Two-loop Five-point Scattering Amplitudes

Yang Zhang

University of Science and Technology of China

EFT&Amplitude USTC
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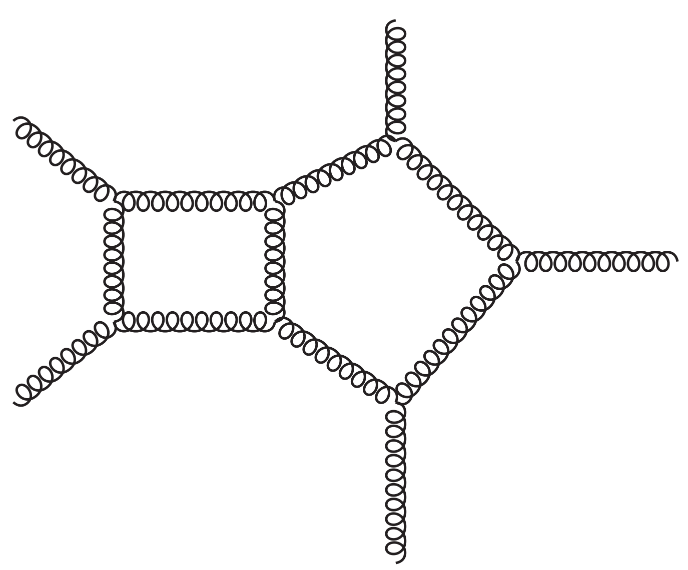
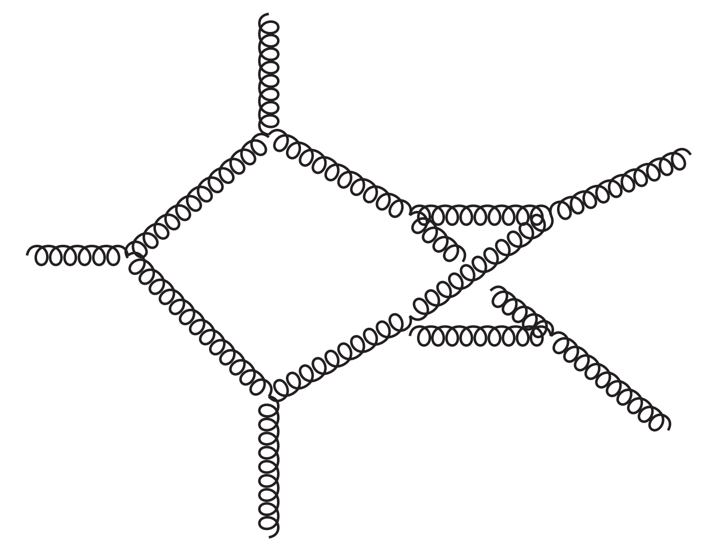
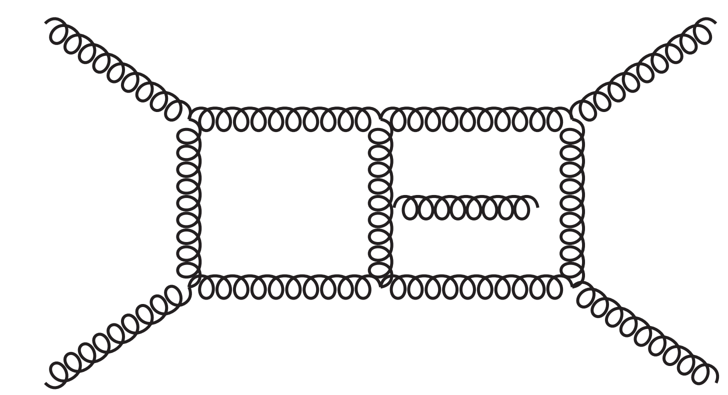
Breakthrough in two-loop 2 -> 3 scattering amplitudes







"All master integrals for three-jet production at NNLO", Phys.Rev.Lett. 123 (2019), no. 4 041603
"Analytic result for a two-loop five-particle amplitude", Phys.Rev.Lett. 122 (2019), no. 12 121602

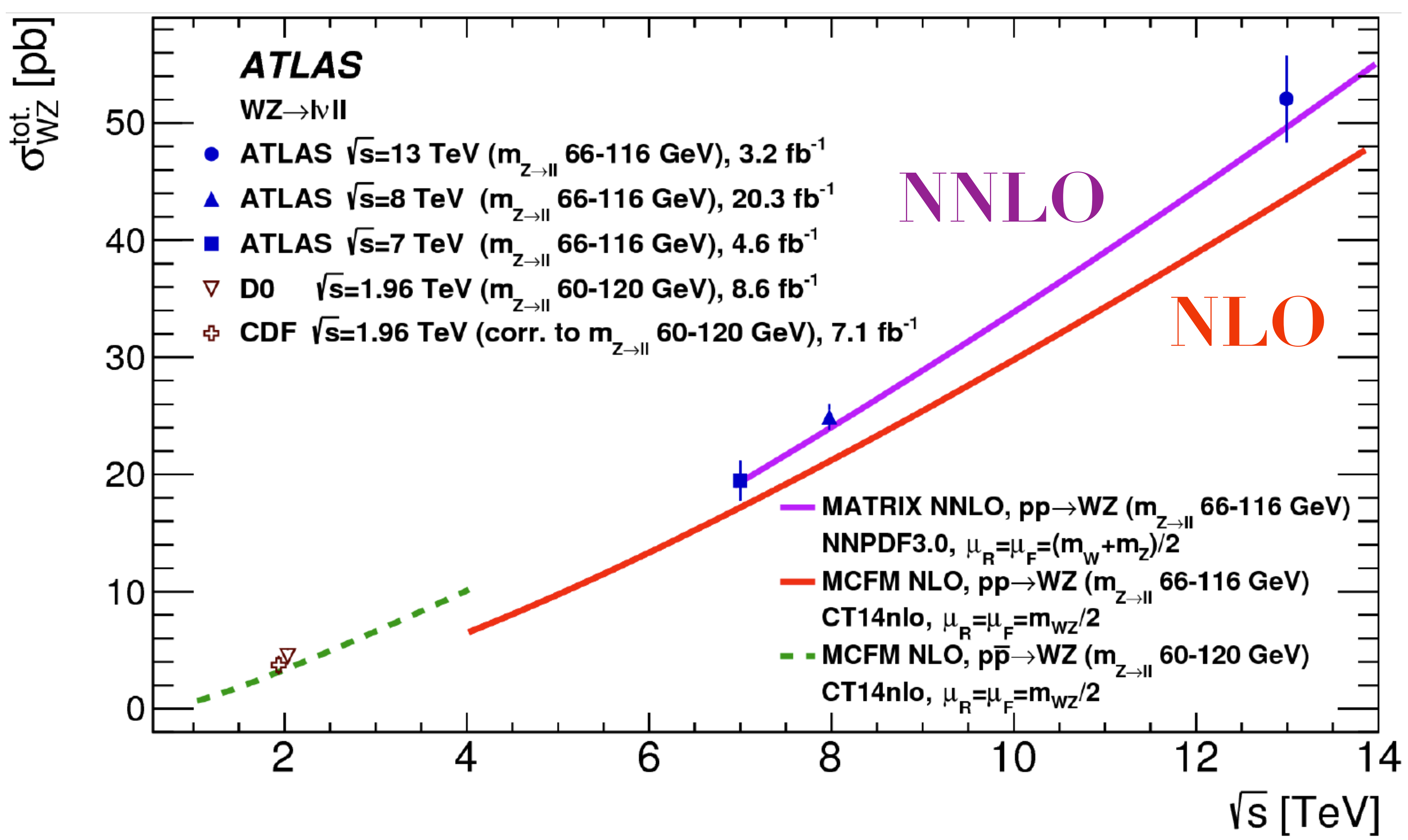
Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia

"Analytic form of the full two-loop five-gluon all-plus helicity amplitude", Phys.Rev.Lett. 123 (2019) no.7, 071601

Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, **YZ**, Zoia

Precision Physics

To interpret the high energy experimental results, to find new physics, next-to-next-to-leading-order (NNLO) cross section computation is needed.



Phys. Lett. B 762 (2016) 1

$$\alpha_s(M_z) = 0.1148 \pm 0.0014(\text{exp}) \pm 0.0018(\text{PDF}) \pm 0.0050(\text{theory})$$

Precision Physics

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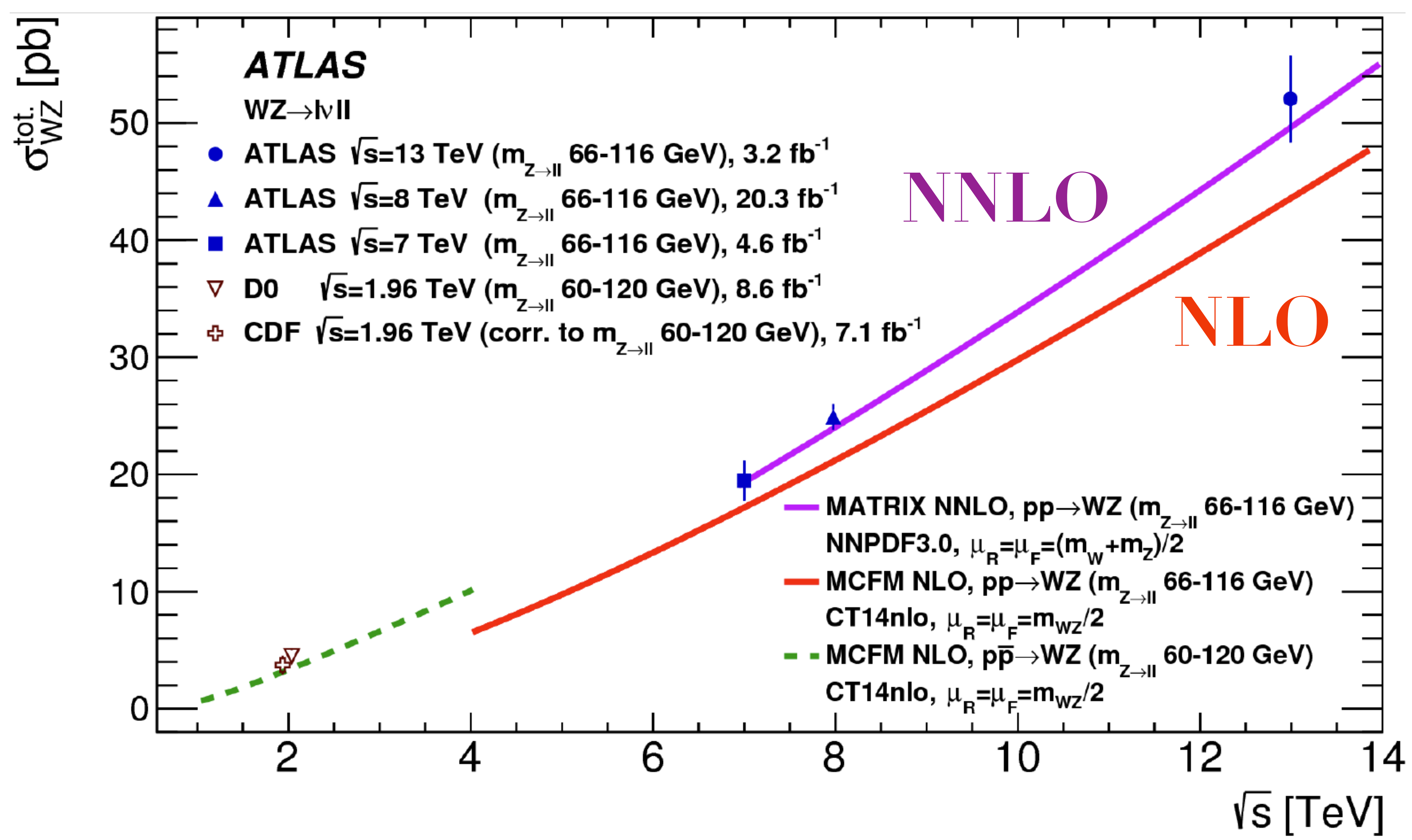


Phys. Lett. B 762 (2016) 1

$$\alpha_s(M_z) = 0.1148 \pm 0.0014(\text{exp}) \pm 0.0018(\text{PDF}) \pm 0.0050(\text{theory})$$

Goals

One bottleneck of NNLO precision physics
is the two-loop scattering amplitude.

To calculate complete two-loop five-point amplitudes
in pQCD/Standard model *analytically*

2g -> 3 jets
quark pair -> quark pair + jet
2g -> Higgs + 2 jets

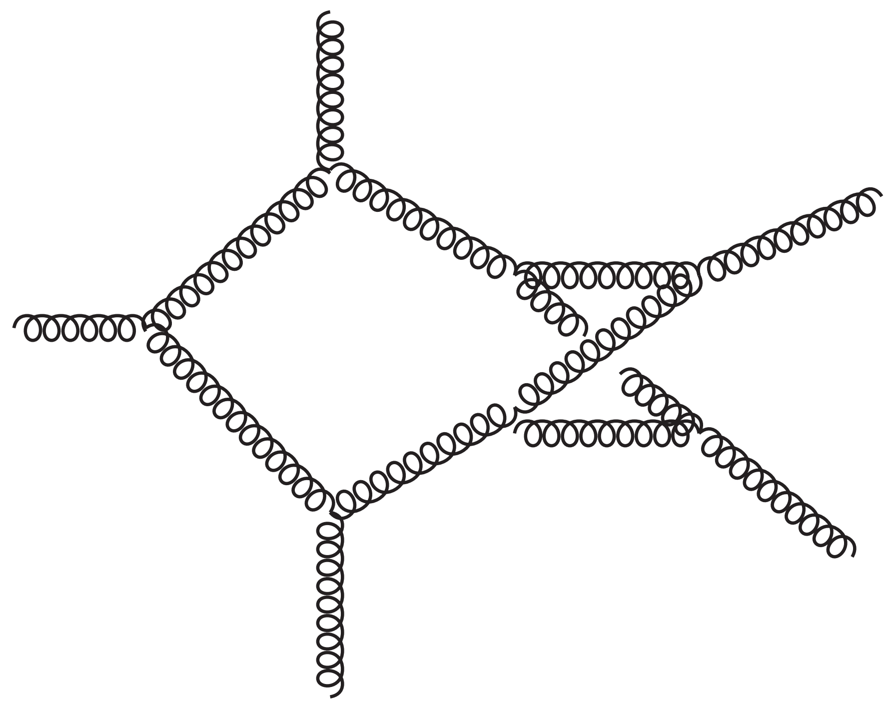
Status: 2-loop 5-point massless amplitudes (no SUSY)

"Gettysburg" for multi-loop scattering amplitude …

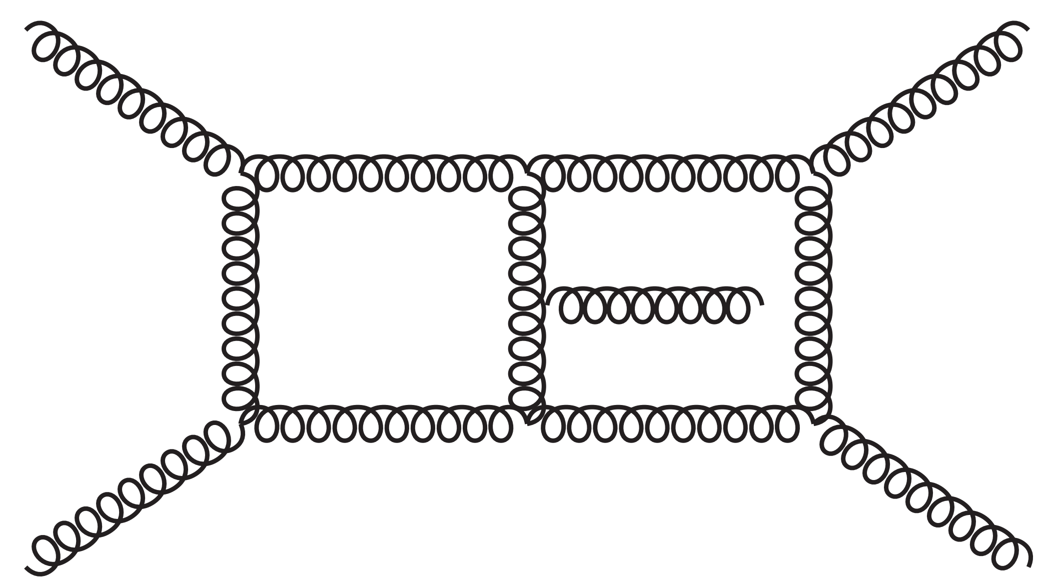
<table>
<tr><th></th><th>Numeric Integrand</th><th>Analytic integrand</th><th>Analytic amplitude</th></tr>
<tr><td>planar +++++ pure-YM</td><td colspan="2">Badger, Hjalte, YZ 2013</td><td>Gehrmann, Henn, Presti 2015</td></tr>
<tr><td>planar all-helicity pure-YM</td><td>Badger, Brønnum-Hansen, Hartanto, Peraro 2017
Abreu, Cordero, Ita, Page, Zeng 2017</td><td>Boels, Jin, Luo 2018</td><td>Abreu, Cordero, Dormans, Ita, Page 2018</td></tr>
<tr><td>planar all-helicity massless quarks</td><td>Badger, Brønnum-Hansen, Hartanto, Peraro 2018
Abreu, Cordero, Ita, Page, Sotnikov 2018</td><td colspan="2">Abreu, Cordero, Dormans, Ita, Page, Sotnikov 2019</td></tr>
<tr><td>nonplanar +++++ pure-YM</td><td colspan="2">Badger, Mogull, Ochirov, O'Connell 2015</td><td>Badger, Gehrmann, Peraro Wasser, Heinrich, Henn, Chicherin, YZ, Zoia 2019</td></tr>
</table>

2-loop 5-point planar integrals	Gehrmann, Henn, Lo Presti, 2015 (Full result)
2-loop 5-point nonplanar integrals	Abreu, Dixon, Herrmann, Page, Zeng 2018 (symbol only) Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia 2018 (Full result)

Difficulty



Two-loop nonplanar



5 Mandelstam variables

31 Symbol letters

Weight 4 functions

square root of Gram determinant

Our techniques

Baikov representation

dlog integral construction

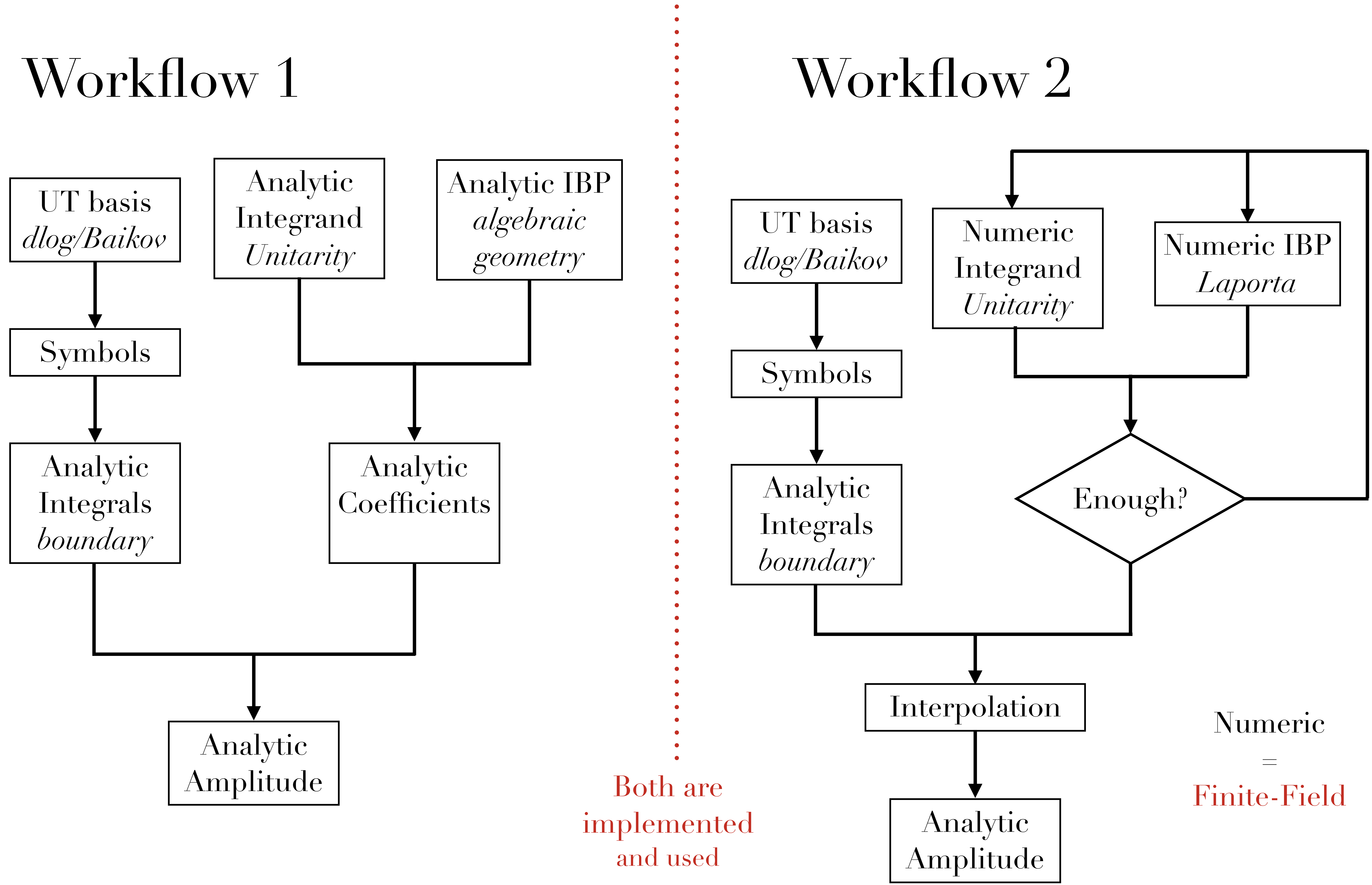
Algebraic geometry methods

Finite field reconstruction

Boundary value consistency

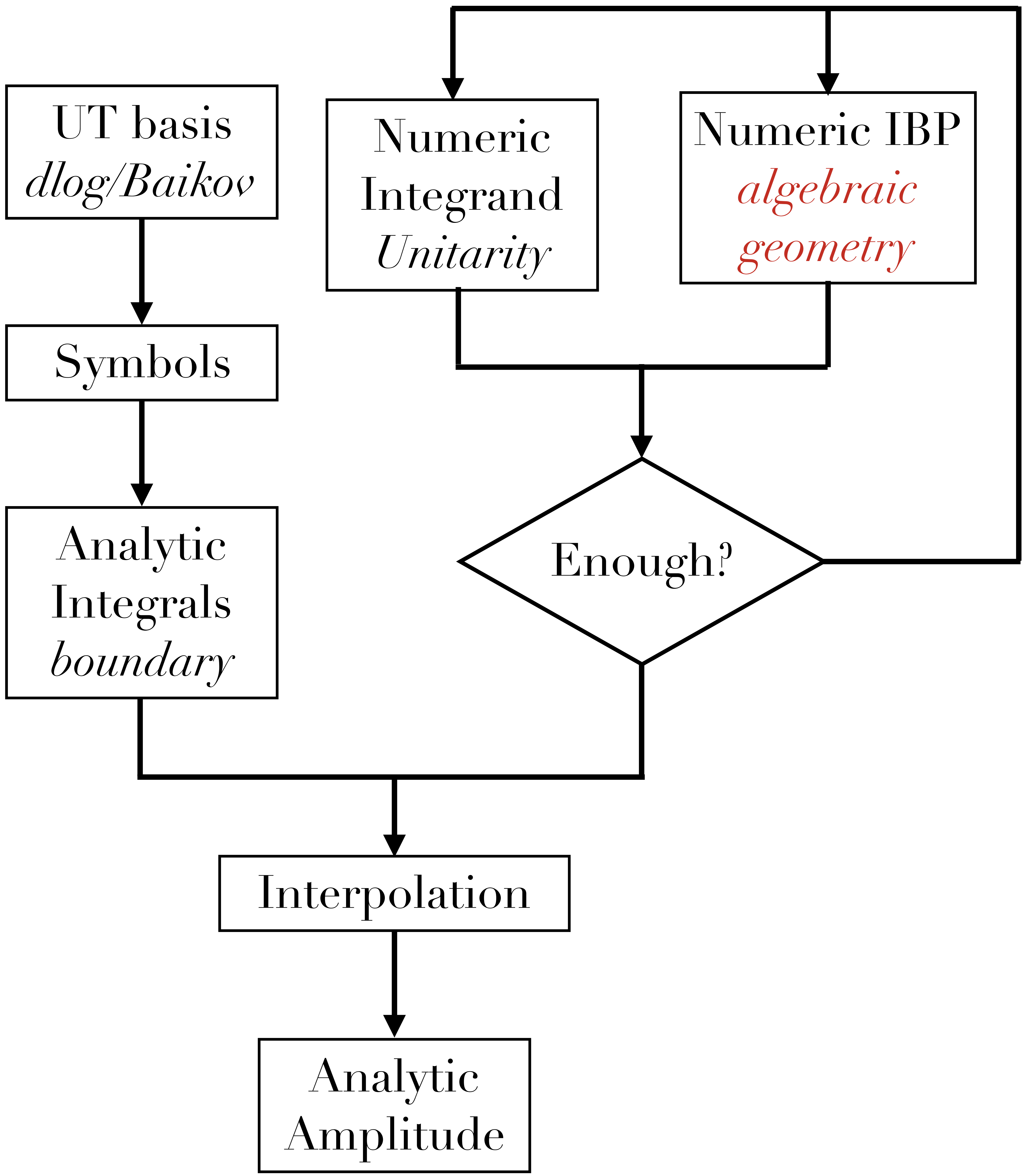
Petri nets

Symbol builder



Workflow 1
UT basis
dlog/Baikov
Analytic
Integrand
Unitarity
Analytic IBP
algebraic
geometry
Symbols
Analytic
Integrals
boundary
Analytic
Coefficients
Analytic
Amplitude
Both are
implemented
and used
Workflow 2
UT basis
dlog/Baikov
Numeric
Integrand
Unitarity
Numeric IBP
Laporta
Symbols
Enough?
Analytic
Integrals
boundary
Interpolation
Analytic
Amplitude
Numeric
=
Finite-Field

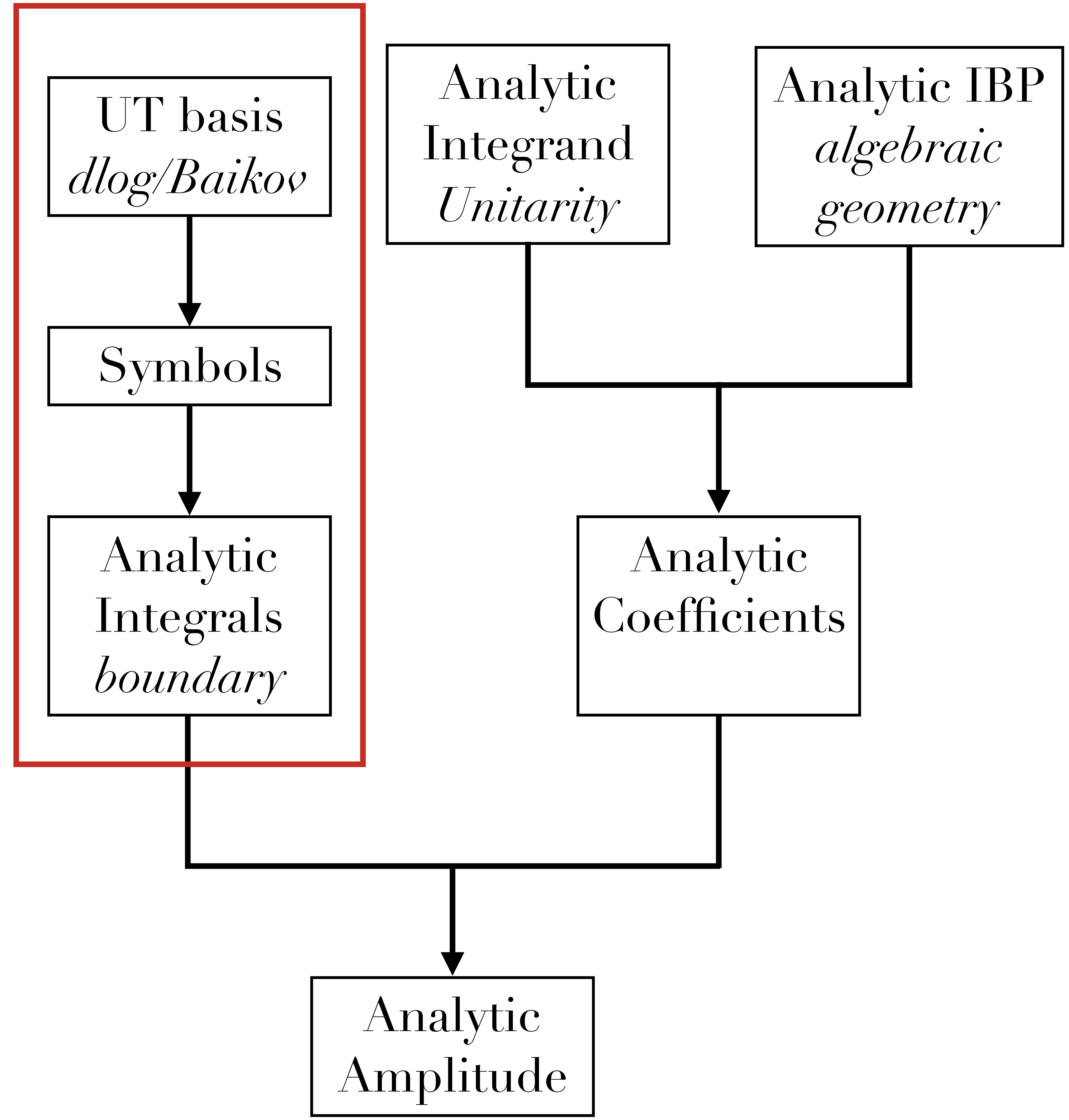
Workflow 3 (obvious)



Under test

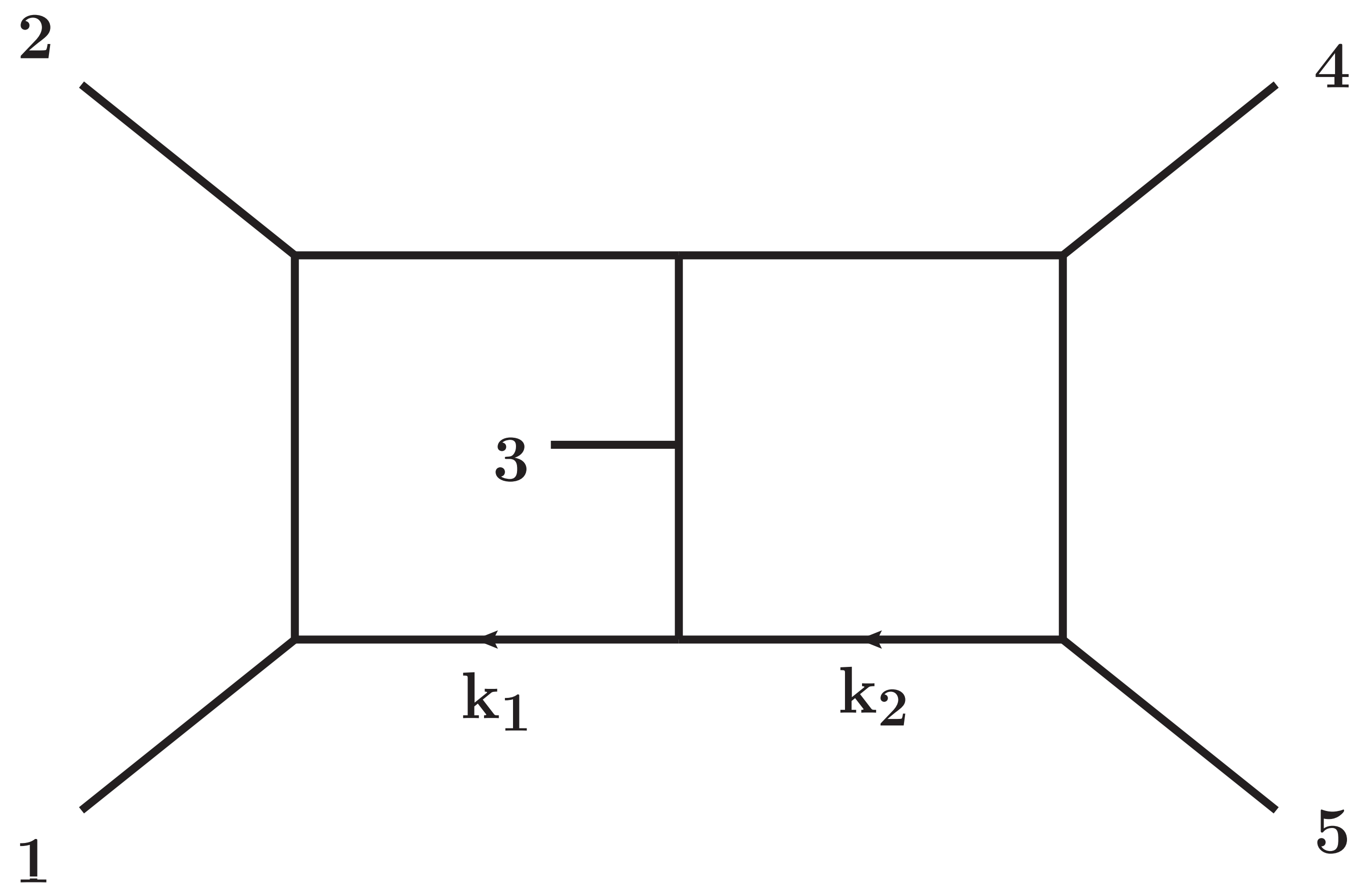
only ~ secs
for reducing
one numeric IBP !

Feynman Integral Analytic Evaluation



Differential equation with
uniform transcendental weights,
Symbols
Boundary value determination

Differential equation for traditional integral basis



108 master integrals

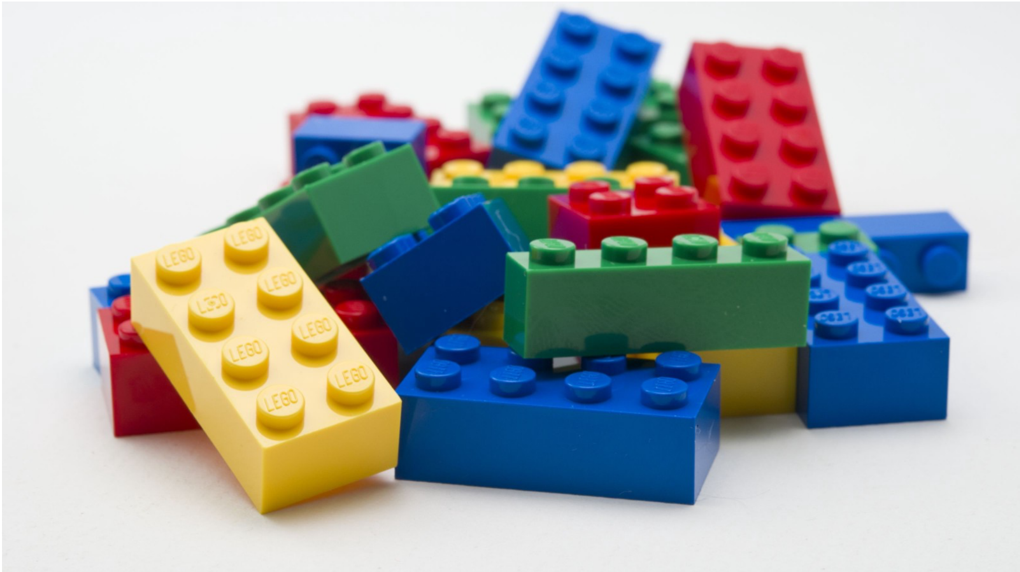
$$\frac{\partial}{\partial x_i} I = A_i I$$

5 Mandelstam variables

1.4 GB !

It took 3 months on Univ. of Zurich cluster,
to compute the five (108,108) matrices

Uniformly transcendental (UT) basis

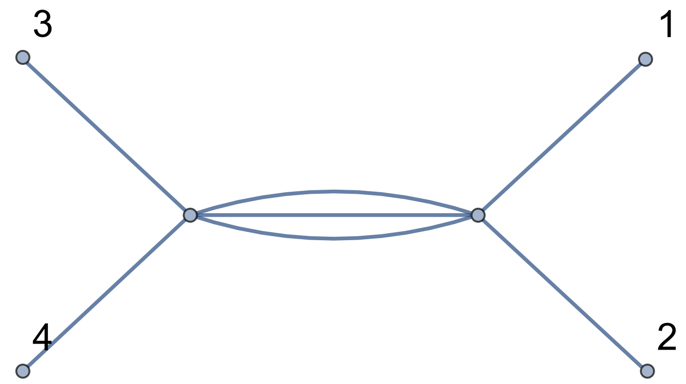


$$\mathcal{T}(\log)=1, \mathcal{T}(\pi)=1, \mathcal{T}(\zeta_n)=n, \mathcal{T}(\mathrm{Li}_n)=n, \ldots, \mathcal{T}(f_1 f_2)=\mathcal{T}(f_1)+\mathcal{T}(f_2)$$

Henn 2013

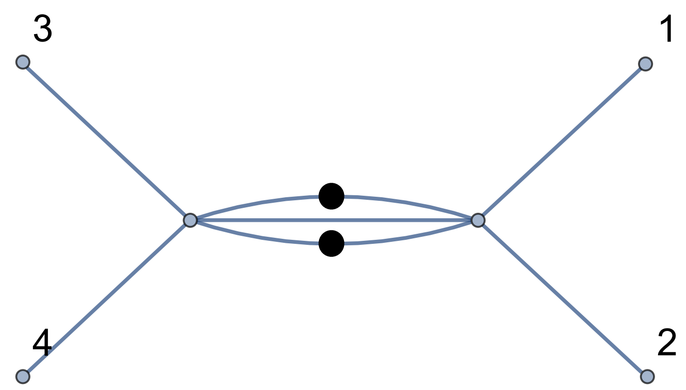
$$I = (\text{overall normalization}) \times \sum_{k=0}^{\infty} \epsilon^k f_k, \quad \mathcal{T}(f_k)=k$$

building blocks (LEGO)
for Feynman integrals



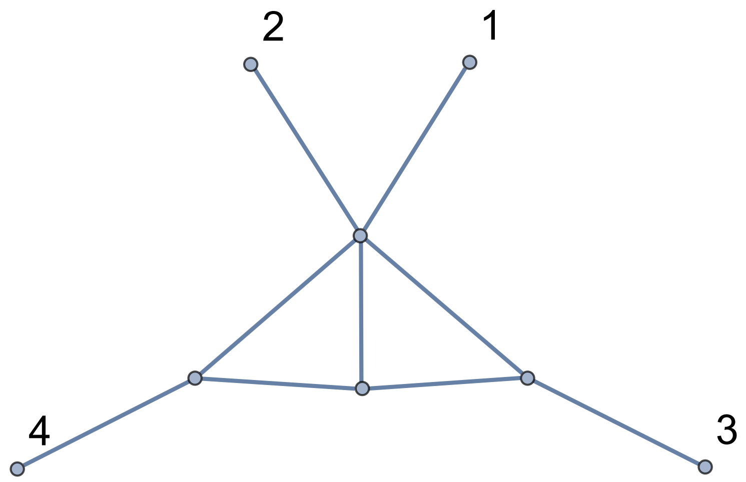
$$(s_{12})^{1-2\epsilon}\left(\frac{1}{4\epsilon}+\frac{13}{8}+\frac{1}{48}\left(345-2\pi^2\right)\epsilon+\frac{1}{96}\left(-256\zeta(3)+2595-26\pi^2\right)\epsilon^2+O\left(\epsilon^3\right)\right)$$

not UT



$$(s_{12})^{-1-2\epsilon}\left(-\frac{1}{\epsilon^2}+\frac{\pi^2}{6}+\frac{32\zeta(3)\epsilon}{3}+\frac{19\pi^4\epsilon^2}{120}+O\left(\epsilon^3\right)\right)$$

UT but not dlog

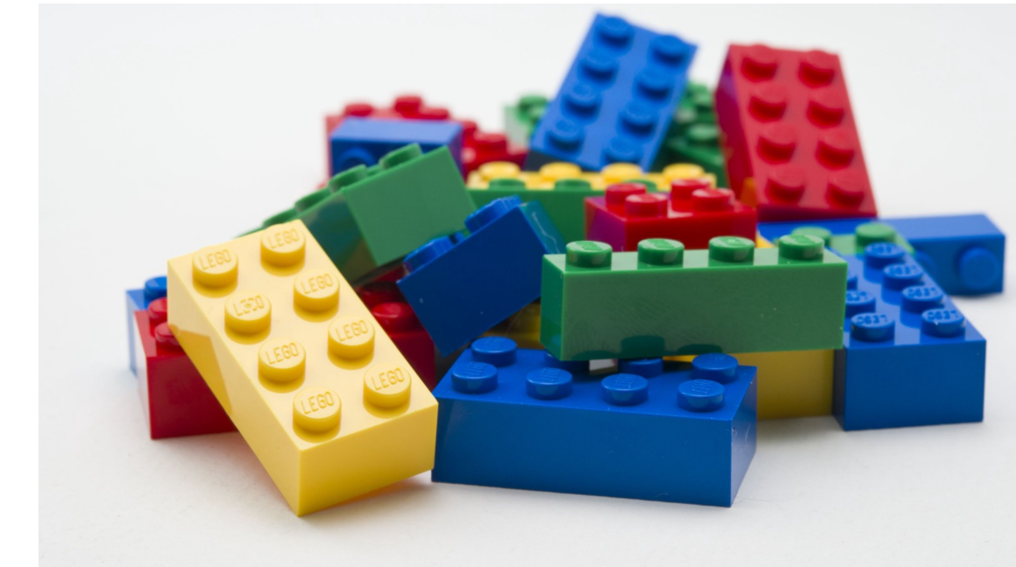


$$(s_{12})^{-1-2\epsilon}\left(-\frac{1}{4\epsilon^4}+\frac{\pi^2}{24\epsilon^2}+\frac{8\zeta(3)}{3\epsilon}+\frac{19\pi^4}{480}+O\left(\epsilon^1\right)\right)$$

UT and dlog

UT basis is also good for numeric computations

Uniformly transcendental (UT) basis



building blocks (LEGO) for analytic amplitude

$$\mathcal{T}(\log) = 1, \mathcal{T}(\pi) = 1, \mathcal{T}(\zeta_n) = n, \mathcal{T}(\mathrm{Li}_n) = n, \ldots, \mathcal{T}(f_1 f_2) = \mathcal{T}(f_1) + \mathcal{T}(f_2)$$

$$I = (\text{overall normalization}) \times \sum_{k=0}^{\infty} \epsilon^k f_k, \quad \mathcal{T}(f_k) = k$$

$$\tilde{I} = T(\epsilon) I, \qquad \frac{\partial}{\partial x_i} \tilde{I} = \epsilon\, A_i \tilde{I}$$

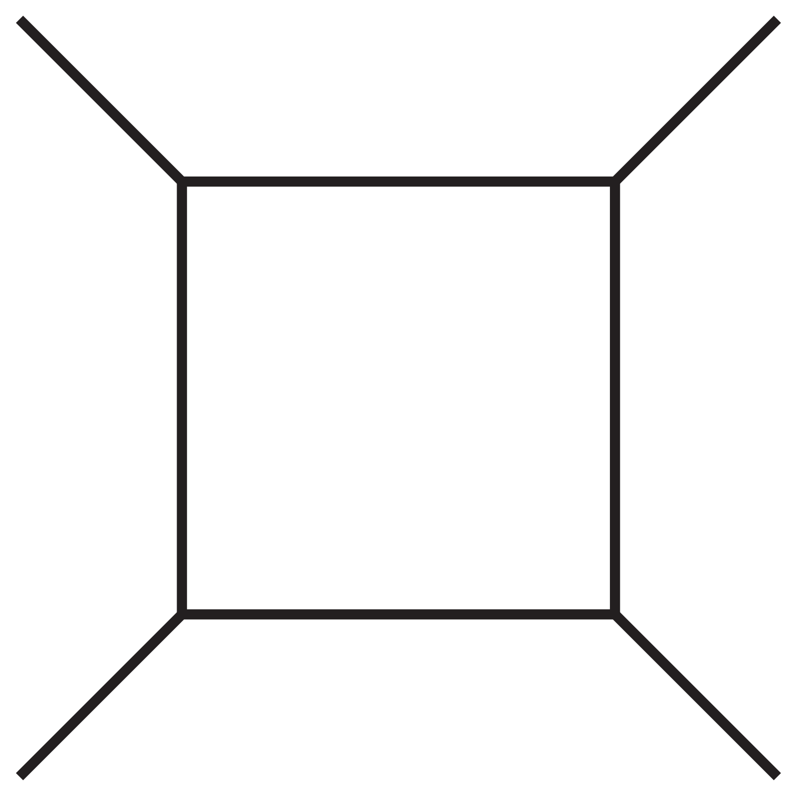
Differential equation in UT basis is extremely simple
Feynman integrals become an iterated integration of rational functions

$$\tilde{I}(x) = P \exp\left(\epsilon \int_{\mathcal{C}} dA\right) \tilde{I}(x_0)$$

Chen's (陈国才) iterated integrals

polylogarithms

To find UT basis: "dlog" approach



Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010

$$st\int d^4 l_1 \frac{1}{D_1 D_2 D_3 D_4} = \int d\log(\frac{F}{D_1}) \wedge d\log(\frac{F}{D_1}) \wedge d\log(\frac{F}{D_3}) \wedge d\log(\frac{F}{D_4})$$

"Usually", a dlog integrand is a a UT integral.

Wasser algorithm for dlog (2017)

Consider the partial fraction in x_1,

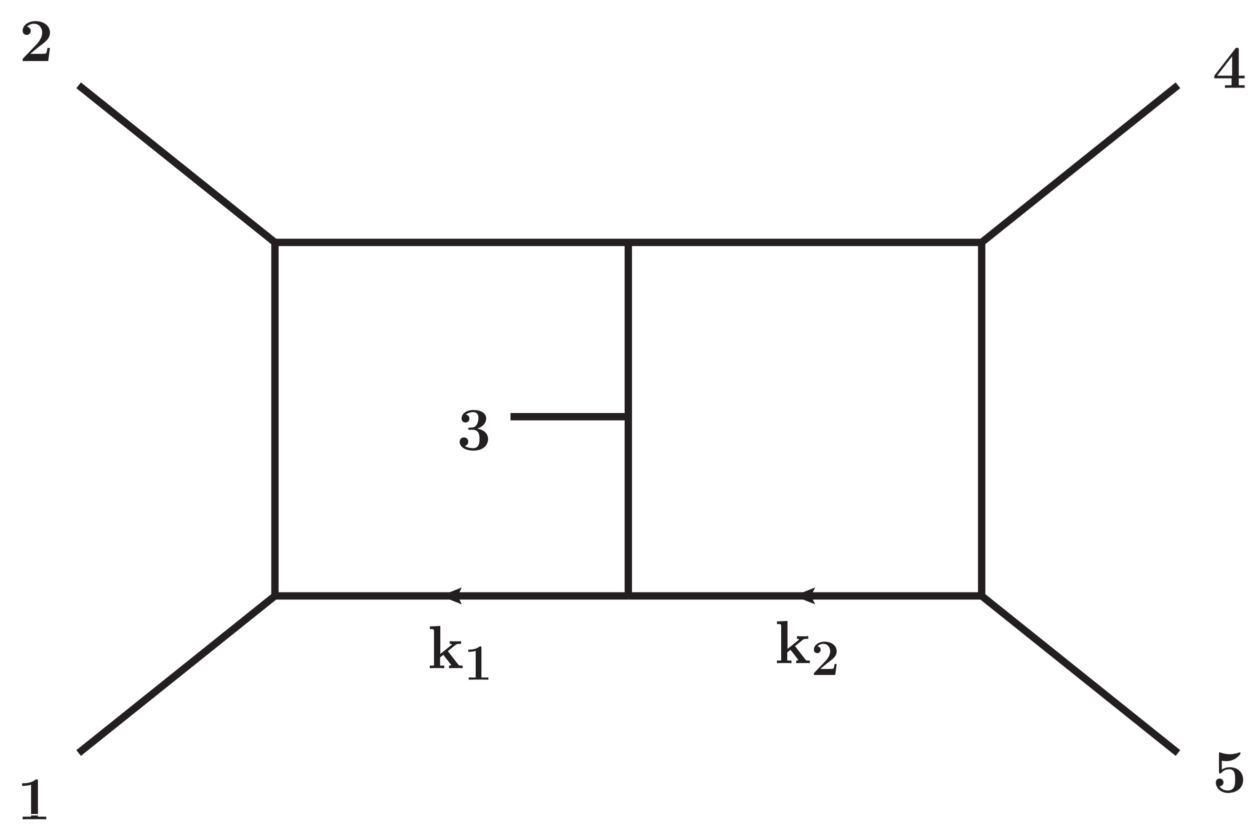
$$\sum_i \frac{dx}{x_1 - a_i} \wedge \Omega_i = \sum_i d\log(x_1 - a_i) \wedge \Omega_i$$

A long ansatz is necessary;
Sometimes this algorithm does not find all dlogs ...

Algebraic geometry approach ...

dlog algorithm from algebraic geometry viewpoint

YZ 2018



$$N = \sum f_\alpha(s_{ij}) \times (\text{scalar product})^\alpha$$

Require that

1. N has 1 or 0 4D leading singularity
2. f must be a polynomial of s_{ij}

$$\sum f_\alpha \times \text{L.S.}[(\text{Scalar Product})^\alpha] = (1, 0, \ldots 1, 0, \ldots)$$

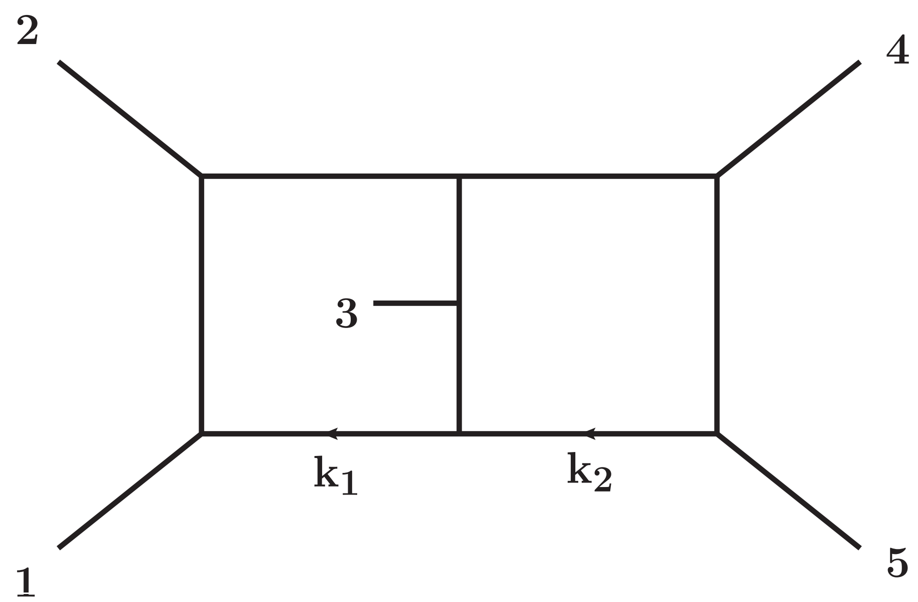
"Lift" problem in Module theory

easily solvable by **Singular** (computational algebraic geometry software)

Find missing dlog integral in Pascal's algorithm

dlog's are not always UT

dlog form analysis ignore the 4D vanishing terms, which are sometimes crucial



Bern, Herrmann, Litsey, Stankowicz, Trnka 2015 8 dlogs on the top

From IBP, there should 9 master integrals

$$B[1]=\langle 13\rangle\langle 24\rangle\Bigg([24][13]\Big(-k_2+\frac{[45]}{[24]}\lambda_5\tilde{\lambda}_2\Big)^2\Big(k_1-p_1-\frac{[23]}{[13]}\lambda_2\tilde{\lambda}_1\Big)^2,$$

$$+[14][23]\Big(-k_2+\frac{[45]}{[14]}\lambda_5\tilde{\lambda}_1\Big)^2\Big(k_1-p_2-\frac{[13]}{[23]}\lambda_1\tilde{\lambda}_2\Big)^2\Bigg),$$

$$B[2]=B[1]\Big|_{\substack{p_1\leftrightarrow p_2,\,p_4\leftrightarrow p_5\\ k_1\to -k_1+p_1+p_2,\,k_2\to -k_2-p_4-p_5}},$$

$$B[3]=B[1]\Big|_{\substack{p_1\leftrightarrow p_5,\,p_2\leftrightarrow p_4\\ k_1\to -k_2,\,k_2\to -k_1}},$$

$$B[4]=B[2]\Big|_{\substack{p_1\leftrightarrow p_5,\,p_2\leftrightarrow p_4\\ k_1\to -k_2,\,k_2\to -k_1}},$$

$$B[5]=B[1]^*,\quad B[6]=B[2]^*,\quad B[7]=B[3]^*,\quad B[8]=B[4]^*$$

NOT UT

for differential equation!

Bottleneck

From 4D leading singularity to D-dim singularity

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, Phys.Rev.Lett. 123 (2019), no. 4 041603

$$(B[1]+B[5]) + \frac{16 s_{45} G_{12}}{\epsilon_5^2} \times (-s_{12}s_{15} + s_{12}s_{23} + 2s_{12}s_{34} + s_{23}s_{34} + s_{15}s_{45} - s_{34}s_{45})$$

"Additives" terms to make a UT

$$G_{11} = G\begin{pmatrix} k_1, p_1, p_2, p_3, p_4 \\ k_1, p_1, p_2, p_3, p_4 \end{pmatrix}$$

$$G_{12} = G\begin{pmatrix} k_1, p_1, p_2, p_3, p_4 \\ k_2, p_1, p_2, p_3, p_4 \end{pmatrix}$$

$$G_{22} = G\begin{pmatrix} k_2, p_1, p_2, p_3, p_4 \\ k_2, p_1, p_2, p_3, p_4 \end{pmatrix} .$$

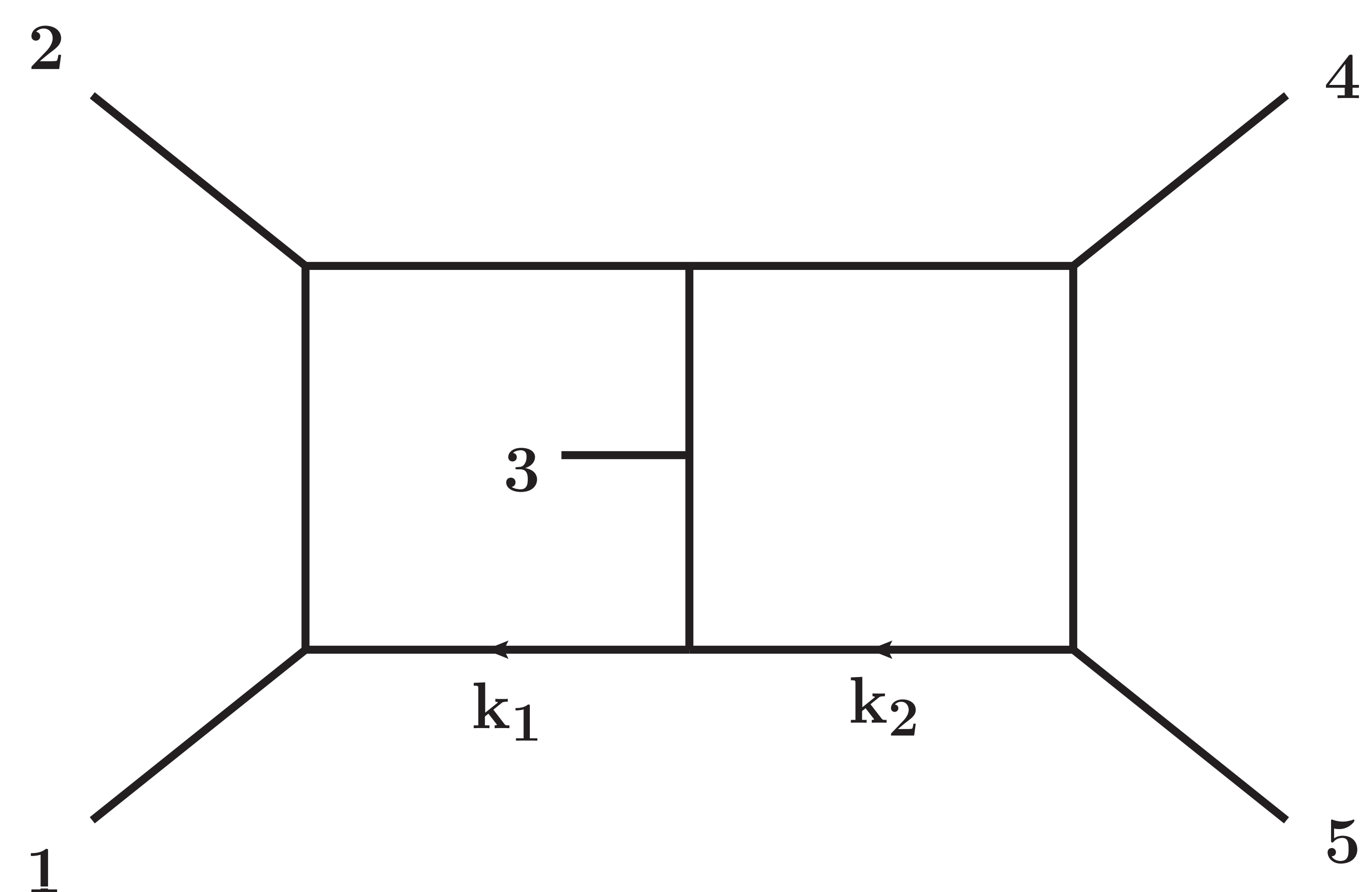
$$\frac{s_{45}}{\epsilon_5}(G_{11}-G_{12}), \quad \frac{s_{12}}{\epsilon_5}(G_{22}-G_{12}), \quad \frac{s_{12}-s_{45}}{\epsilon_5}G_{12}$$

4D vanishing terms, UT

Determined by Baikov representation, D-dimensional cuts

$$G(p_1 \ldots p_E)^{\frac{E+1-D}{2}} \int dz_1 \ldots dz_m \; G(k_1, k_2, p_1 \ldots p_E)^{\frac{L+E+1-D}{2}} \frac{1}{z_1^{\alpha_1} \ldots z_m^{\alpha_m}}$$

All UT basis found



$$\frac{\partial}{\partial x_i} I = A_i I$$

1.4 GB

$$\tilde{I} = T(\epsilon) I, \qquad \frac{\partial}{\partial x_i} \tilde{I} = \epsilon \tilde{A}_i \tilde{I}$$

5 MB

Now it is possible to solve differential equation

Further decomposition

Symbol: Goncharov, Spradlin, Vergu and Volovich

$$d\tilde{I}(s_{ij};\epsilon) = \epsilon dA(s_{ij})\tilde{I}(s_{ij};\epsilon)$$

$$d\tilde{I}(s_{ij};\epsilon) = \epsilon\left(\sum_{k=1}^{31} a_k d\log W_k(s_{ij})\right)\tilde{I}(s_{ij};\epsilon)$$

31 (108,108) matrices with rational number entries

symbol letters

$$\begin{aligned}
W_1 &= v_1\,, & W_6 &= v_3+v_4\,, & W_{11} &= v_1-v_4\,, & W_{16} &= v_1+v_2-v_4\,,\\
W_2 &= v_2\,, & W_7 &= v_4+v_5\,, & W_{12} &= v_2-v_5\,, & W_{17} &= v_2+v_3-v_5\,,\\
W_3 &= v_3\,, & W_8 &= v_5+v_1\,, & W_{13} &= v_3-v_1\,, & W_{18} &= v_3+v_4-v_1\,,\\
W_4 &= v_4\,, & W_9 &= v_1+v_2\,, & W_{14} &= v_4-v_2\,, & W_{19} &= v_4+v_5-v_2\,,\\
W_5 &= v_5\,, & W_{10} &= v_2+v_3\,, & W_{15} &= v_5-v_3\,, & W_{20} &= v_5+v_1-v_3\,,
\end{aligned}$$

$$v_1 = s_{12},\ v_2 = s_{23},\ v_3 = s_{34},\ v_4 = s_{45},\ v_5 = s_{15}$$

$$\begin{aligned}
W_{21} &= v_3+v_4-v_1-v_2\,, & W_{26} &= \frac{v_1v_2-v_2v_3+v_3v_4-v_1v_5-v_4v_5-\sqrt{\Delta}}{v_1v_2-v_2v_3+v_3v_4-v_1v_5-v_4v_5+\sqrt{\Delta}}\,,\\
W_{22} &= v_4+v_5-v_2-v_3\,, & W_{27} &= \frac{-v_1v_2+v_2v_3-v_3v_4-v_1v_5+v_4v_5-\sqrt{\Delta}}{-v_1v_2+v_2v_3-v_3v_4-v_1v_5+v_4v_5+\sqrt{\Delta}}\,,\\
W_{23} &= v_5+v_1-v_3-v_4\,, & W_{28} &= \frac{-v_1v_2-v_2v_3+v_3v_4+v_1v_5-v_4v_5-\sqrt{\Delta}}{-v_1v_2-v_2v_3+v_3v_4+v_1v_5-v_4v_5+\sqrt{\Delta}}\,,\\
W_{24} &= v_1+v_2-v_4-v_5\,, & W_{29} &= \frac{v_1v_2-v_2v_3-v_3v_4-v_1v_5+v_4v_5-\sqrt{\Delta}}{v_1v_2-v_2v_3-v_3v_4-v_1v_5+v_4v_5+\sqrt{\Delta}}\,,\\
W_{25} &= v_2+v_3-v_5-v_1\,, & W_{30} &= \frac{-v_1v_2+v_2v_3-v_3v_4+v_1v_5-v_4v_5-\sqrt{\Delta}}{-v_1v_2+v_2v_3-v_3v_4+v_1v_5-v_4v_5+\sqrt{\Delta}}\,,
\end{aligned}$$

$$W_{31} = \sqrt{\Delta}\,.$$

Solving canonical differential equation

$$\tilde{I}(s_{ij},\epsilon) = \epsilon^{-4}\sum_{m}^{\infty}\epsilon^{m}\tilde{I}^{(m)}(s_{ij})$$

leading terms are rational numbers

$$\epsilon^{4}\tilde{I}(s_{ij},\epsilon) = B^{(0)} + \epsilon\left(B^{(1)} + \int_{\gamma} dA(s_{ij})B^{(0)}\right) + \epsilon^{2}\left(B^{(2)} + \int_{\gamma} dA(s_{ij})\left(B^{(1)} + \int_{\gamma'} dA(s_{ij})B^{(0)}\right)\right) + \ldots$$

boundary value

$$\epsilon^{4}\tilde{I}(e_{ij},\epsilon) = \sum_{m=0}^{\infty}\epsilon^{m}B^{(m)}$$

boundary point

we choose the boundary point for a physical region

$$\{e_{12}, e_{23}, e_{34}, e_{45}, e_{15}\} = \{3, -1, 1, 1, -1\}$$

Boundary value

- Many integrals (from sub-diagrams) are known analytically
- "dlog" integral with $\varepsilon<0$, is FINITE even if a symbol letter vanishes

These two conditions usually determine a boundary value analytically.

All 2-loop 5-point massless integrals are analytically evaluated
Goncharov polylogarithms

$$G(\underbrace{0,\ldots,0}_{k};z) = \frac{1}{k!}\big(\log z\big)^k, \qquad G(a_1,\ldots,a_k;z) = \int_0^z \frac{dt}{t-a_1}G(a_2,\ldots,a_k;t)$$

implemented in **Ginac**

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, Phys.Rev.Lett. 123 (2019), no. 4 041603

Why analytic integrals?

numeric evaluation with **pySecDec**

- Must be evaluated in 6-2**ε** dim, then converted back to 4-2**ε** dim by IBPs
- GPU is necessary

NVIDIA Tesla V100 GPUs

×8

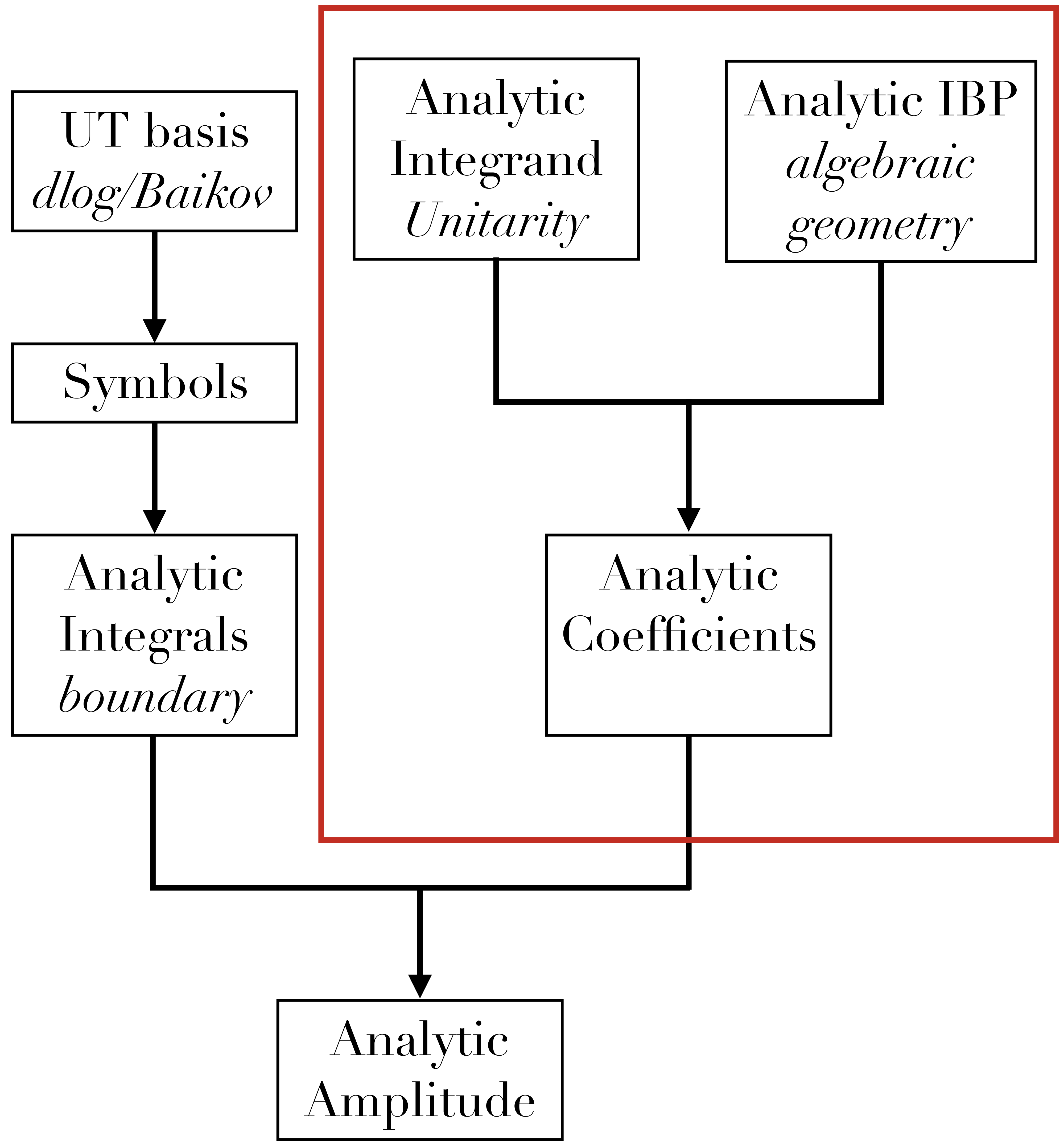
1 week to get one numeric point
error estimated to be ~0.5%

Analytic with our result

~ minutes with one CPU to get 50 digits, for one point

Assembly of Amplitudes

IBP with algebraic geometry
Finite field reconstruction



Integral reduction

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = 0$$

Chetyrkin, Tkachov 1981
Laporta 2001

Integration-by-Parts (IBP) reduction

Laporta

FIRE (Smirnov)
Reduze2 (von Manteuffel, Studerus)
LiteRed (Lee)
Kira (Maierhofer, Usovitsch, Uwer)

IBP with algebraic geometry

syzygy (Gluza,Kajda, Kosower 2010)
module intersection (Larsen, YZ 2016)

Non-derivative approach

η expansion for Feynman integrals, Liu Ma 2018
Intersection theory, Mastrolia et al. 2018

Module Intersection

module intersection (Larsen, YZ 2016)

IBPs in Baikov Rep.

$$0 = \left(\prod_{i=1}^{k} \int dz_i\right) \sum_{j=1}^{k} \frac{\partial}{\partial z_j} \left(a_j(z) \det(S)^{\frac{D-L-E-1}{2}} \frac{1}{z_1 \ldots z_m} \right)$$

Polynomials!

Require

1. no shifted exponent: $\sum_{j=1}^{k} a_j(z) \frac{\partial F}{\partial z_j} + \beta(z) F = 0$ These $\big(a_1(z), \ldots a_k(z)\big)$ form a module $M_1 \subset R^k$.

2. no doubled propagator: $a_i(z) \in \langle z_i \rangle, \quad 1 \le i \le m$ These $\big(a_1(z), \ldots a_k(z)\big)$ form a module $M_2 \subset R^k$.

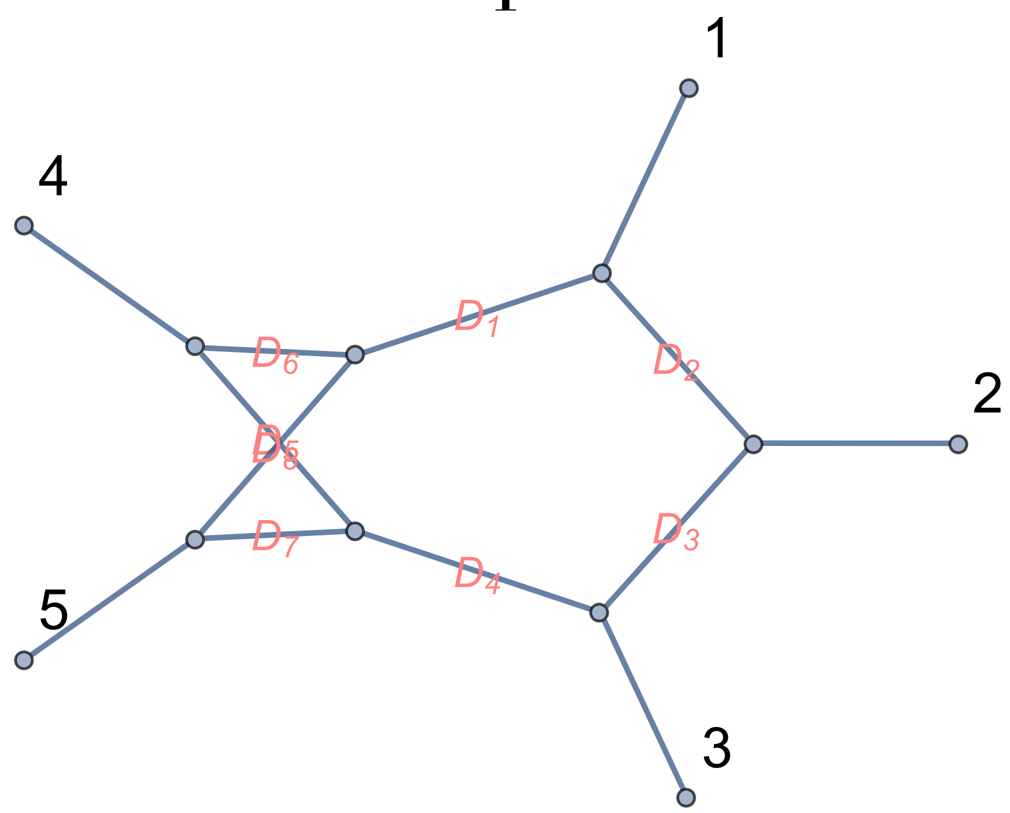
$$\boxed{M_1 \cap M_2}$$

Intersection of two modules

Solvable by **Singular** with the localization trick
Dramatically reduce the number of IBP relations

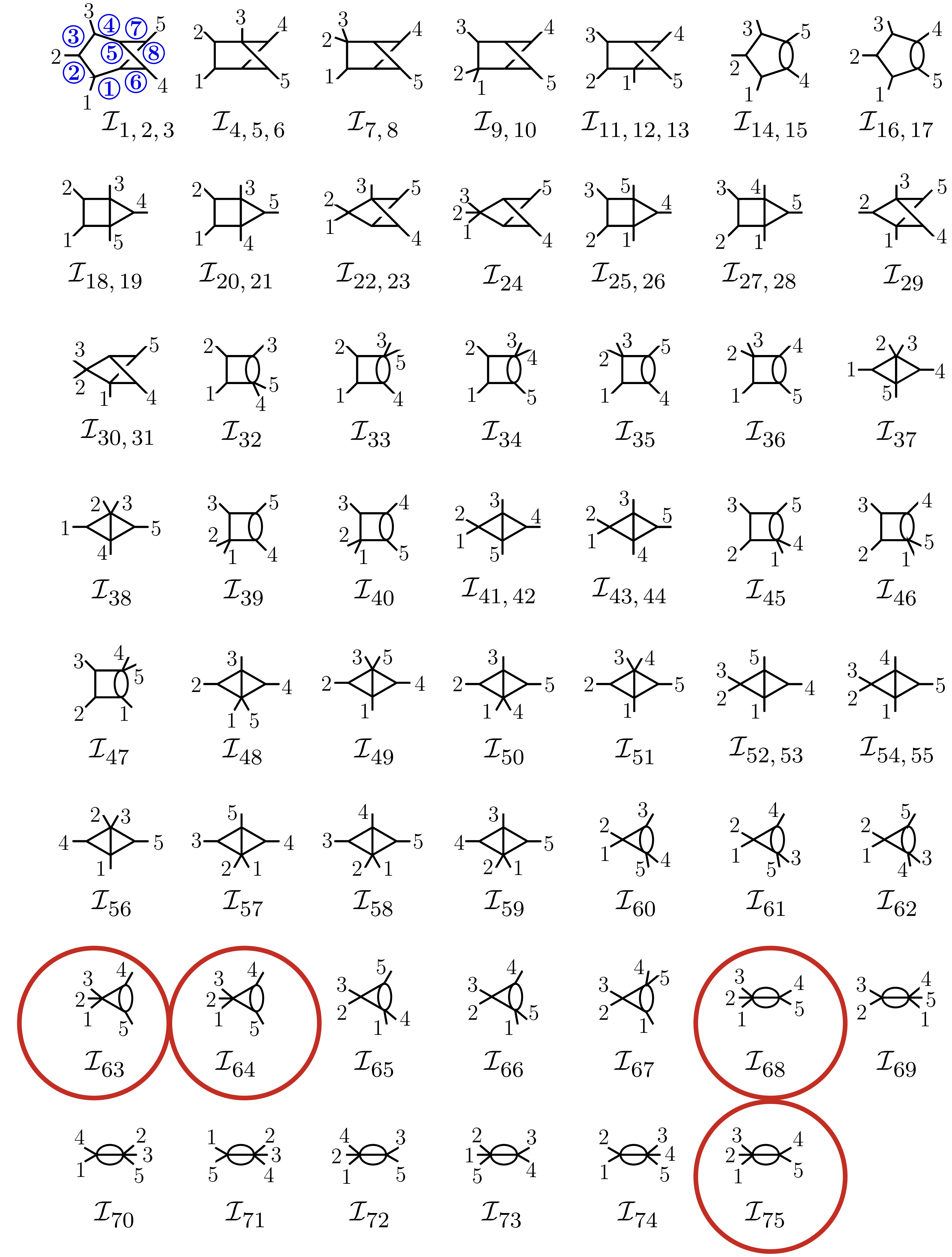
Module Intersection IBP

5 scales + spacetime dim



up to degree 4

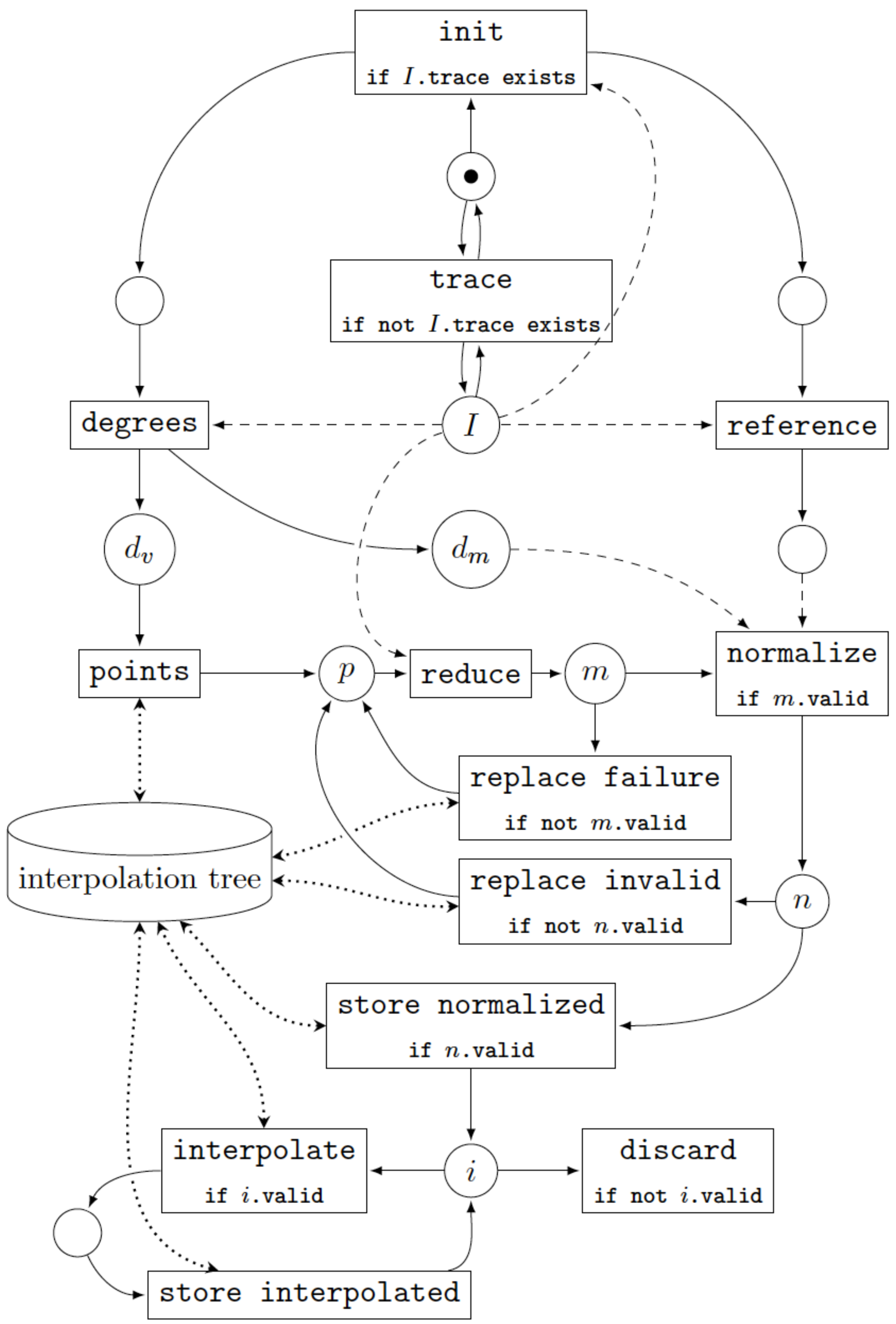
IBP



Boehm, Schoenemann, Georgoudis, Larsen, YZ
JHEP 1809 (2018) 024

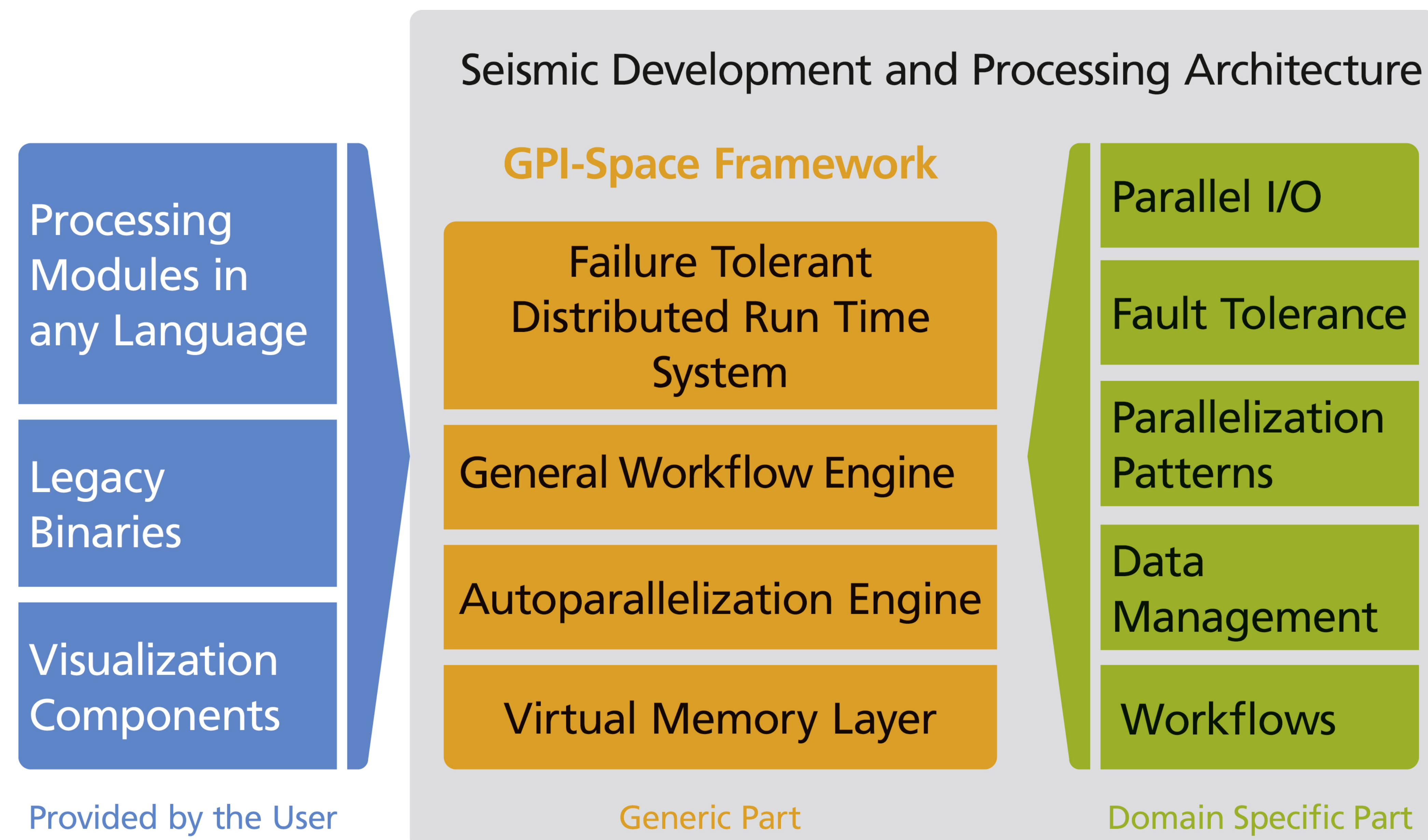
could not be done with FIRE or Kira

Module Intersection + Petri Net



Petri Net: a graphic rep. of discrete event dynamic system (Carl Adam Petri)

Automated with the framework GPI-space



Fraunhofer Institute for Industrial Mathematics
Kaiserslautern

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, YZ
2019

Color structure for five-gluon amplitudes

$$\mathcal{A}_5^{(1)} = \sum_{\lambda=1}^{12} N_c A_\lambda^{(1,0)} T_\lambda + \sum_{\lambda=13}^{22} A_\lambda^{(1,1)} T_\lambda$$

$$\mathcal{A}_5^{(2)} = \sum_{\lambda=1}^{12} \left(N_c^2 A_\lambda^{(2,0)} + A_\lambda^{(2,2)} \right) T_\lambda + \sum_{\lambda=13}^{22} N_c A_\lambda^{(2,1)} T_\lambda$$

$$\begin{aligned}
T_1 &= [\mathrm{Tr}(12345) - \mathrm{Tr}(15432)], & T_7 &= [\mathrm{Tr}(12543) - \mathrm{Tr}(13452)], \\
T_2 &= [\mathrm{Tr}(14325) - \mathrm{Tr}(15234)], & T_8 &= [\mathrm{Tr}(14523) - \mathrm{Tr}(13254)], \\
T_3 &= [\mathrm{Tr}(13425) - \mathrm{Tr}(15243)], & T_9 &= [\mathrm{Tr}(13524) - \mathrm{Tr}(14253)], \\
T_4 &= [\mathrm{Tr}(12435) - \mathrm{Tr}(15342)], & T_{10} &= [\mathrm{Tr}(12534) - \mathrm{Tr}(14352)], \\
T_5 &= [\mathrm{Tr}(14235) - \mathrm{Tr}(15324)], & T_{11} &= [\mathrm{Tr}(14532) - \mathrm{Tr}(12354)], \\
T_6 &= [\mathrm{Tr}(13245) - \mathrm{Tr}(15423)], & T_{12} &= [\mathrm{Tr}(13542) - \mathrm{Tr}(12453)],
\end{aligned}$$

Edison, Naculich, 2012

and

$$\begin{aligned}
T_{13} &= \mathrm{Tr}(12)\,[\mathrm{Tr}(345) - \mathrm{Tr}(543)], & T_{18} &= \mathrm{Tr}(13)\,[\mathrm{Tr}(245) - \mathrm{Tr}(542)], \\
T_{14} &= \mathrm{Tr}(23)\,[\mathrm{Tr}(451) - \mathrm{Tr}(154)], & T_{19} &= \mathrm{Tr}(24)\,[\mathrm{Tr}(351) - \mathrm{Tr}(153)], \\
T_{15} &= \mathrm{Tr}(34)\,[\mathrm{Tr}(512) - \mathrm{Tr}(215)], & T_{20} &= \mathrm{Tr}(35)\,[\mathrm{Tr}(412) - \mathrm{Tr}(214)], \\
T_{16} &= \mathrm{Tr}(45)\,[\mathrm{Tr}(123) - \mathrm{Tr}(321)], & T_{21} &= \mathrm{Tr}(41)\,[\mathrm{Tr}(523) - \mathrm{Tr}(325)], \\
T_{17} &= \mathrm{Tr}(51)\,[\mathrm{Tr}(234) - \mathrm{Tr}(432)], & T_{22} &= \mathrm{Tr}(52)\,[\mathrm{Tr}(134) - \mathrm{Tr}(431)],
\end{aligned}$$

Nonplanar N=4 amplitude (symbols)

Integrand: Carrasco-Johansson, 2011

Park-Taylor factor → weight-w function

$$A_{\lambda}^{(2,k)} = \frac{1}{\epsilon^4} \sum_{w=0}^{4} \epsilon^w \sum_{i=1}^{6} \mathrm{PT}_i f_{w,i}^{(k,\lambda)} + \mathcal{O}(\epsilon) \,,$$

"Most" terms in this amplitude are determined by the infrared structure (Catani's formula), only the weight-4 double trace term is non-trivial.

only 500 KB

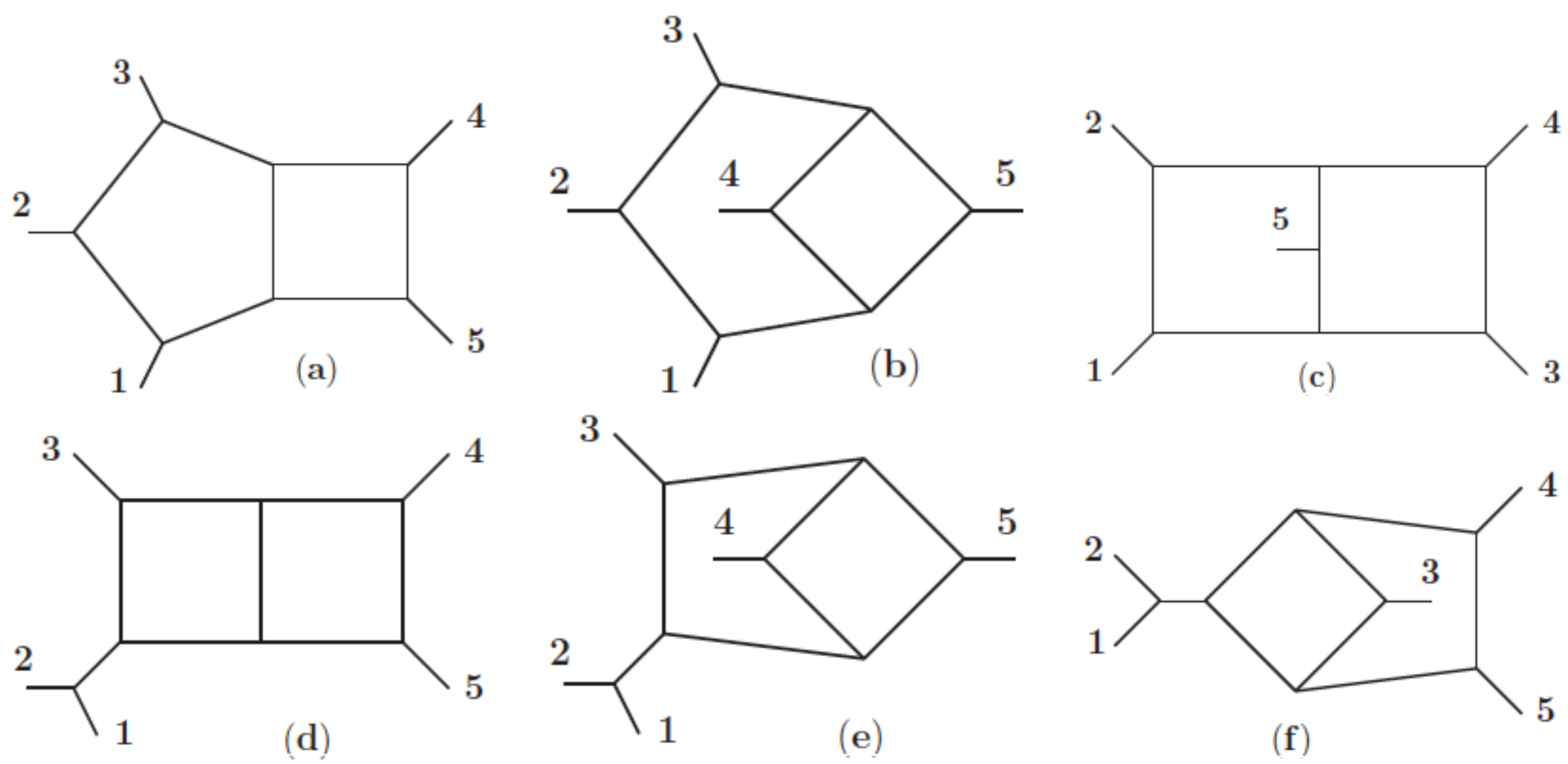
$$\text{hard function} = \sum_{S_5} \mathrm{PT}_1 T_{13} g_{\text{seed}}$$

PhysRevLett.122.121602
Gehrmann, Henn, Chicherin
Wasser, **YZ**, Zoia

Nonplanar N=8 Supergravity amplitude (symbol)

Bern-Carrasco-Johansson relation 2008

Carrasco-Johansson, 2011



$$(N_i^{\text{sYM}})^2 = N_i^{\text{Sugra}}$$

$$\mathcal{M}_5^{(2)} = \frac{1}{\epsilon^2} \sum_{j=1}^{515} r_j^{(2)} \sum_{w=0}^{2} \epsilon^w \, g_j^{(w)} + \mathcal{O}(\epsilon)\,,$$

After infrared subtraction

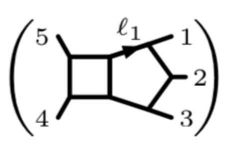
$$\boxed{\mathcal{H}_5^{(2)} = \sum_{S_5} r_{\text{seed}} \, h_5^{(2)}\,,}$$

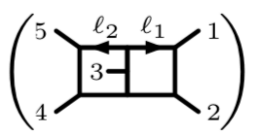
key formula
inspired by
my Ph.D. thesis
supervised by Henry Tye

$$r_{\text{seed}} = s_{12}s_{23}s_{34}s_{45}\text{PT}(12345)\text{PT}(21435)\,,$$

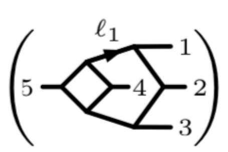
2-loop 5-point +++++ pure-YM amplitude

Badger, Frellesvig, **YZ**, 2013
Badger, Mogull, Ochirov, O'Connell 2015

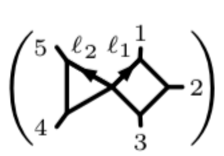
$\Delta_{431} = \Delta\Big($  $\Big) = -\dfrac{s_{12}s_{23}s_{45}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}\left(\mathrm{tr}_+(1345)(\ell_1+p_5)^2 + s_{15}s_{34}s_{45}\right),$

$\Delta_{332} = \Delta\Big($  $\Big) = \dfrac{s_{12}s_{45}F_1}{4\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}$

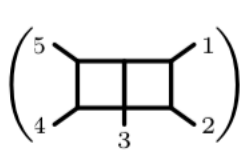
$$\times\Big(s_{23}\mathrm{tr}_+(1345)\big(2s_{12} - 4\,\ell_1\cdot(p_5-p_4) + 2(\ell_1-\ell_2)\cdot p_3\big) - s_{34}\mathrm{tr}_+(1235)\big(2s_{45} - 4\,\ell_2\cdot(p_1-p_2) - 2(\ell_1-\ell_2)\cdot p_3\big) - 4s_{23}s_{34}s_{15}(\ell_1-\ell_2)\cdot p_3\Big),$$

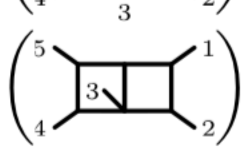
$\Delta_{422} = \Delta\Big($  $\Big) = -\dfrac{s_{12}s_{23}s_{45}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}$

$$\times\left(\mathrm{tr}_+(1345)\left(\ell_1\cdot(p_5-p_4) - \frac{s_{45}}{2}\right) + s_{15}s_{34}s_{45}\right).$$

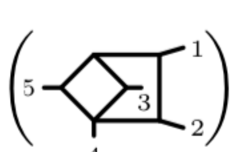
$\Delta_{430} = \Delta\Big($  $\Big) = -\dfrac{s_{12}\mathrm{tr}_+(1345)}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{13}}(2(\ell_1\cdot\omega_{123}) + s_{23})$

$$\times\left(F_2 + F_3\frac{(\ell_1+\ell_2)^2 + s_{45}}{s_{45}}\right),$$

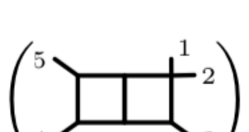
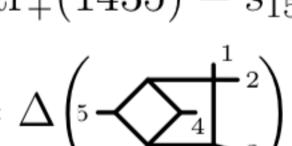
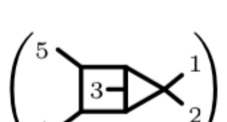
$\Delta_{331;5L_1} = \Delta\Big($  $\Big) = \dfrac{s_{12}s_{23}s_{34}s_{45}s_{51}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5},$

$\Delta_{331;5L_2} = \Delta\Big($  $\Big) = -\dfrac{s_{12}s_{45}F_1}{4\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}$

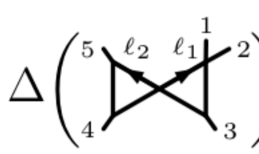
$$\times\left(2s_{23}s_{34}s_{15} - s_{23}\mathrm{tr}_+(1345) + s_{34}\mathrm{tr}_+(1235)\right),$$

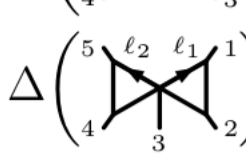
$\Delta_{322;5L_1} = \Delta\Big($  $\Big) = -\dfrac{s_{12}F_1}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}$

$$\times\left(s_{23}s_{45}\mathrm{tr}_+(1435) - s_{15}s_{34}\mathrm{tr}_+(2453)\right),$$

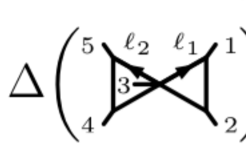
$\Delta_{331;M_1} = \Delta\Big($  $\Big) = \Delta_{322;M_1} = \Delta\Big($  $\Big) = \Delta_{232;M_1} = \Delta\Big($  $\Big)$

$$= -\frac{s_{34}s_{45}^2\mathrm{tr}_+(1235)F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5},$$

$\Delta_{330;M_1} = \Delta\Big($  $\Big) = -\dfrac{(s_{45}-s_{12})\mathrm{tr}_+(1345)}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{13}}\left(F_2 + F_3\dfrac{(\ell_1+\ell_2)^2 + s_{45}}{s_{45}}\right),$

$\Delta_{330;5L_1} = \Delta\Big($  $\Big) = -\dfrac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$

$$\times\Bigg\{\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)\times\left(F_2 + F_3\frac{4(\ell_1\cdot p_3)(\ell_2\cdot p_3) + (\ell_1+\ell_2)^2(s_{12}+s_{45}) + s_{12}s_{45}}{s_{12}s_{45}}\right)$$
$$+ F_3\Bigg[(\ell_1+\ell_2)^2 s_{15} + \mathrm{tr}_+(1235)\left(\frac{(\ell_1+\ell_2)^2}{2s_{35}} - \frac{\ell_1\cdot p_3}{s_{12}}\left(1 + \frac{2(\ell_2\cdot\omega_{543})}{s_{35}} + \frac{s_{12}-s_{45}}{s_{35}s_{45}}(\ell_2-p_5)^2\right)\right)$$
$$+ \mathrm{tr}_+(1345)\left(\frac{(\ell_1+\ell_2)^2}{2s_{13}} - \frac{\ell_2\cdot p_3}{s_{45}}\left(1 + \frac{2(\ell_1\cdot\omega_{123})}{s_{13}} + \frac{s_{45}-s_{12}}{s_{12}s_{13}}(\ell_1-p_1)^2\right)\right)\Bigg]\Bigg\},$$

$\Delta_{330;5L_2} = \Delta\Big($  $\Big) = \dfrac{F_3}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{12}}$

$$\times\Bigg((s_{45}-s_{12})\mathrm{tr}_+(1245) - \left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)2(\ell_1\cdot p_3)$$
$$- \frac{s_{45}\mathrm{tr}_+(1235)}{s_{35}}\left(2(\ell_2\cdot\omega_{543}) + \frac{s_{12}-s_{45}}{s_{45}}(\ell_2-p_5)^2\right) + \frac{s_{12}\mathrm{tr}_+(1345)}{s_{13}}\left(2(\ell_1\cdot\omega_{123}) + \frac{s_{45}-s_{12}}{s_{12}}(\ell_1-p_1)^2\right)\Bigg).$$

2-loop 5-point +++++ pure-YM amplitude

numerator degree-5 IBP needed (impossible by current analytic IBP method)
indirect finite-field fitting for the amplitude (after IBP) is applicable

All weight-3, weight-4 part of the amplitude cancels out

$$\mathcal{H}^{(2)} = \sum_{S_5/S_{T_1}} T_1\, \mathcal{H}_1^{(2)} + \sum_{S_5/S_{T_{13}}} T_{13}\, \mathcal{H}_{13}^{(2)}$$

$$\mathcal{H}_1^{(2,0)} = \sum_{S_{\mathcal{T}_1}} \left\{ -\kappa \frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} I_{123;45} + \kappa^2 \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \left[5\, s_{12}s_{23} + s_{12}s_{34} + \frac{\mathrm{tr}_+^2(1245)}{s_{12}s_{45}} \right] \right\},$$

$$\mathcal{H}_{13}^{(2,1)} = \sum_{S_{\mathcal{T}_{13}}} \left\{ \kappa \frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} \left[I_{234;15} + I_{243;15} - I_{324;15} - 4\, I_{345;12} - 4\, I_{354;12} - 4\, I_{435;12} \right] - 6\,\kappa^2 \left[\frac{s_{23}\,\mathrm{tr}_-(1345)}{s_{34}\,\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} - \frac{3}{2} \frac{[12]^2}{\langle 34\rangle\langle 45\rangle\langle 53\rangle} \right] \right\},$$

$$I_{123;45} = \mathrm{Li}_2\left(1 - s_{12}/s_{45}\right) + \mathrm{Li}_2\left(1 - s_{23}/s_{45}\right) + \log^2\left(s_{12}/s_{23}\right) + \pi^2/6\,.$$

Summary

- Systematic way of finding UT basis
- Novel approach of determining integral boundary condition
- Novel practical IBP reduction methods
- New amplitudes calculated
  - 2-loop 5-point N=4 Super-Yang-Mills amplitude
  - 2-loop 5-point N=8 Supergravity amplitude
  - 2-loop 5-point +++++ YM amplitude

… …

Towards a revolution of
2-loop, 2 to 3 scattering amplitude computation

Infrared structure

(Catani's dipole formula 98)

$$A(s_{ij},\epsilon) = \mathbf{Z}(s_{ij},\epsilon)A^f(s_{ij},\epsilon) \qquad \mathbf{Z}(s_{ij},\epsilon) = \exp g^2\left(\frac{\mathbf{D}_0}{2\epsilon^2} - \frac{\mathbf{D}}{2\epsilon}\right)$$

$$\mathbf{D}_0 = \sum_{i\neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j\,, \quad \mathbf{D} = \sum_{i\neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j \log\left(-\frac{s_{ij}}{\mu^2}\right),$$

$\mathbf{T}_i$ is the adjoint action of $su(N_c)$ Lie algebra.

More references

UT integral search

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Gluza, Kajda, Kosower, 1009.0472
Larsen, **YZ**, 1511.01071
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