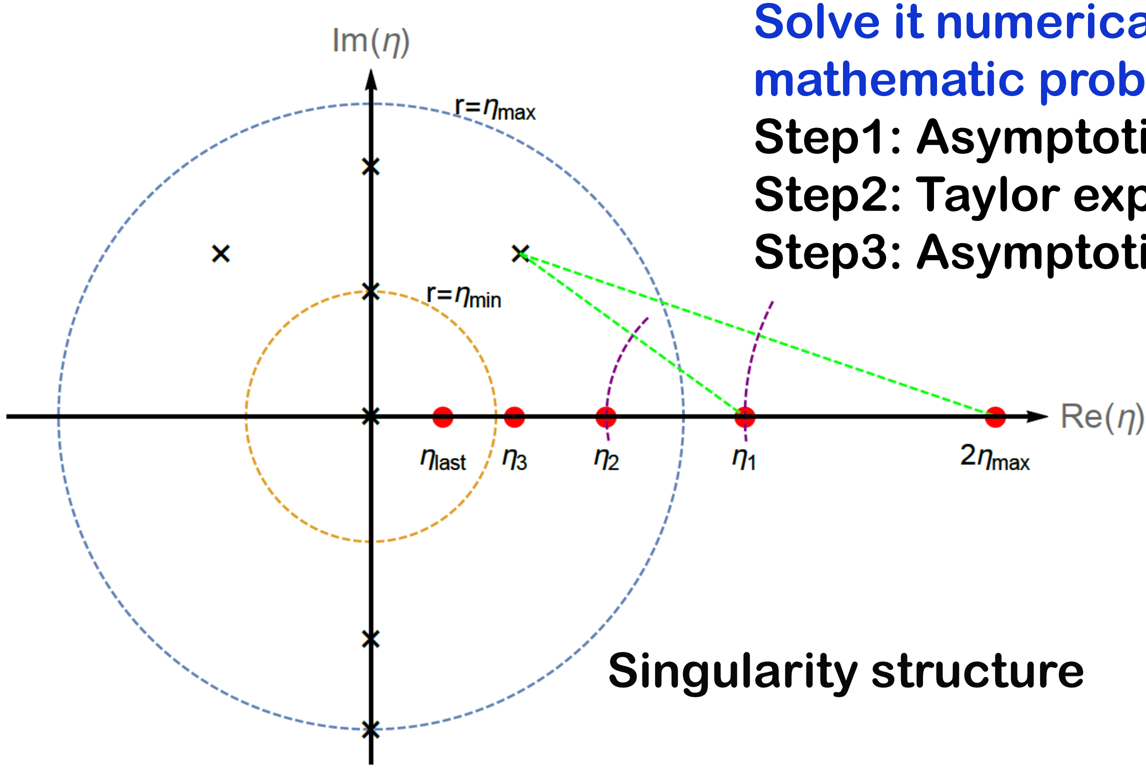
$$\frac{\partial}{\partial \eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta)$$ **with known** $\vec{I}(D;\infty)$

Solve it numerically: a well-studied mathematic problem

Step1: Asymptotic expansion at $\eta = \infty$

Step2: Taylor expansion at analytical points

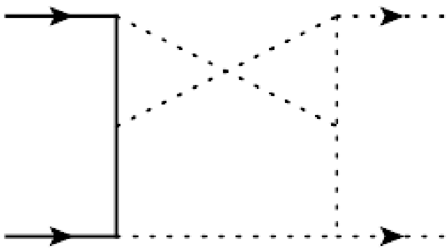
Step3: Asymptotic expansion at $\eta = 0$



Singularity structure

Example

➢ **2-loop non-planar sector for $\mathrm{Q} + \bar{\mathrm{Q}} \to g + g$**



- **168 master integrals**

Feng, Jia, Sang, 1707.05758

- **Traditional method sector decomposition: $O(10^4)$ CPU core-hour**
- **Our method: a few minutes**

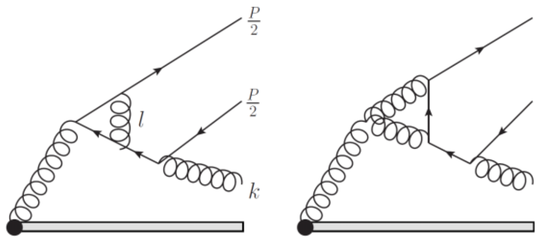
MIs can be thought as special functions, and DEs tell us how to evaluate these special functions

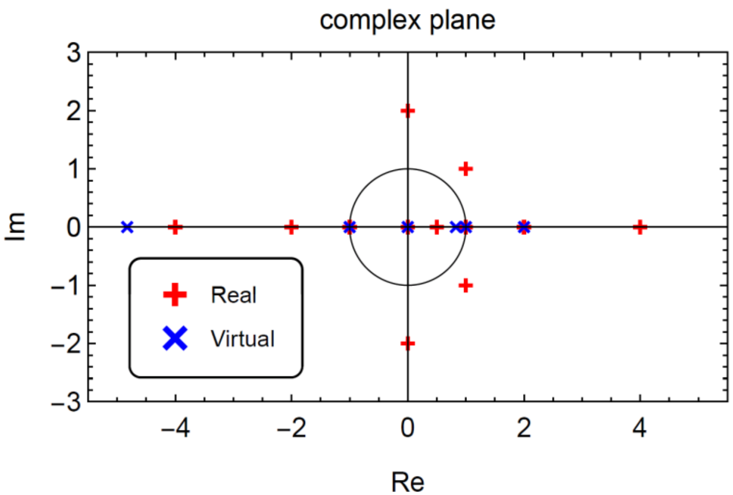
Practical use

➢ Use η expansion to determine MIs

FFs of $g \to Q\bar{Q}({}^1S_0^{[1,8]}) + X$

Zhang, Wang, Liu, Ma, Meng, Chao, 1810.07656





$$\frac{\mathrm{d}\boldsymbol{I}(\epsilon,z)}{\mathrm{d}z} = A(\epsilon,z)\boldsymbol{I}(\epsilon,z)$$

$$d_{\mathrm{NLO}}^{[1]}(z) = \frac{\alpha_s^3}{2\pi N_c m_Q^3} \times \left(d^{[1]}(z) + \ln\left(\frac{\mu_r^2}{4m_Q^2}\right) b_0\, d_{\mathrm{LO}}^{(0)}(z) + \ln\left(\frac{\mu_f^2}{4m_Q^2}\right) f(z) \right),$$

$$d_{\mathrm{NLO}}^{[8]}(z) = \frac{\alpha_s^3 (N_c^2-4)}{4\pi N_c (N_c^2-1) m_Q^3} \times \left(d^{[8]}(z) + \ln\left(\frac{\mu_r^2}{4m_Q^2}\right) b_0\, d_{\mathrm{LO}}^{(0)}(z) + \ln\left(\frac{\mu_f^2}{4m_Q^2}\right) f(z) \right)$$

$$d^{[1/8]}(z) = \begin{cases} -\frac{N_c}{2z} + \sum_{i=0}^{2}\sum_{j=0}^{\infty} \ln^i z\,(2z)^j \left(A_{ij}^f n_f + A_{ij}^{[1/8]} N_c + \frac{A_{ij}^N}{N_c} \right), & \text{for } 0 < z < \frac{1}{4} \\ \sum_{j=0}^{\infty} (2z-1)^j \left(B_j^f n_f + B_j^{[1/8]} N_c + \frac{B_j^N}{N_c} \right), & \text{for } \frac{1}{4} \le z \le \frac{3}{4} \\ \sum_{i=0}^{3}\sum_{j=0}^{\infty} \ln^i(1-z)\,(2-2z)^j \left(C_{ij}^f n_f + C_{ij}^{[1/8]} N_c + \frac{C_{ij}^N}{N_c} \right), & \text{for } \frac{3}{4} < z < 1 \end{cases}$$

of z. To obtain about 150-digit precision for any value of z, we will attach an ancillary file for the arXiv preprint in future, in which these coefficients will be calculated up to $j = 500$ with 150 digits for each coefficient.

- **Use η expansion at $z = \frac{1}{4}, \frac{3}{4}$ to obtain 200-digit precision**
- **Combine η expansion and numerical Des w.r.t. kinematic variables**

Outline

I. Introduction

II. A New Representation

III. Reduction

IV. Outlook

Summary

- A new (series) representation: Feynman integrals are fully determined by vacuum integrals
- A general strategy to do reduction
- A general strategy to evaluate MIs
- Two–loop examples: our method is correct and efficient
- Application to fragmentation function: correct and helpful

Future plan: practical application

- A package to do systematic reduction
  - Express all FIs as linear combinations of MIs
- A package to calculate MIs
  - Can be thought as a multi-loop version of “looptools”

Future plan: formal development

- **Introduce auxiliary masses to Lagrangian**

$$\widetilde{\mathcal{L}}_{\mathrm{QED}}(\lambda) = \mathcal{L}_{\mathrm{QED}} + \lambda\bar{\psi}\psi + \lambda^2 A^2$$

  - Construct "heavy particle effective theory" at $\lambda \to \infty$ (similar to but not the same as HQET)
  - Evaluate scattering amplitudes as power expansion in $\frac{1}{\lambda^2}$
  - Each term should be very compact, including only vacuum integrals

- **Recover physical results at $\lambda \to 0$ by analytical continuation**

Thank you!

Remarks: generality

- As η presents in all FIs, our method can be used for any problem

- We can find out any existing relation between FIs
  - Relations due to IBP, LI, symmetries, accidental relations, non-linear relations …

Remarks: efficiency of reduction

- Cost of setting up analytical reduction relations: linear in the number of target FIs
  - Set up one reduction relation for each FI
  - Each reduction relation can be obtained in a short time
  - The cost for each relation is insensitive to the number of scales: two-loop $gg \to t\bar{t}H$ is similar to $5g$
- Cost of numerically diagonalizing reduction relations: linear in the number of target FIs
  - Reduction relations are block-diagonalized
  - # of equations equals to # of target FIs
  - Do it at each phase space point (floating numbers)

Remarks: efficiency of evaluating MIs

- **Our strategy is to numerically solve DEs w.r.t. η and kinematic variables**
  - Increase the efficiency
  - Determine analytical structure
  - Cost is linearly dependent on the required number of effective digits

Remarks: infrared divergences

- **No IR divergence when η is finite**
  - η **plays the role as an IR regulator**

- **IR divergences come out as $\eta \to 0^+$**
  - ϵ **becomes the IR regulator after taking this limit**

Remarks: number of MIs

- **Number of MIs at finite η is larger than the number of MIs at $\eta \to 0^+$**
  - It is not a problem become the number is still small, and much smaller than the number of target Fis
  - Also these MIs can be calculated within our method
- **DEs w.r.t. η provide constraints as $\eta \to 0^+$**
  - Number of MIs at $\eta \to 0^+$ can be minimized

Remarks: effect of η

- Do reduction relations become more complicated with η?
  - No! Just the opposite!
  - The mass dimension of reductions relations becomes smaller